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ON THE PSEUDO-FIBONACCI AND PSEUDO-LUCAS QUATERNIONS

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Abstract. There are a lot of quaternion numbers that are related to the Fibonacci and Lucas numbers or their generalizations have been described and extensively explored. The coefficients of these quaternions have been chosen from terms of Fibonacci and Lucas numbers. In this study, we define two new quaternions that are pseudo-Fibonacci and pseudo-Lucas quaternions. Then, we give their Binet-like formulas, generating functions, certain binomial sums and Honsberg-like, d'Ocagne-like, Catalan-like and Cassini-like identities.

1. Introduction

Ferns introduced the pseudo-Fibonacci and pseudo-Lucas sequences in 1968 as novel generalizations of the Fibonacci and Lucas sequences as follows:
First, consider two recurrence relations

$$\Phi_{n+1} = \Phi_n + \Psi_n, \tag{1}$$

$$\Psi_{n+1} = \Phi_{n+1} + \gamma\Phi_n \tag{2}$$

with initial conditions $\Phi_1 = 1$ and $\Psi_1 = 1$ in which γ is a positive integer. $\{\Phi_n\}$ and $\{\Psi_n\}$ are pseudo-Fibonacci and pseudo-Lucas numbers, respectively (see [1]). Actually, by eliminating first the Φ_n's and then the Ψ_n's, from (1) and (2), the following pseudo-Fibonacci and pseudo-Lucas sequences are obtained

$$\Phi_{n+2} = 2\Phi_{n+1} + \gamma\Phi_n, \tag{3}$$

$$\Psi_{n+2} = 2\Psi_{n+1} + \gamma\Psi_n \tag{4}$$

with initial conditions $\Phi_0 = 0$, $\Phi_1 = 1$ and $\Psi_0 = 1$, $\Psi_1 = 1$, respectively. Therefore, although having different initial values, the two numbers described by (3) and (4)

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satisfy the same recurrence relations.
A Binet-like formula for each of the pseudo-Fibonacci sequence $\{\Phi_n\}$ and pseudo-Lucas sequence $\{\Psi_n\}$ is

$$\Phi_n = \frac{A^n - B^n}{A - B}, \tag{5}$$

$$\Psi_n = \frac{A^n + B^n}{A + B} \tag{6}$$

where A and B are $1 + \sqrt{1+\gamma}$ and $1 - \sqrt{1+\gamma}$, respectively.
Thus, it is apparent that

$$AB = -\gamma, \qquad A + B = 2, \qquad A - B = 2\sqrt{1+\gamma}$$
$$A^2 = 2A + \gamma, \qquad B^2 = 2B + \gamma$$
$$A^2 + B^2 = 2(2+\gamma), \qquad A^2 - B^2 = 4\sqrt{1+\gamma}$$

(see [1]).
Quaternions applied to mechanics in three-dimensional space were first discovered by Hamilton (1805–1865) for the purpose of expanding complex numbers. According to Hamilton, a quaternion is the product of two directed lines divided by three, or alternatively, the product of two vectors. A quaternion is defined by

$$q = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$$

where a_0, a_1, a_2 and a_3 are real numbers and $e_0 = 1$, $e_1 = i$, $e_2 = j$ and $e_3 = k$ are the standard basis in $\mathbb{R}^4$ (see [2]).
The quaternion multiplication is defined using the rules:

$$e_0^2 = 1, \quad e_1^2 = e_2^2 = e_3^2 = -1$$

$$e_1 e_2 = -e_2 e_1 = e_3, \quad e_2 e_3 = -e_3 e_2 = e_1 \quad \text{and} \quad e_3 e_1 = -e_1 e_3 = e_2.$$

This algebra is associative and non-commutative.
Let $q = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$ and $p = b_0 e_0 + b_1 e_1 + b_2 e_2 + b_3 e_3$ be any two quaternions. Then addition and subtraction them are

$$q \mp p = (a_0 \mp b_0)e_0 + (a_1 \mp b_1)e_1 + (a_2 \mp b_2)e_2 + (a_3 \mp b_3)e_3$$

and for $k \in \mathbb{R}$, the multiplication by scalar is

$$kq = ka_0 e_0 + ka_1 e_1 + ka_2 e_2 + ka_3 e_3$$

and the conjugate and norm of a quaternion are

$$\overline{q} = a_0 e_0 - a_1 e_1 - a_2 e_2 - a_3 e_3$$

and

$$N(q) = q\overline{q} = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

More studies about the quaternions can be found in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

2. The Pseudo-Fibonacci and Pseudo-Lucas Quaternions

In this part, we define two new quaternions that are the pseudo-Fibonacci and pseudo-Lucas quaternions. Then, we give their Binet-like formulas, generating functions, certain binomial sums and Honsberg-like, d'Ocagne-like, Catalan-like and Cassini-like identities.

Definition 2.1. *The pseudo-Fibonacci quaternion* $\left\{\hat{\Phi}_n\right\}_{n\geq 0}$ *is defined by*

$$\hat{\Phi}_n = \Phi_n e_0 + \Phi_{n+1}e_1 + \Phi_{n+2}e_2 + \Phi_{n+3}e_3 \tag{7}$$

where Φ_n *is the nth pseudo-Fibonacci numbers.*

Definition 2.2. *The pseudo-Lucas quaternion* $\left\{\hat{\Psi}_n\right\}_{n\geq 0}$ *is defined by*

$$\hat{\Psi}_n = \Psi_n e_0 + \Psi_{n+1}e_1 + \Psi_{n+2}e_2 + \Psi_{n+3}e_3 \tag{8}$$

