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UPPER BOUNDS FOR RADIUS PROBLEMS INVOLVING RATIOS OF ANALYTIC FUNCTIONS

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Abstract. In recent years, the problem of finding the sharp radii bounds for certain properties in geometric function theory has attracted several researchers. However, there are several instances where only lower bounds for the radius problems have been established. In this paper, we have worked in a similar direction to compute the upper bounds in these cases which coincides with the conjectured values. Moreover, explicit functions are provided which yield that these bounds are attainable.

1. Introduction

For $\alpha \in [0, 1)$, let $\mathcal{P}(\alpha)$ be the class of complex-valued analytic functions p defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $p(0) = 1$ and satisfying $\operatorname{Re} p(z) > \alpha$ for all $z \in \mathbb{D}$. Set $\mathcal{P} := \mathcal{P}(0)$. Let $\mathcal{A}$ be the class of analytic functions f defined in $\mathbb{D}$ with $f(0) = 0 = f'(0) - 1$ and $\mathcal{S}$ be its subclass consisting of univalent functions. By making use of the concept of subordination, Ma and Minda [6] integrated several subclasses of functions which map $\mathbb{D}$ onto a starlike domain and defined the class $\mathcal{S}^*(\phi)$ (for a specific ϕ) consisting of functions $f \in \mathcal{A}$ with $zf'(z)/f(z) \prec \phi(z)$ for all $z \in \mathbb{D}$, where the function ϕ is univalent, with positive real part that maps $\mathbb{D}$ onto a domain symmetric with respect to real axis and starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. Given two subsets $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{A}$, the $\mathcal{F}_2$-radius of the class $\mathcal{F}_1$, denoted by $\mathcal{R}_{\mathcal{F}_2}(\mathcal{F}_1) \in (0, 1]$ is the largest R such that for every $f \in \mathcal{F}_1$, $r^{-1}f(rz) \in \mathcal{F}_2$ for each $r \leq R$. By making use of the class $\mathcal{P}(\alpha)$, Lecko *et al.* [5] introduced the class

$$\mathcal{G} := \left\{ f \in \mathcal{A} : \frac{f}{g} \in \mathcal{P} \ , \ \frac{g}{zp} \in \mathcal{P}\left(\frac{1}{2}\right) \text{ for some } g \in \mathcal{A},\, p \in \mathcal{P} \right\}$$

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and determined the $\mathcal{S}^*(\phi)$-radius for several choices of ϕ. However, we are concerned specifically with the five choices of ϕ, namely

$$\phi_{PAR}(z) := 1 + \left(\frac{2}{\pi^2}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right),$$

$\phi_e(z) := e^z$, $\phi_C = 1 + (4/3)z + (1/4)z^2$, $\phi_{\mathbb{C}}(z) := 1 + \sqrt{z} - z^2/2$ and $\phi_R(z) = 1 + (zk + z^2)/(k^2 - kz)$, $k = \sqrt{2}+1$. These classes were investigated in [4,8–10,14]. Lecko *et al.* [5] calculated lower radii bounds $\mathcal{R}_{\mathcal{S}^*(\phi)}(\mathcal{G})$ for the classes $\mathcal{S}^*_{PAR} := \mathcal{S}^*(\phi_{PAR})$ [5, Theorem 3(ii), p. 9], $\mathcal{S}^*_e$ [5, Theorem 4(ii), p. 10], $\mathcal{S}^*_C$ [5, Theorem 5(ii), p. 11], $\mathcal{S}^*_{\mathbb{C}}$ [5, Theorem 7(ii), p. 14] and $\mathcal{S}^*_R$ [5, Theorem 8(ii), p. 15]. However these obtained bounds were not sharp. In Section 2, we compute the upper bounds of $\mathcal{R}_{\mathcal{S}^*(\phi)}(\mathcal{G})$ for $\phi_{PAR}, \phi_e, \phi_C, \phi_{\mathbb{C}}$ and ϕ_R, which coincide with the conjectured values given by Lecko [5, p. 21].

In 2019, Cho *et al.* [3] introduced and studied the class $\mathcal{S}^*_{sin} := \mathcal{S}^*(1 + \sin z)$. In the last section, we determine the upper bounds of $\mathcal{S}^*_{sin}$-radius for the classes $\mathcal{H}_i$ $(i = 1, 2, 3)$ given by

$$\mathcal{H}_i = \left\{ f \in \mathcal{A} : \left|\frac{f(z)}{g(z)} - 1\right| < 1 \text{ for some } g \in \mathcal{A} \text{ with } \operatorname{Re}\left(\frac{g(z)}{\psi_i(z)}\right) > 0 \right\},$$

where the functions $\psi_i \in \mathcal{A}$ are given by $z/(1-z)^2$ and $z/(1+z)$ for $i = 1, 2$ respectively, and

$$\mathcal{H}_3 = \left\{ f \in \mathcal{A} : \left|\frac{f(z)}{g(z)} - 1\right| < 1 \text{ for some } g \in \mathcal{A} \text{ with } \operatorname{Re}\left(\frac{g(z)}{\zeta(z)}\right) > 0, \zeta \in \mathcal{S}^*(\alpha) \right\}.$$

