

Generation of Ion-acoustic Super-periodic Waves in Ultra-relativistic Degenerate Quantum Plasma

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Abstract

In this article, the pseudopotential technique of Sagdeev is applied to study the evolution of large-amplitude ion acoustic waves in electron-positron-ion (e-p-i) plasma. The electrons and positrons energies are taken in the ultra-relativistic region. Periodic, rarefactive and compressive solitary and super-periodic waves are predicted in the plasma system. All probable phase plots, including super-periodic, periodic, homoclinic trajectories are obtained using the bifurcation analysis of dynamical system. The effects of phase speed of ion acoustic traveling wave, chemical potential and other plasma parameters on the characteristic properties of the periodic wave, super periodic wave and the Sagdeev pseudopotential are investigated. The findings of this paper may be valuable in understanding the fundamental features of nonlinear waves in dense celestial bodies such as neutron stars and white dwarfs.

Keywords: bifurcation analysis, super-nonlinear wave, degenerate unmagnetized ultra-relativistic plasma, Sagdeev's pseudopotential approach.

Introduction

Recently, quantum degenerate plasma (QDP) attracted the attention of many researchers (P. A. Markowich et al, 1990; L. K. Ang et al, 2003; T. C. Killian, 2006; M. Marklund and P.K. Shukla, 2006; S. H. Glenzer et al, 2007; G. Brodin et al, 2008; M. Marklund et al, 2008). From quantum mechanical point of view, the

inter-particle distance for degenerate particles is shorter than the de-Broglie wavelength. As examples of QDPs, white dwarfs are mostly made up of massive components like hydrogen, helium, carbon, and oxygen at core, with lighter elements near the crust (S. L. Shapiro et al, 1983; D. Koester and G. Chanmugam, 1990; A. A. Mamun and P. K. Shukla, 2010). Furthermore, White dwarf interiors were observed and theoretically simulated to feature a dense solid enveloped by degenerate particles

(such as electrons and positrons) and ions (D. Koester and Astron, 2002; R. S. Fletcher et al, 2002; E. Garcia-Berro et al, 2010; A. Witze, 2014; A. Vanderburg et al, 2015; S. Sultana and R. Schlickeiser, 2018). In those interstellar objects, the degenerate electron number density is very high. Chandrasekhar provides a mathematical explanation for the equation of state for degenerate electrons in such compact objects for two limits, namely the nonrelativistic and ultra-relativistic limits. Regarding this, Chandrasekhar obtained the mathematical criteria for the white dwarf (S. Chandrasekhar, 1931; S. Chandrasekhar, 1931; S. Chandrasekhar et al, 1935) using Fermi–Dirac statistics for electrons. In the case of ultra-relativistic degenerate dense plasma, which exist in interstellar compact objects like white dwarfs, the total energy of the fermions (electrons) is much larger than relativistic energy i.e., $E \gg mc^2$.

Evidently, the dense e-p-i plasma is one of the most important and interesting types of QDPs. Many researchers are interested in studying the excitation of nonlinear acoustic modes (F. Haas et al, 2003; S. A. Khanand and W. Masood, 2008; S. Mahmood and A. Mushtaq, 2008; E. F. El-Shamy et al, 2009; W. Masood et al, 2009; S. K. El-Labany et al, 2010; F. Haas, 2016; S. Islam et al, 2017) in QDP since it is essential for understanding the main features of compact components in astrophysical environment (F. C. Michel, 1991; S. Ali et al, 2007; A. Rahman et al, 2013; E. F. El-Shamy, 2015; E. F. El-Shamy et al, 2016). QDP model with zero-temperature Fermi gas can be suitable to describe nonlinear acoustic waves (F. Haas et al, 2003; S. A. Khanand and W. Masood, 2008; S. Mahmood and A. Mushtaq, 2008; E. F. El-Shamy et al, 2009; W. Masood et al, 2009; S. K. El-Labany et al, 2010; F. Haas, 2016; S. Islam et al, 2017). Based on the assumption of nonzero temperature, the theoretical simulation of QDP models has been improved by Dubinov and his research team (A. E. Dubinov, 2007; A. E. Dubinov and A. A. Dubinova, 2008; A. E. Dubinov and M. A. Sazonkin, 2009; A. E. Dubinov et al, 2010; A. E. Dubinov and M. A. Sazonkin, 2010; A. E. Dubinov and I. N. Kitaev, 2014). Later, the modified model has been used in the study of propagation (A. El-Depsy and M. M. Selim, 2016), instability (E. F. El-Shamy et al, 2020), and overtaking collisions (E. F. El-Shamy et al, 2020) of nonlinear isothermal ion-acoustic waves in the

QDPs, where they are stated to be more realistic quantum plasma models.