Here, $\mathcal{S}^*(\alpha)$ is the class of starlike functions of order α, for $0 \le \alpha < 1$. The classes $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ were studied by Sebastian and Ravichandran [12], Ahmad El-Faqeer *et al.* [1] and Madhumitha and Ravichandran [7]. Sebastian and Ravichandran [12, Theorem 2.2(vi), p. 91] and Ahmad El-Faqeer *et al.* [1, Theorem 2.2(vi), p. 523] determined the non-sharp bounds for the classes $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. However, $\mathcal{S}^*_{sin}$-radius was not computed by Madhumitha and Ravichandran [7, Theorem 2.2, p. 10].

The following lemmas will be needed for the investigation,

Lemma 1.1. *[2, Lemma 4, p. 182] If $p \in \mathcal{P}(1/2)$, then*

$$\operatorname{Re}\left(\frac{zp'(z)}{p(z)}\right) \ge -\frac{|z|}{1+|z|} \quad for \quad |z| \le \frac{1}{3}.$$

Lemma 1.2. *[13, Lemma 2, p. 239] If $p \in \mathcal{P}(\alpha)$, $0 \le \alpha < 1$, then*

$$\operatorname{Re}\left(\frac{zp'(z)}{p(z)}\right) \le \left|\frac{zp'(z)}{p(z)}\right| \le \frac{2r(1-\alpha)}{(1-r)(1+(1-2\alpha)r)}, \quad |z| = r.$$

2. Radius constants for $\mathcal{G}$

In this section, we will compute the upper bounds of $\mathcal{S}^*(\phi)$-radius for class $\mathcal{G}$ for five different choices of ϕ. The function $f_0 : \mathbb{D} \to \mathbb{C}$ defined by

$$f_0(z) = \frac{z(1+z)^2}{(1-z)^3} \quad \text{with} \quad g_0(z) = \frac{z(1+z)}{(1-z)^2} \text{ and } p_0(z) = \frac{1+z}{1-z} \tag{1}$$

belongs to the class $\mathcal{G}$ as $p_0 \in \mathcal{P}$, $g_0/zp_0 \in \mathcal{P}(1/2)$ and $f_0/g_0 \in \mathcal{P}$. Also, $zf_0'(z)/f_0(z) = (1+5z)/(1-z^2)$.

Theorem 2.1. *The upper bounds of $\mathcal{S}^*(\phi)$-radius for the class $\mathcal{G}$, are given by the following table:*

S. No.	$\mathcal{S}^*(\phi)$	$\mathcal{R}_{\mathcal{S}^*(\phi)}(\mathcal{G}) \leq r_\phi$
(a)	$\mathcal{S}^*_{PAR}$	$r_{PAR} = 5 - 2\sqrt{6} \approx 0.1010$
(b)	$\mathcal{S}^*_e$	$r_e = \dfrac{5e}{2} - \dfrac{\sqrt{4-4e+25e^2}}{2} \approx 0.127622$
(c)	$\mathcal{S}^*_C$	$r_C = \dfrac{15-\sqrt{217}}{2} \approx 0.13454$
(d)	$\mathcal{S}^*_{\leftmoon}$	$r_{\leftmoon} = \dfrac{(5-\sqrt{41-12\sqrt{2}})(\sqrt{2}+1)}{2} \approx 0.118317.$
(e)	$\mathcal{S}^*_R$	$r_R = \dfrac{(5-\sqrt{81-40\sqrt{22}})(\sqrt{2}+1)}{4} \approx 0.0345119$

Proof. Let $f \in \mathcal{G}$ with associated functions $g \in \mathcal{A}$ and $p \in \mathcal{P}$. Then the functions $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ defined by $p_1 = f/g$ and $p_2 = g/zp$ belong to the classes $\mathcal{P}$ and $\mathcal{P}(1/2)$ respectively. Moreover $f(z) = zp(z)p_1(z)p_2(z)$, which gives

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = 1 + \operatorname{Re}\left(\frac{zp_1'(z)}{p_1(z)}\right) + \operatorname{Re}\left(\frac{zp_2'(z)}{p_2(z)}\right) + \operatorname{Re}\left(\frac{zp'(z)}{p(z)}\right). \tag{2}$$

Using Lemmas 1.1 and 1.2 in (2), we obtain

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq 1 - \frac{r}{1+r} - \frac{4r}{1-r^2} = \frac{1-5r}{1-r^2} \quad \text{for} \quad |z| = r < \frac{1}{3}. \tag{3}$$

(a) Let $\Omega_{PAR} = \phi_{PAR}(\mathbb{D}) = \{w \in \mathbb{C} : \operatorname{Re} w > |w-1|\} = \{a+ib : 2a - b^2 - 1 > 0\}$. Note that a necessary condition for $zf'(z)/f(z)$ to lie inside Ω_{PAR} is $\operatorname{Re}(zf'(z)/f(z)) > 1/2$. Consequently, (3) yields

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{1-5r}{1-r^2} > \frac{1}{2}$$

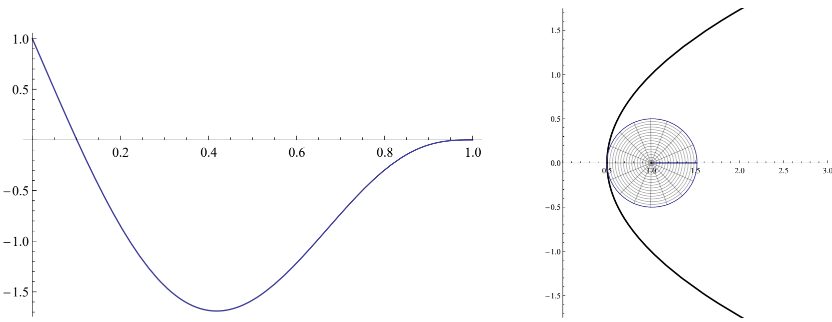
which holds provided $r < r_{PAR} := 5 - 2\sqrt{6}$. Thus $\mathcal{R}_{\mathcal{S}^*_{PAR}}(\mathcal{G}) \leq r_{PAR}$. In order to show that this bound is attainable, we consider the function f_0 given by (1). For this, we prove that $w = zf_0'(z)/f_0(z) \in \Omega_{PAR}$ for $|z| < r_{PAR}$. For $z = re^{it}$ and $u = \cos t$, a straightforward calculation gives

$$2a - b^2 - 1 = \frac{h_{PAR}(r,u)}{(1+r^2-2ru)^2(1+r^2+2ru)^2},$$

where $w = a+ib$ and $h_{PAR}(r,u) = -(1 - 23r^2 - 50r^4 - 27r^6 - r^8 + 10ru - 10r^3u - 30r^5u - 10r^7u + 21r^2u^2 + 46r^4u^2 + 29r^6u^2 - 20r^3u^3 + 60r^5u^3 + 4r^4u^4)$. The problem now reduces to show that the function $h_{PAR}(r,u) > 0$ for $r < r_{PAR}$ and $u \in [-1,1]$. Observe that the roots of $h_{PAR}(r,u) = 0$ in $(0,1)$ are increasing as a function of $u \in [-1,1]$. As a result, it follows that $h_{PAR}(r,u) > 0$ for $-1 \leq u \leq 1$ if and only if

$$h_{PAR}(r,-1) = 1 - 10r - 2r^2 + 30r^3 - 30r^5 + 2r^6 + 10r^7 - r^8 > 0,$$

which gives $r < r_{PAR}$ (Figure 1(a)). Thus $f_0(r_{PAR}z)/r_{PAR} \in \mathcal{S}^*_{PAR}$ as shown in Figure 1(b).



(a) $h_{PAR}(r, -1)$ for $r \in (0, 1)$ (b) $zf_0'(z)/f_0(z)$

FIGURE 1. $\mathcal{R}_{\mathcal{S}^*_{PAR}}(\mathcal{G})$

(b) Consider $\Omega_e = \phi_e(\mathbb{D}) = \{w \in \mathbb{C} : |\log w| < 1\}$. Observe that $\operatorname{Re}(zf'(z)/f(z)) > 1/e$ is a necessary condition for $zf'(z)/f(z)$ to lie inside Ω_e. In view of Lemmas 1.1 and 1.2 in (3), we obtain $(1 - 5r)/(1 - r^2) > 1/e$ which implies that $r < r_e := (5e - \sqrt{4 - 4e + 25e^2})/2$. To establish $f_0(r_e z)/r_e \in \mathcal{S}^*_e$, consider the function f_0 given by (1). Geometrical considerations show that $zf_0'(z)/f_0(z)$ lies inside Ω_e for $|z| < r_e$ (Figure 2) and

$$\frac{zf_0'(z)}{f_0(z)} = \frac{1}{e} \quad \text{at } z = -r_e.$$

Thus $f_0(r_e z)/r_e \in \mathcal{S}^*_e$.

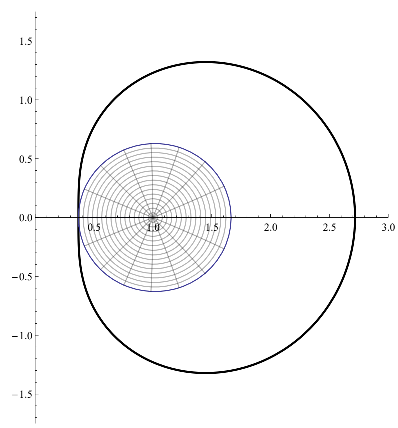


FIGURE 2. $\mathcal{R}_{\mathcal{S}^*_e}(\mathcal{G})$

(c) Let $\Omega_C = \phi_C(\mathbb{D}) = \left\{w = a + ib : (9a^2 + 9b^2 - 18a + 5)^2 < 16(9a^2 + 9b^2 - 6a + 1)\right\}$. In this case, if $w = zf'(z)/f(z) \in \Omega_C$ then it is necessary that $\operatorname{Re}(zf'(z)/f(z)) > 1/3$. Using the similar analysis carried out in the previous parts, it follows that $(1 - 5r)/(1 - r^2) > 1/3$ for $r < r_C := (15 - \sqrt{217})/2$. Therefore $\mathcal{R}_{\mathcal{S}^*_C}(\mathcal{G}) \le r_C$.