However, many earlier studies are restricted to the physical nature of the periodic and nonlinear solitary traveling waves in the QDPs. Recently, Dubinov et al. (A. E. Dubinov et al, 2012) investigated a new class of nonlinear waves in plasmas which are super-nonlinear waves. They presented a topological classification of such super-nonlinear waves based on the dynamical system of the Hamiltonian. Numerous researchers lately become interested in studying these structures in various plasma systems. A multi-component nature of the plasma was revealed to be a crucial condition for the existence of these super-nonlinear waves. Superperiodic waves and supersolitons can exist in a three-species plasma. In this direction, the well-known bifurcation method is essential to investigate nonlinear ion acoustic periodic and super-periodic waves for several plasma models (A. Saha, 2017), in particular QDPs. Interestingly, the bifurcation analysis has been widely used to investigate the physical nature of nonlinear waves for various plasma models in different plasma situations (U.K. Samanta et al, 2013; A. Saha and P. Chatterjee, 2014; B. Pal et al, 2015; A. Saha and P. Chatterjee, 2015; A. Saha et al, 2015; M. M. Selim et al, 2015; D.T. Patrice et al, 2017; S.K. El-Labany et al, 2018). Hence, the key objective of the simulation model in this study is to numerically obtain and solve the Hamiltonian system in order to characterize the physical behavior of nonlinear ion acoustic periodic and super-periodic waves in QDPs. We also studied the influence of different plasma parameters on these nonlinear waves and the Sagdeev pseudopotential. Moreover, the implications of the present work in white dwarfs are investigated.

This paper is arranged as: Equation of state for degenerate e-p Fermi gas is introduced in Section 2. Basic quantum hydrodynamics fluid equations are presented in Section 3. The description of the dynamical system is shown in Section 4. At the end, Section 5 contains the findings and comments. The conclusion is finally described in Section 6.

Equation of state for degenerate electron-positron Fermi gas

The density of states of a relativistic

Fermi gas with kinetic energy E in a volume V is given by (A. Saha et al, 2019)

$$G(E) = \frac{g_s V E}{2\pi^2 \hbar^3 c^3} \sqrt{E^2 - m^2 c^4}, \quad (1)$$

where $g_s (= 2s + 1)$ is the degenerate factor of states, s is the half-integer spin of fermions, m is the mass of the relativistic moving particle and c is the speed of light. The density of states for ultra-relativistic fermions, $E \gg mc^2$, can be expanded as

$$G(E) = \frac{V}{\pi^2 \hbar^3 c^3} \left(E^2 - \frac{1}{2} m^2 c^4 + \dots \right). \quad (2)$$

Hence, the number of particles of Fermi gas in this case is expressed as (A. Saha et al, 2019)

$$N(\mu, T) = \int_0^\infty \frac{G(E) dE}{e^{\frac{1}{k_B T}(E-\mu)} + 1}, \quad (3)$$

where μ is the chemical potential of fermions which equals to the Fermi energy at temperature, $T=0$ and K_B is the Boltzmann constant. Eq. (2) can be substituted into Eq. (3) to yield

$$n(\mu, T) = \frac{N(\mu, T)}{V} = \frac{1}{\pi^2 \hbar^3 c^3} \int_0^\infty \frac{\left(E^2 - \frac{1}{2} m^2 c^4 + \dots \right) dE}{e^{\frac{1}{k_B T}(E-\mu)} + 1}. \quad (4)$$

By using the Sommerfeld expansion and after some mathematical manipulations, the number density of the fermions can be obtained as (A. E. Dubinov and A. A. Dubinova, 2008; A. E. Dubinov and I. N. Kitaev, 2014; A. Saha et al, 2019; A. E. Dubinov et al, 2012)

$$n(\mu, T) = \frac{8\pi}{(hc)^3} \left(\frac{\mu^3}{3} + \left(\frac{\pi^2}{3} (K_B T)^2 - \frac{1}{2} m^2 c^4 \right) \mu + \dots \right) \quad (5)$$