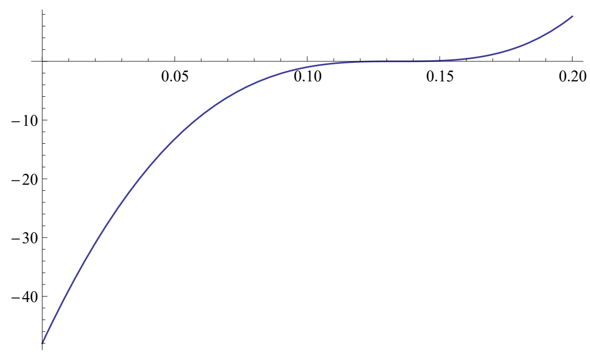
Furthermore, consider the expression $(9a^2+9b^2-18a+5)^2-16(9a^2+9b^2-6a+1)$ where $w = zf_0'(z)/f_0(z) = a+ib$. For $z = re^{it}$ and $u = \cos t$, this simplifies to

$$\frac{h_C(r,u)}{(1+r^2-2ru)^2(1+r^2+2ru)^2},$$

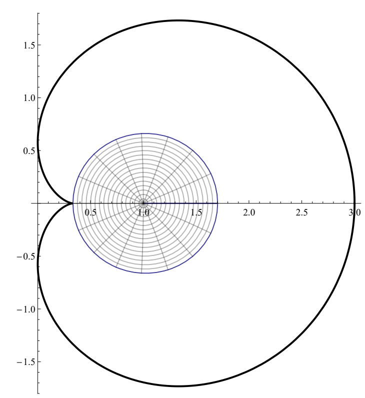
where $h_C(r,u) = 3(-16-1800r^2+13299r^4-466r^6+3r^8-320ru-1040r^3u+12380r^5u+140r^7u+6944r^4u^2+2732r^6u^2+1280r^3u^3+1600r^5u^3+256r^4u^4)$. The roots of the equation $h_C(r,u)=0$ in $(0,1)$ are increasing as a function of $u \in [-1,1]$. Hence $h_C(r,u) < 0$ for $u \in [-1,1]$ if and only if

$$h_C(r,-1) = -3(2-r)(1+3r)(2-15r+r^2) < 0$$

for $r < r_C$ (Figure 3(a)). Therefore, $f_0(r_C z)/r_C \in \mathcal{S}_C^*$. The image of the subdisk $|z| < r_C$ under the function $zf_0'(z)/f_0(z)$ is illustrated in Figure 3(b).



(a) $h_C(r,u)$ for $r \in (0, 0.2)$



(b) $zf_0'(z)/f_0(z)$

FIGURE 3. $\mathcal{R}_{\mathcal{S}_C^*}(\mathcal{G})$

(d) Let $\Omega_{\leftmoon} = \phi_{\leftmoon}(\mathbb{D}) = \left\{w \in \mathbb{C} : |w^2-1| < 2|w|\right\}$. A necessary condition for $w = zf'(z)/f(z)$ to lie inside $\Omega_{\leftmoon}$ is $\operatorname{Re}(zf'(z)/f(z)) > 2(\sqrt{2}-1)$. From (3), we obtain $(1-5r)/(1-r^2) > 2(\sqrt{2}-1)$ provided $r < r_{\leftmoon} := (5-\sqrt{41-12\sqrt{2}})(\sqrt{2}+1))/2$. This bound is attainable. To see this, consider the expression $|w^2-1|^2 -$

$4|w|^2 = 1 + a^4 - 2b^2 + b^4 + 2a^2(b^2 - 3)$ where $w = zf_0'(z)/f_0(z) = a + ib$ and f_0 is given by (1). If $z = re^{it}$ and $u = \cos t$, then

$$1 + a^4 - 2b^2 + b^4 + 2a^2(b^2 - 3) = \frac{h_{☾}(r,u)}{(1 + r^2 - 2ru)^2(1 + r^2 + 2ru)^2},$$

where $h_{☾}(r,u) = p(r,u)q(r,u)$ with $p(r,u) = -2 - 25r^2 + r^4 - 20ru + 10r^3u$ and $q(r,u) = 2 - 21r^2 + r^4 - 10r^3u - 8r^2u^2$. We observe that

$$\frac{\partial p(r,u)}{\partial u} = 10r(r^2 - 2) < 0$$

and thus $p(r,u)$ is a decreasing function of u for each $r \in (0,1)$. Therefore $p(r,u) \leq p(r,-1)$. But $p(r,-1) = -2 + 20r - 25r^2 - 10r^3 + r^4 < 0$, if $0 < r < r_{☾}$ (Figure 4(a)) which yields $p(r,u) < 0$ for $0 < r < r_{☾}$ and for all $u \in [-1,1]$.