Based on equation (5), the ultra-relativistic degenerate electron and positron distributions are roughly obtained as (A. Rahman et al, 2013; A. E. Dubinov, 2007; A. E. Dubinov and M. A. Sazonkin, 2010)

$$n_j(\mu_j, T_j) \approx \frac{8\pi}{(hc)^3} \left[\frac{\mu_j^3}{3} + \left(\frac{\pi^2}{3} (K_B T_j)^2 - \frac{1}{2} m_j^2 c^4 \right) \mu_j \right], \quad (6)$$

where $j = e$ and p for electron and positron, respectively. In the present study, we suppose that the compression-expansion wave process is isothermal i.e., the Fermi temperature for electrons and positrons, $T_{Fj} = \text{constant}$. In this situation, degenerate plasma is regarded to be collisionless and perfect (A. E. Dubinov and A. A. Dubinova, 2008; A. E. Dubinov and M. A. Sazonkin, 2009; A. E. Dubinov et al, 2010; A. E. Dubinov and I. N. Kitaev, 2014).

Furthermore, thermodynamic equilibrium can be produced due to uncorrelated Coulomb interaction between particles (A. E. Dubinov et al, 2010; A. E. Dubinov and M. A. Sazonkin, 2010).

Essentially, by considering zero mass of electrons and positrons ($m_j \rightarrow 0$) in the momentum equations, and performing some algebraic manipulation, the relationship between the chemical potential of electrons/positrons and the electrostatic wave potential ϕ can be obtained as (A. E. Dubinov et al, 2010; A. E. Dubinov and M. A. Sazonkin, 2010)

$$\mu_j = \mu_{0j} \pm e\phi. \quad (7)$$

Substituting the chemical potential μ_j in Eq. (6), we get

$$n_j = \frac{8\pi}{(hc)^3} \left[\frac{(\mu_{0j} \pm e\phi)^3}{3} + \left(\frac{\pi^2}{3} (K_B T_j)^2 - \frac{1}{2} m_j^2 c^4 \right) (\mu_{0j} \pm e\phi) \right], j = e, p. \quad (8)$$

In Eq. (8), μ_{0j} and $e\phi$ can be scaled by the Fermi energy, E_{Fj} while n_j is scaled by n_{0j} (i.e., $\mu_{0j} \rightarrow \frac{\mu_{0j}}{E_{Fj}}$, $\phi \rightarrow \frac{e\phi}{E_{Fe}}$ and $n_j \rightarrow \frac{n_j}{n_{0j}}$, where $E_{Fj} = (3\pi^2 n_{0j})^{\frac{1}{3}} \hbar c$). Hence, the unperturbed number densities of electrons and positrons can be obtained, substituting $\phi = 0$, in Eq. (8) to get

$$n_{0j} = \frac{8\pi E_{Fj}^3}{(\hbar c)^3} \left[\frac{\mu_{0j}^3}{3} + \mu_{0j} \left(\frac{\pi^2}{3} \frac{(K_B T_j)^2}{E_{Fj}^2} - \frac{m_j^2 c^4}{2E_{Fj}^2} \right) \right] = \frac{8\pi E_{Fj}^3}{(\hbar c)^3} C_{1j}, \quad (9)$$

where $C_{1j} = \left(\frac{\mu_{0j}^3}{3} + \mu_{0j} \left(\frac{\pi^2}{3} \frac{(K_B T_j)^2}{E_{Fj}^2} - \frac{m_j^2 c^4}{2E_{Fj}^2} \right) \right)$.

Governing equations of plasma model

In this article we considered an unmagnetized plasma system, consists of an ultrarelativistic degenerate inertialess electrons and non-degenerate inertial cold positive and negative ions as well as positrons. The ions are treated as classical fluid. The dynamics of the nonlinear ion acoustic wave (IAWs) spreading in such system can be studied based on; the continuity equations:

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} (n_i u_i) = 0, \quad (10a)$$

$$\frac{\partial n_n}{\partial t} + \frac{\partial}{\partial x} (n_n u_n) = 0. \quad (10b)$$