In order to show that $h_{☾}(r,u) < 0$ for $0 < r < r_{☾}$, it suffices to show that $q(r,u) > 0$ there, for all $u \in [-1,1]$. As

$$\frac{\partial q(r,u)}{\partial u} = -2r^2(8u + 5r),$$

therefore $q(r,u)$ attains its local maxima at $u_0 = -5r/8$. Also, $q(r,u)$ is increasing for $u \in [-1,u_0)$ and decreasing for $(u_0,1]$. Also, we see that $q(r,-1) - q(r,1) = 20r^3 > 0$. Consequently, it follows that $q(r,u)$ attains its local minima at $u = 1$. As $q(r,1) = 2 - 29r^2 - 10r^3 + r^4 > 0$ for $0 < r < s_0$ where

$$s_0 = \frac{5 - 5\sqrt{2} + \sqrt{83 - 54\sqrt{2}}}{2} \approx 0.252145$$

(Figure 4(b)), therefore $q(r,u) \geq q(r,-1) > 0$ for $0 < r < s_0$. Since $r_{☾} < s_0$, we conclude that $h_{☾}(r,u) < 0$ for $u \in [-1,1]$ and $0 < r < r_{☾}$. Hence $f_0(r_{☾} z)/r_{☾} \in \mathcal{S}^*_{☾}$. The Figure 4(c) depicts the image of $zf_0'(z)/f_0(z)$ under the subdisk $|z| < r_{☾}$.

(e) An analytic function $f \in \mathcal{S}^*_R$ if and only if $zf'(z)/f(z) \prec \phi_R(z)$. This infers that a necessary condition for $zf'(z)/f(z) \prec \phi_R(z)$ is $\mathrm{Re}(zf'(z)/f(z)) > 2(\sqrt{2} - 1)$. By (3), we have $(1 - 5r)/(1 - r^2) > 2(\sqrt{2} - 1)$ which gives $r < r_R := (\sqrt{81 - 40\sqrt{2}})(\sqrt{2} + 1)/4$. For the function f_0 given by (1), Figure 5 depicts that the quantity $zf_0'(z)/f_0(z)$ lies inside $\phi_R(\mathbb{D})$ for $|z| < r_R$ and

$$\frac{zf_0'(z)}{f_0(z)} = 2(\sqrt{2} - 1) \quad \text{at } z = -r_R.$$

Thus $f_0(r_R z)/r_R \in \mathcal{S}^*_R$. □

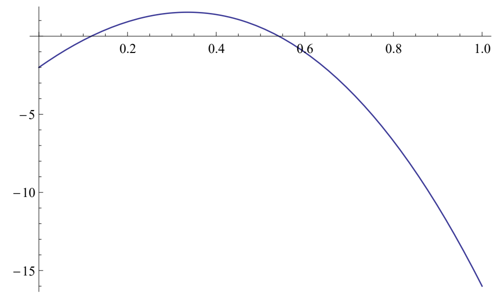
3. Radius constants for $\mathcal{H}_i$

In this section, we determine the upper bounds of $\mathcal{S}^*_{sin}$-radius for the classes $\mathcal{H}_i$ for $i = 1,2,3$. Apart from Lemmas 1.1 and 1.2, we shall make use of the following lemma to prove our results.

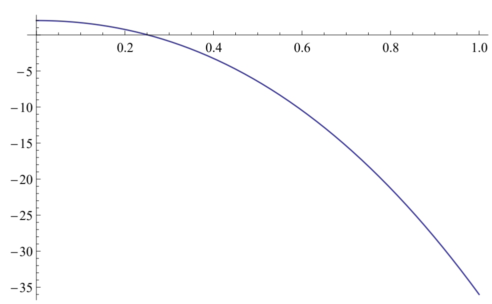
Lemma 3.3. *[11, Lemma 2.1, p. 267] If $p \in \mathcal{P}(\alpha)$, $0 \leq \alpha < 1$, then*

$$\left| p(z) - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{(1 - \alpha)r}{1 - r^2} \quad |z| = r.$$

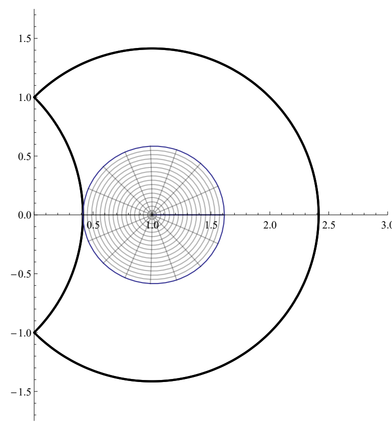
Theorem 3.2. *The upper bounds of $\mathcal{S}^*_{sin}$-radius for the classes $\mathcal{H}_i$ are given by the following table:*