The momentum equations:

$$\left(\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x}\right) = -\frac{\partial \phi}{\partial x}, \tag{11a}$$

$$\left(\frac{\partial u_n}{\partial t} + u_n \frac{\partial u_n}{\partial x}\right) = M \frac{\partial \phi}{\partial x}, \tag{11b}$$

where $M = \frac{m_{ip}}{m_{in}}$, and m_{ip} is the mass of positive ions and m_{in} is the mass of negative ions, respectively. Poisson's equation for this system can be expressed as

$$\frac{\partial^2 \phi}{\partial x^2} = (\beta n_e - \alpha n_p - n_i + \gamma n_n), \tag{12}$$

where $\beta = \frac{n_{oe}}{n_{oi}}$, $\gamma = \frac{n_{on}}{n_{oi}}$ and $\alpha = \frac{n_{op}}{n_{oi}}$. Here, n_s, u_s ($s = i, n$) are the number density and velocity of the inertial cold, positive and negative ions respectively. Based upon Eqs. (8) and (9), the normalized relativistic degenerate electrons and positrons number densities, respectively are expressed as

$$n_e = \frac{1}{C_{1e}} \left(C_{1e} + C_{2e}\phi + \mu_{0e}\phi^2 + \frac{\phi^3}{3} \right), \tag{13a}$$

$$n_p = \frac{1}{C_{1p}} \left(C_{1p} - C_{2p}\sigma_F\phi + \mu_{0p}\sigma_F^2\phi^2 - \sigma_F^3\frac{\phi^3}{3} \right), \tag{13b}$$

where

$$C_{2j} = \mu_{0j}^2 + \left(\frac{\pi^2 (K_B T_j)^2}{3 E_{Fj}^2} - \frac{m_j^2 c^4}{2 E_{Fj}^2} \right), \tag{9}$$

$j = e, p,$

and $\sigma_F = \frac{T_{Fp}}{T_{Fe}}$. All physical quantities in Eqs.

(10) to (12) are scaled as $n_i \rightarrow \frac{n_i}{n_{oi}}, n_e \rightarrow \frac{n_e}{n_{oe}}, n_p \rightarrow \frac{n_p}{n_{op}}, u_i \rightarrow \frac{u_i}{C_F}, \phi \rightarrow \frac{e\phi}{E_{Fe}}, \mu_j \rightarrow \frac{\mu_j}{E_{Fj}}, t \rightarrow t\omega_i, x \rightarrow \frac{x}{\lambda_F}$, where $C_F = \sqrt{E_{Fe}/m_i}$ is the ion

Fermi acoustic speed, $\omega_i = (\sqrt{4\pi e^2 n_{oi}/m_i})$ is the ion plasma frequency and $\lambda_F = \sqrt{E_{Fe}/4\pi e^2 n_{oi}}$ is the Fermi Debye radius. At equilibrium, the charge neutrality condition reads $\beta = \alpha + 1 - \gamma(n_{oe}/n_{oi} = n_{op}/n_{oi} + 1 - n_{on}/n_{oi})$ Here, n_{oi}, n_{on}, n_{oe} , and n_{op} are the unperturbed number densities of positive ions, negative ions, ultra-relativistic degenerate electrons and positrons, respectively.

Propagation of superperiodic waves is characterized by large amplitudes. We are dealing with large-amplitude modes through the arbitrary amplitude Sagdeev pseudopotential analysis.

First step in Sagdeev pseudopotential method is using the transformation, $\xi = x - vt$, one can obtain the planar dynamical system for the plasma system expressed by Eqs. (10) to (12) as

$$-v \frac{dn_i}{d\xi} + \frac{d}{d\xi} (n_i u_i) = 0, \tag{15a}$$

$$-v \frac{dn_n}{d\xi} + \frac{d}{d\xi} (n_n u_n) = 0, \tag{15b}$$

$$\left(-v \frac{du_i}{d\xi} + u_i \frac{du_i}{d\xi}\right) = -\frac{d\phi}{d\xi}, \tag{16a}$$

$$\left(-v \frac{du_n}{d\xi} + u_n \frac{du_n}{d\xi}\right) = M \frac{d\phi}{d\xi}, \tag{16b}$$

$$\frac{d^2 \phi}{d\xi^2} = (\beta n_e - \alpha n_p - n_i + \gamma n_n), \tag{17}$$

where v indicates the phase speed of ion acoustic traveling wave. Integrating Eqs. (15), (16) and applying the boundary conditions $u_i = 0, u_n = 0, n_i = 1, n_n = 1$ and $\phi=0$ as $\xi \rightarrow \pm\infty$, one can obtain:

$$n_i = \left(1 - \frac{2\phi}{v^2}\right)^{-\frac{1}{2}} \approx 1 + \frac{1}{v^2}\phi + \frac{3}{2v^4}\phi^2 + \frac{5}{2v^6}\phi^3, \tag{18a}$$

$$n_n = \left(1 + \frac{2M\phi}{v^2}\right)^{-\frac{1}{2}} \approx 1 - \frac{M}{v^2}\phi + \frac{3M^2}{2v^4}\phi^2 - \frac{5M^3}{2v^6}\phi^3. \tag{18b}$$