(a) $p(r,-1)$ for $r \in (0,1)$



(b) $q(r,1)$ for $r \in (0,1)$



(c) $zf_0'(z)/f_0(z)$

FIGURE 4. $\mathcal{R}_{\mathcal{S}^*_{☾}}(\mathcal{G})$

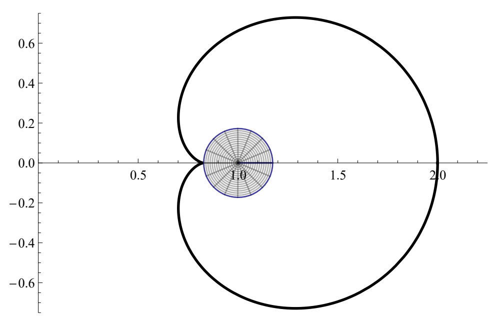


FIGURE 5. $\mathcal{R}_{\mathcal{S}^*_R}(\mathcal{G})$

Proof. Let $f \in \mathcal{H}_i$ for $i = 1, 2$. Then there exists an analytic function g such that the function $q = g/f \in \mathcal{P}(1/2)$. Observe that a necessary condition for $zf'(z)/f(z) \prec \phi_{sin}(\mathbb{D})$ is $\mathrm{Re}(zf'(z)/f(z)) < 1 + \sin 1$.

S. No.	$\mathcal{H}_i$	$\mathcal{R}_{\mathcal{S}^*_{sin}}(\mathcal{H}_i) \leq r_i$
(a)	$\mathcal{H}_1$	$r_1 = (\csc 1)(\sqrt{4+\sin^2 1} - 2) \approx 0.201801$
(b)	$\mathcal{H}_2$	$r_2 = \dfrac{\sqrt{27 - 2\cos 2 + 4\sin 1} - 5}{2(1+\sin 1)} \approx 0.158985$

For (a), the function $p_1 : \mathbb{D} \to \mathbb{C}$ defined by $p_1(z) = g(z)(1+z)/z$ is a member of class $\mathcal{P}$. Note that

$$\frac{zf'(z)}{f(z)} = \frac{zp_1'(z)}{p_1(z)} - \frac{zq'(z)}{q(z)} + \frac{1}{1+z}. \tag{4}$$

By making use of Lemmas 1.1 and 1.2 in (4), we obtain

$$\begin{aligned} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left(\frac{zp_1'(z)}{p_1(z)}\right) - \operatorname{Re}\left(\frac{zq'(z)}{q(z)}\right) + \operatorname{Re}\left(\frac{1}{1+z}\right) \\ &\leq \frac{2r}{1-r^2} + \frac{r}{1+r} + \frac{1}{1-r} \quad \text{for} \quad |z| < \frac{1}{3} \\ &= \frac{1+4r-r^2}{1-r^2} < 1 + \sin 1, \end{aligned}$$

which yields $r < r_1 := (\csc 1)(\sqrt{4+\sin^2 1} - 2)$. Thus $\mathcal{R}_{\mathcal{S}^*_{sin}}(\mathcal{H}_1) \leq r_1$. Consider the function

$$f_1(z) = \frac{z(1-z)^2}{(1+z)^2} \quad \text{with} \quad g_1(z) = \frac{z(1-z)}{(1+z)^2}$$

belonging to the class $\mathcal{H}_1$. Figure 6 depicts that the value $zf_1'(z)/f_1(z)$ lies inside $\phi_{sin}(\mathbb{D})$ for $|z| < r_1$ and

$$\frac{zf_1'(z)}{f_1(z)} = 1 - \sin 1 \quad \text{at } z = r_1.$$

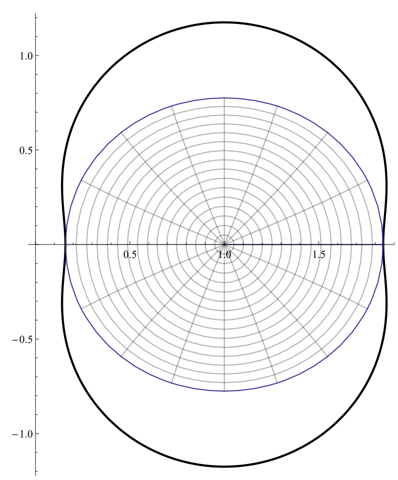


FIGURE 6. $\mathcal{R}_{\mathcal{S}^*_{sin}}(\mathcal{H}_1)$

(b) Let $f \in \mathcal{H}_2$. Then the function $p_2 : \mathbb{D} \to \mathbb{C}$ defined as $p_2(z) = g(z)(1-z)^2/z$ is a member of $\mathcal{P}$. For $f(z) = zp_2(z)/(q(z)(1-z)^2)$, note that

$$\frac{zf'(z)}{f(z)} = \frac{zp_2'(z)}{p_2(z)} - \frac{zq'(z)}{q(z)} + \frac{1+z}{1-z}. \tag{5}$$

Using Lemmas 1.1 and 1.2, (5) takes the form

$$\begin{aligned}\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left(\frac{zp_4'(z)}{p_4(z)}\right) - \operatorname{Re}\left(\frac{zq'(z)}{q(z)}\right) + \operatorname{Re}\left(\frac{1+z}{1-z}\right) \\ &\leq \frac{2r}{1-r^2} + \frac{r}{1+r} + \frac{1+r}{1-r} \quad \text{for} \quad |z| < \frac{1}{3} \\ &= \frac{1+5r}{1-r^2} < 1 + \sin 1,\end{aligned}$$

which holds for $r < r_2 := (\sqrt{27 - 2\cos 2 + 4\sin 1} - 5)/(2 + 2\sin 1)$. This gives $\mathcal{R}_{\mathcal{S}^*_{sin}}(\mathcal{H}_2) \leq r_2$. Observe that the bound r_2 can be attained. This can be seen by considering the function

$$f_2(z) = \frac{z(1+z)^2}{(1-z)^3} \quad \text{with} \quad g_2(z) = \frac{z(1+z)}{(1-z)^3}.$$

Then $f_2 \in \mathcal{H}_2$ and the fact that $zf_2'(z)/f_2(z)$ lies inside $\phi_{sin}(\mathbb{D})$ for $|z| < r_2$ is illustrated in Figure 7 wherein

$$\frac{zf_2'(z)}{f_2(z)} = 1 + \sin 1 \quad \text{at } z = r_2.$$

Hence $f_2(r_2 z)/r_2 \in \mathcal{S}^*_{sin}$. □

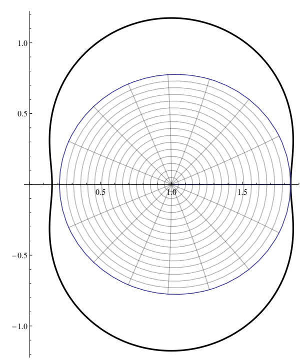


FIGURE 7. $\mathcal{R}_{\mathcal{S}^*_{sin}}(\mathcal{H}_2)$

Theorem 3.3. *The upper bound of $\mathcal{S}^*_{sin}$-radius for the class $\mathcal{H}_3$ is given by*

$$r_\alpha = \begin{cases} s_\alpha, & \text{for } 0 \leq \alpha \leq 1/2 \\ t_\alpha, & \text{for } 1/2 \leq \alpha \leq 1 \end{cases}$$

where

$$s_\alpha := \frac{5 - 2\alpha - \sqrt{27 - 20\alpha + 4\alpha^2 - 2\cos 2 + 4\sin 1}}{2(2\alpha - \sin 1 - 1)}$$

and

$$t_\alpha := \frac{2\alpha - 5 + \sqrt{27 - 20\alpha + 4\alpha^2 - 2\cos 2 - 4\sin 1 + 8\alpha \sin 1}}{2(2\alpha + \sin 1 - 1)}.$$

Proof. Let $f \in \mathcal{H}_3$. Then the functions $h_1, h_2 : \mathbb{D} \to \mathbb{C}$ defined by $h_1 = g/f \in \mathcal{P}(1/2)$ and $h_2 = g/\zeta \in \mathcal{P}$ satisfy $f(z) = \zeta(z)h_2(z)/h_1(z)$ and therefore

$$\frac{zf'(z)}{f(z)} = \frac{zh_2'(z)}{h_2(z)} - \frac{zh_1'(z)}{j_1(z)} + \frac{z\zeta'(z)}{\zeta(z)}. \tag{6}$$

For $zf'(z)/f(z) \prec \phi_{sin}(z)$, one of the necessary condition is $\operatorname{Re}(zf'(z)/f(z)) < 1 + \sin 1$. In accordance with Lemmas 1.1, 1.2 and 3.3 in (6), we obtain

$$\begin{aligned}
\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left(\frac{zh_2'(z)}{h_2(z)}\right) - \operatorname{Re}\left(\frac{zh_1'(z)}{h_1(z)}\right) + \operatorname{Re}\left(\frac{z\zeta'(z)}{\zeta(z)}\right) \\
&\leq \frac{2r}{1-r^2} + \frac{r}{1+r} + \frac{1+r-2\alpha r}{1-r} \quad \text{for} \quad |z| < \frac{1}{3} \\
&= \frac{1 + (5-2\alpha)r - 2\alpha r^2}{1-r^2} < 1 + \sin 1,
\end{aligned}$$

which yields $r < s_\alpha$. In order to prove the sharpness, consider the functions

$$f_\alpha(z) = \frac{z(1+z)^2}{(1-z)^{3-2\alpha}}, \quad g_\alpha(z) = \frac{z(1+z)}{(1-z)^{3-2\alpha}} \quad \text{and} \quad \zeta_\alpha(z) = \frac{z}{(1-z)^{2-2\alpha}}.$$