The higher order terms in Eq. (18) are ignored. Since $v > 1$; the coefficient $\frac{1}{v} < 1$, only first four terms are taken into account. Substitute n_e, n_p, n_n and n_i from Eqs. (13) and (18) into Poisson equation, (17), to get

$$\frac{d^2 \phi}{d\xi^2} = \left(\beta \frac{C_{2e}}{C_{1e}} + \alpha \sigma_F \frac{C_{2p}}{C_{1p}} - \frac{1}{v^2} - \frac{\gamma M}{v^2} \right) \phi + \tag{19}$$

$$\left(\beta \frac{\mu_{0e}}{C_{1e}} - \alpha \sigma_F^2 \frac{\mu_{0p}}{C_{1p}} - \frac{3}{2v^4} + \frac{3\gamma M^2}{2v^4} \right) \phi^2 +$$

$$\left(\beta \frac{1}{3C_{1e}} + \alpha \sigma_F^3 \frac{1}{3C_{1p}} - \frac{5}{2v^6} - \frac{5\gamma M^3}{2v^6} \right) \phi^3,$$

or

$$\frac{d^2 \phi}{d\xi^2} = a \phi + b \phi^2 + c \phi^3, \tag{20}$$

where

$$a = \left(\beta \frac{C_{2e}}{C_{1e}} + \alpha \sigma_F \frac{C_{2p}}{C_{1p}} - \frac{1}{v^2} - \frac{\gamma M}{v^2} \right), \quad b =$$

$$\left(\beta \frac{\mu_{0e}}{C_{1e}} - \alpha \sigma_F^2 \frac{\mu_{0p}}{C_{1p}} - \frac{3}{2v^4} + \frac{3\gamma M^2}{2v^4} \right), \text{ and}$$

$$c = \left(\beta \frac{1}{3C_{1e}} + \alpha \sigma_F^3 \frac{1}{3C_{1p}} - \frac{5}{2v^6} - \frac{5\gamma M^3}{2v^6} \right).$$

Description of the dynamical system

To confirm the possibility of finding solitary, periodic and superperiodic wave solutions, we applied the bifurcation analysis. The analysis of phase plane using bifurcation theory gives

significant features of the dynamical system (DS). It is known that each trajectory in the phase portrait gives a traveling wave solution. Hence the formula. (20) can be expressed as DS of the form (S. H. Strogatz, 2007)

$$\frac{d\phi}{d\xi} = z, \tag{21-a}$$

$$\frac{dz}{d\xi} = \phi(a + b\phi + c\phi^2). \tag{21-b}$$

The DS in Eqs. (21) corresponds to the Hamiltonian of the system. The Hamiltonian function can be expressed as

$$\frac{1}{2}z^2 + V(\phi) = h, \tag{22}$$

where h is the integration constant. The value of the constant of integration h is evaluated from the boundary conditions $\phi \rightarrow 0, \frac{d\phi}{d\xi} \rightarrow 0, \frac{d^2\phi}{d\xi^2} \rightarrow 0$, at $\xi \rightarrow \pm\infty$. The value of integration constant h affects the type obtained waves.

Furthermore, the Sagdeev pseudopotential is determined as

$$V(\phi) = -\left(\frac{a}{2}\phi^2 + \frac{b}{3}\phi^3 + \frac{c}{4}\phi^4\right). \tag{23}$$

It is clear that the DS (21) has three equilibrium points; $P_0(\phi_0, 0), P_1(\phi_1, 0)$ and $P_2(\phi_2, 0)$, where

$$\phi_0 = 0 \text{ and } \phi_{1,2} = -\frac{b}{2c} \pm \sqrt{\left(\frac{b}{2c}\right)^2 - \frac{a}{c}}. \tag{24}$$

These points are derived by solving the next equations;

The Jacobin matrix can be obtained by expressing the dynamical system (21) as

$$\frac{d\phi}{d\xi} = F(\phi, z) = z, \tag{25-a}$$

$$\frac{dz}{d\xi} = G(\phi, z) = \phi(a + b\phi + c\phi^2). \tag{25-b}$$

Then:

$$J = \begin{pmatrix} \frac{\partial F(\phi, z)}{\partial \phi} & \frac{\partial F(\phi, z)}{\partial z} \\ \frac{\partial G(\phi, z)}{\partial \phi} & \frac{\partial G(\phi, z)}{\partial z} \end{pmatrix}. \tag{26}$$

From Eq. (25) the Jacobian matrix can be derived as

$$J = \begin{pmatrix} 0 & 1 \\ a + 2b\phi + 3c\phi^2 & 0 \end{pmatrix}, \tag{27}$$

and the determinant is

$$M = DetJ(\phi_i, 0) = -(a + 2b\phi_i + 3c\phi_i^2), \tag{28}$$

where $i = 0, 1, 2$. If $M < 0$, then $P_i(\phi_i, 0)$ is a saddle node while for $M > 0, P_i(\phi_i, 0)$ is a center (A. E. Dubinov et al, 2011).

Results and discussions

It is important to note that the Hamiltonian

system depends mostly on the physical parameters in the suggested model. Based on the dynamical system bifurcation analysis in Eq. (21), we study the bifurcation of solitary, periodic, and super-periodic waves using symbolic computation. The phase graphs of the dynamical system vary based on the critical points and enclosed separatrix layers (A. E. Dubinov et al, 2011). Every trajectory in the dynamical system's phase portrait corresponds to a single travelling wave solution. The trajectories are classified as super-periodic, SPT(m,n), periodic, PT(m,n), superhomoclinic, SHT(m,n), and homoclinic, HT(m,n), where m indicates the number of centers surrounded by the trajectory and n is the number of separatrices enclosed by that trajectory (A. E. Dubinov et al, 2011). As a result, in order to analyze the bifurcation in the system, Eq. (21), we must first acquire all possible trajectories based on plasma characteristics. Figure (1-a), displays the phase plot of the nonlinear dynamical system (21) for $v= 2.2, \mu_0= 0.1, M= 0.75, \gamma = 0.1$ and $\delta = 0.3$ where δ is the ratio of unperturbed number density of protons to unperturbed number density of electrons ($\delta = n_{0p}/n_{0e}$) and γ is ratio of unperturbed number density of negative ions to unperturbed number density of positive ions ($\gamma = n_{0n}/n_{0i}$). In this case, we get three forms of qualitatively different nonlinear trajectories, which are periodic trajectories (PT_{1,0}), homoclinic trajectories (HT_{1,0}) and super-periodic trajectory (SPT_{2,1}). It is essential to note that for the homoclinic trajectories (HT_{1,0}), one can get solitary wave solutions, and for periodic trajectories (PT_{1,0}), one can get periodic wave solutions (A. Saha et al, 2015) which are shown in Figs. (1-c) and (1-d). There is also a super-periodic wave solution, as seen in Fig. (1-e) which is corresponding to the super-periodic trajectory (SPT_{2,1}). The plot of Sagdeev pseudopotential change with ϕ for the same parameters is shown in Fig. (1-b). It is proved that the nonlinear dynamical system contains three critical points $P_0(\phi_0, 0), P_1(\phi_1, 0)$ and $P_2(\phi_2, 0)$. In Fig. (1-b), the pseudopotential has three fixed points; two centers; $P_1(\phi_1, 0)$ and $P_2(\phi_2, 0)$ and one one saddle; $P_0(\phi_0, 0)$, ensuring the presence of a periodic ion-acoustic super-nonlinear wave (SNW) in examined system (A. E. Dubinov et al, 2011). Local minima of potential graphs suggest solitary solutions if $V(\phi) \rightarrow 0$ when $\phi \rightarrow \pm\infty$ (A. E. Dubinov, et al, 2012). In Fig. (1-b), the wave

solution in the positive zone of ϕ reveals a compressive soliton, whereas the wave solution in the negative region of ϕ refers to a rarefactive soliton. The depth of the potential well represents the steepness of the solitons, and the width of the potential well represents the amplitude of the solitons, revealing important information on the structure of solitary waves. The effect of chemical potential, μ_0 , on Sagdeev pseudopotential $V(\phi)$ is displayed in Fig. (2-a). The figure shows that the depth of Sagdeev potential is decreasing with the growth in chemical potential, μ_0 , indicating that the steepness of the solitary wave shrinks. On the other hand, the wave width broadens in the negative region of ϕ and shrinks in the positive one with increasing μ_0 . Hence the amplitude of the rarefactive solitary waves grows with growing of μ_0 , while the amplitude of the compressive solitary wave decreases with increasing μ_0 .