Then $f_\alpha \in \mathcal{H}_3$ and $zf_\alpha'(z)/f_\alpha(z) = \phi_{sin}(1) = 1 + \sin 1$ at $z = s_\alpha$. However the bound s_α is not sharp for the whole range of α. For instance, $zf_\alpha'(z)/f_\alpha(z)$ does not map the sub-disk $|z| < s_\alpha$ inside $\phi_{sin}(\mathbb{D})$ for $\alpha = 3/4$. This is illustrated in Figure 8(a). As a result, we will employ another necessary condition $\operatorname{Re}(zf'(z)/f(z)) > 1 - \sin 1$ for the subordination $zf'(z)/f(z) \prec \phi_{sin}(z)$ to hold. By making use of Lemmas 1.2 and 3.3 in (6), we obtain

$$\begin{aligned}
\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &\geq \frac{1 - r + 2\alpha r}{1+r} - \frac{2r}{1-r^2} - \frac{r}{1-r} \\
&= \frac{1 - (5-2\alpha)r - 2\alpha r^2}{1-r^2} > 1 - \sin 1,
\end{aligned}$$

which gives $r < t_\alpha$. In this case, $zf_\alpha'(z)/f_\alpha(z) = \phi_{sin}(-1) = 1 - \sin 1$ at $z = -t_\alpha$ and the values $zf_\alpha'(z)/f_\alpha(z)$ does not lie in $\phi_{sin}(\mathbb{D})$ for $\alpha = 1/4$ (Figure 8(b)). Now we will compare s_α and t_α to compute the desired sharp bound. Clearly Figure 8(c) shows that $\min\{s_\alpha, t_\alpha\} = s_\alpha$ for $0 < \alpha \leq 1/2$ and $\min\{s_\alpha, t_\alpha\} = t_\alpha$ for $1/2 \leq \alpha < 1$. □

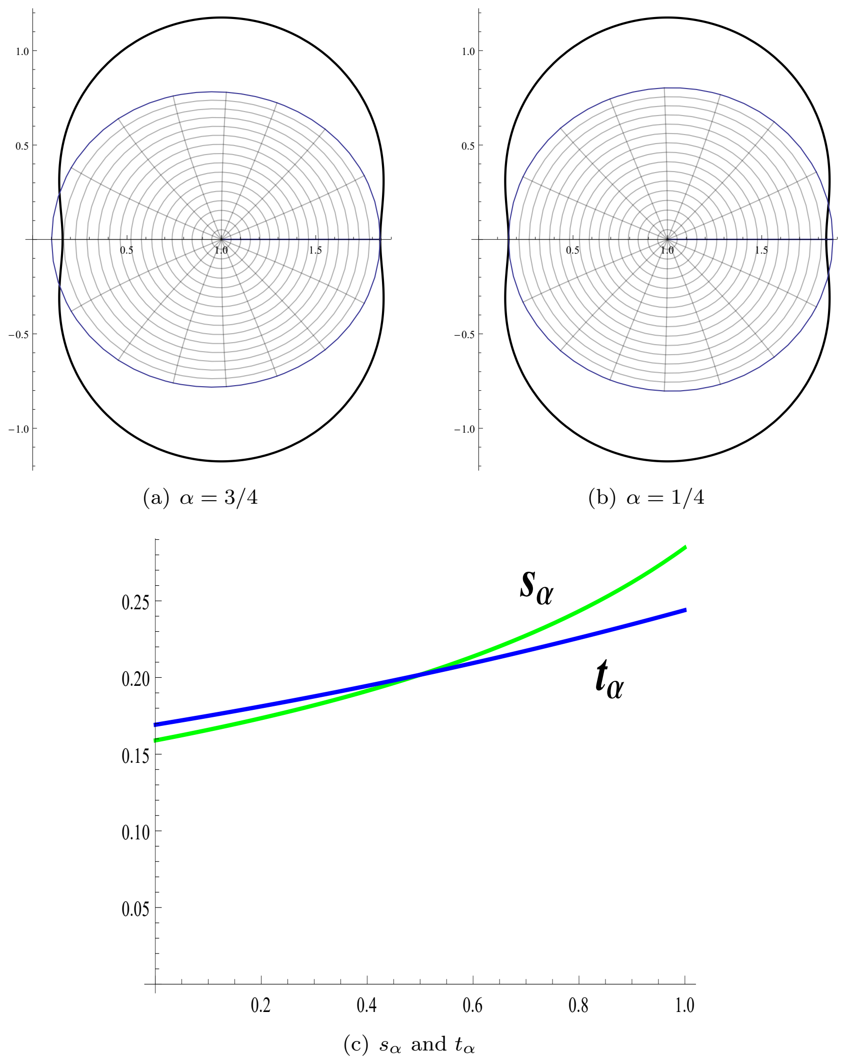


Figure 8. $\mathcal{R}_{\mathcal{S}^*_{sin}}(\mathcal{H}_3)$

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