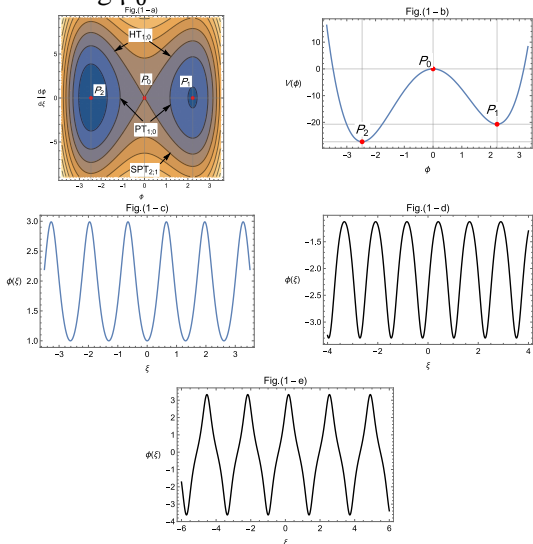


Fig. 1 (a) Phase portrait of the dynamical system, (b) Sagdeev pseudopotential $V(\phi)$, (c) periodic wave around $P_1(\phi_1,0)$ and (d) periodic wave around $P_2(\phi_2,0)$, (e) super-periodic wave with $v = 2.2$, $\mu_0 = 0.1$, $M = 0.75$, $\gamma = 0.1$ and $\delta = 0.3$.

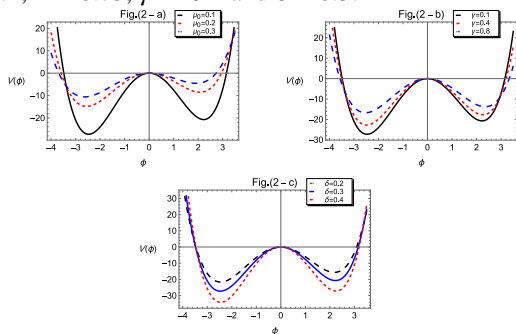


Fig. 2 Influence of (a) chemical potential μ_0 , (b) $\gamma (= n_{0n}/n_{0i})$ (c) $\delta (= n_{0p}/n_{0e})$ on Sagdeev pseudopotential $V(\phi)$.

The influence of γ on Sagdeev pseudopotential $V(\phi)$ is displayed in Fig. (2-b). This figure shows that the depth of Sagdeev potential is diminishing with the growth in γ , leading to a reduction in the steepness of the solitary wave. It is shown that the wave width broadens with increasing γ , indicating an amplification of the amplitude of the compressive solitary wave. Fig. (2-c) displays the effect δ on Sagdeev pseudopotential $V(\phi)$. It is observed that the depth of pseudopotential $V(\phi)$ grows as δ increases, and hence the steepness of the solitary waves rises. It is shown that the width of pseudopotential $V(\phi)$ is enhanced for the positive values of ϕ and shrinks for negative values of ϕ , Fig. (2-c). Hence, the amplitude of the compressive solitary waves is amplified with growing δ . In contrast, the amplitude of the rarefactive solitary wave decreases.

The effect of the chemical potential μ_0 , γ and δ , on the periodic wave around $P_1(\phi_1,0)$ are displayed in Fig. 3, respectively. It can be interpreted from Fig. (3-a) that the periodic wave's width grows as the chemical potential μ_0 increases; indicating an increase in the dispersion of the wave. On the other hand, the amplitude diminishes as μ_0 increases indicating a decrease in energy and an increase in the nonlinearity of the system.

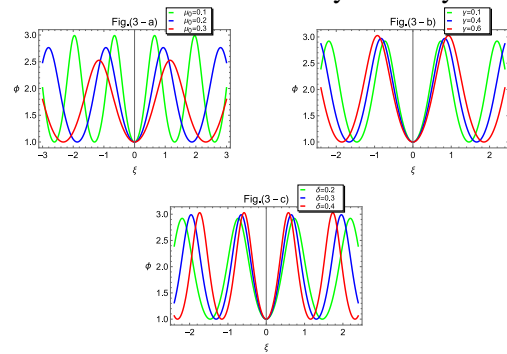


Fig. 3. Influence of (a) chemical potential μ_0 , (b) $\gamma (= n_{0n}/n_{0i})$ (c) $\delta (= n_{0p}/n_{0e})$ on periodic wave around $P_1(\phi_1,0)$.

In Fig. (3-b), it is noticed that when γ is increased, the amplitude of the periodic wave around $P_1(\phi_1, 0)$ is amplified which means that the energy of plasma increases and the nonlinearity decreases. It is shown that the width of the periodic wave is broadened, and this means an increase in the dispersion of the wave. Fig. (3-c) shows that when δ is increased, the amplitude of the periodic wave around $P_1(\phi_1, 0)$ is increasing, which means an enhancement in the energy of the plasma,

indicating a decrease in the nonlinearity of the system. On other hand, the width decreases with higher values of δ , indicating a decrease in the dispersion of the wave.

The effects of μ_0 , γ and δ on the super-periodic wave are displayed in Figs. (4). Figure (4-a) shows that the growth in the chemical potential μ_0 shows a reduction in width of superperiodic waves, which indicates a decrease in wave dispersion. Further, the amplitude of superperiodic waves rises with increasing values of chemical potential μ_0 ; this behavior refers to a growth in the energy of the system. From the physical point of view, the nonlinearity decreases with higher values of chemical potential.

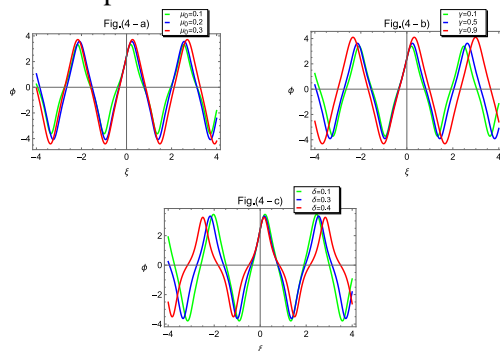


Fig. 4. Influence of (a) chemical potential μ_0 , (b) $\gamma (= n_{0n}/n_{0i})$ (c) $\delta (= n_{0p}/n_{0e})$ on superperiodic wave.

In Fig. (4-b), it is shown that the width of the super-periodic wave is broadened with higher values of γ , and this means an increase in the dispersion of the wave. By increasing γ , also the amplitude of the superperiodic wave is amplified and this represents an increase in the energy of the plasma indicating a reduction in the nonlinearity of the system. It is shown in Fig.(4-c) that the width of the super-periodic wave is enhanced with higher values of δ , which means an increase in the dispersion of the wave. On the other hand, by increasing δ , the amplitude of the super-periodic wave decreases, indicating a reduction in energy and an enhancement in nonlinearity of the system.

Conclusion

In a degenerate ultrarelativistic plasma composed of electrons, positrons, and positive ions, bifurcation analysis of IA periodic and superperiodic waves was examined. All

potential phase charts, including periodic, homoclinic, and superperiodic trajectories, have been successfully displayed using bifurcation analysis of the dynamical system. In ultra-relativistic degenerate four species quantum plasma, superperiodic waves have been found under specific boundary conditions and for definite range of plasma parameters. Furthermore, rarefactive and compressive solitary waves are shown to exist. The effect of different plasma characteristics on the periodic wave, super periodic wave, and Sagdeev pseudopotential features is investigated. The chemical potential, the ratio of unperturbed number density of negative ions to unperturbed number density of positive ions, and the ratio of unperturbed number density of protons to unperturbed number density of electrons have all been found to have a significant effect on the energy and nonlinearity of the system. The findings of this work could help researchers better understand the underlying properties of nonlinear ion-acoustic periodic and superperiodic waves in dense objects like white dwarfs and neutron stars.

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الملخص العربي

عنوان البحث: توليد موجات صوتية أيونية دورية فائقة في البلازما الكمية المتحللة فائقة النسبية

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في هذه المقالة، يتم تطبيق تقنية الجهد لساغديف لدراسة تطور الموجات الصوتية الأيونية ذات السعة الكبيرة في بلازما مكونه من الإلكترونات والأيونات والبوزيترونات. يتم أخذ طاقات الإلكترونات والبوزيترونات في المنطقة النسبية الفائقة. يتم التنبؤ بموجات سولوتونية تضاعفية وتخلخلية وموجات دورية وموجات فائقة الدورية في نظام البلازما. يتم الحصول على جميع مخططات الطور المحتملة، بما في ذلك المسارات فائقة الدورية والدورية والمتجانسة باستخدام تحليل التشعب للنظام الديناميكي. تمت دراسة تأثيرات سرعة الطور للموجة الصوتية الأيونية والجهد الكيميائي ومعاملات البلازما الأخرى على الخصائص المميزة للموجة الدورية والموجة الدورية الفائقة وجهد ساغديف. قد تكون نتائج هذه الورقة قيمة في فهم السمات الأساسية للموجات غير الخطية في الأجرام السماوية الكثيفة، مثل النجوم النيوترونية والأقزام البيضاء.