where Ψ_n *is the nth pseudo-Lucas number.*

Here, e_1, e_2 and e_3 correspond to the basis vectors for real quaternions. As a result of these definitions, we may derive new sequences using the pseudo-Fibonacci and pseudo-Lucas. These new sequences are called the pseudo-Fibonacci and pseudo-Lucas quaternion sequences. For $n \geq 0$, they are possible to write, respectively,

$$\hat{\Phi}_{n+2} = 2\hat{\Phi}_{n+1} + \gamma\hat{\Phi}_n, \tag{9}$$

$$\hat{\Psi}_{n+2} = 2\hat{\Psi}_{n+1} + \gamma\hat{\Psi}_n \tag{10}$$

with initial conditions

$$\hat{\Phi}_0 = e_1 + 2e_2 + (4+\gamma)e_3,$$

$$\hat{\Phi}_1 = e_0 + 2e_1 + (4+\gamma)e_2 + (8+4\gamma)e_3$$

and

$$\hat{\Psi}_0 = e_0 + e_1 + (2+\gamma)e_2 + (4+3\gamma)e_3,$$

$$\hat{\Psi}_1 = e_0 + (2+\gamma)e_1 + (4+3\gamma)e_2 + (\gamma^2+8\gamma+8)e_3,$$

respectively.

Theorem 2.1. *The Binet-like formulas of the pseudo-Fibonacci and pseudo-Lucas quaternions are, respectively,*

$$\hat{\Phi}_n = \frac{\hat{A}A^n - \hat{B}B^n}{A-B}, \tag{11}$$

$$\hat{\Psi}_n = \frac{\hat{A}A^n + \hat{B}B^n}{A+B} \tag{12}$$

where $\hat{A} = e_0 + Ae_1 + A^2e_2 + A^3e_3$ *and* $\hat{B} = e_0 + Be_1 + B^2e_2 + B^3e_3$.

Proof. From equalities (7) and (8), the Binet-like formulas (5) and (6) for the pseudo-Fibonacci and pseudo-Lucas numbers, respectively. The proof of the equality (11), we write

$$\begin{aligned}\hat{\Phi}_n &= \Phi_n e_0 + \Phi_{n+1} e_1 + \Phi_{n+2} e_2 + \Phi_{n+3} e_3 \\ &= \frac{A^n - B^n}{A-B} e_0 + \frac{A^{n+1} - B^{n+1}}{A-B} e_1 + \frac{A^{n+2} - B^{n+2}}{A-B} e_2 + \frac{A^{n+3} - B^{n+3}}{A-B} e_3 \\ &= \frac{(e_0 + Ae_1 + A^2 e_2 + A^3 e_3)A^n - (e_0 + Be_1 + B^2 e_2 + B^3 e_3)B^n}{A-B}\end{aligned}$$

The proof of the equality (12) is similar to the first proof.
Thus, the proof is completed. □

Theorem 2.2. *The generating functions for the pseudo-Fibonacci and pseudo-Lucas quaternions are, respectively,*

1. $G_{\hat{\Phi}}(x) = \dfrac{e_1 + 2e_2 + (4+\gamma)e_3 + (e_0 + \gamma e_2 + 2\gamma e_3)x}{1 - 2x - \gamma x^2}$
2. $G_{\hat{\Psi}}(x) = \dfrac{e_0 + e_1 + (2+\gamma)e_2 + (4+3\gamma)e_3 + (-e_0 + \gamma e_1 + \gamma e_2 + (\gamma^2 + 2\gamma)e_3)x}{1 - 2x - \gamma x^2}$

Proof. Let

$$G_{\hat{\Phi}}(x) = \sum_{n=0}^{\infty} \hat{\Phi}_n x^n = \hat{\Phi}_0 + \hat{\Phi}_1 x + \hat{\Phi}_2 x^2 + \cdots + \hat{\Phi}_n x^n + \ldots$$

and

$$G_{\hat{\Psi}}(x) = \sum_{n=0}^{\infty} \hat{\Psi}_n x^n = \hat{\Psi}_0 + \hat{\Psi}_1 x + \hat{\Psi}_2 x^2 + \cdots + \hat{\Psi}_n x^n + \ldots$$

be generating functions of the pseudo-Fibonacci and pseudo-Lucas quaternions, respectively. We have

1. Multiply both of sides of the equality by the term $-2x$ such as

   $$-2xG_{\hat{\Phi}}(x) = -2\hat{\Phi}_0 x - 2\hat{\Phi}_1 x^2 - 2\hat{\Phi}_2 x^3 - \cdots - 2\hat{\Phi}_n x^{n+1} - \ldots$$

   and multiply by the term $-\gamma x^2$ such as

   $$-\gamma x^2 G_{\hat{\Phi}}(x) = -\gamma\hat{\Phi}_0 x^2 - \gamma\hat{\Phi}_1 x^3 - \gamma\hat{\Phi}_2 x^4 - \cdots - \gamma\hat{\Phi}_n x^{n+2} - \ldots$$

   Then, we write

   $$\begin{aligned}(1 - 2x - \gamma x^2)G_{\hat{\Phi}}(x) &= \hat{\Phi}_0 + (\hat{\Phi}_1 - 2\hat{\Phi}_0)x + (\hat{\Phi}_2 - 2\hat{\Phi}_1 - \gamma\hat{\Phi}_0)x^2 + \ldots \\ &\quad + (\hat{\Phi}_n - 2\hat{\Phi}_{n-1} - \gamma\hat{\Phi}_{n-2})x^n + \ldots\end{aligned}$$

   Now, by using

   $$\hat{\Phi}_0 = e_1 + 2e_2 + (4+\gamma)e_3,$$
   $$\hat{\Phi}_1 = e_0 + 2e_1 + (4+\gamma)e_2 + (8+4\gamma)e_3,$$

   and

   $$\hat{\Phi}_n - 2\hat{\Phi}_{n-1} - \gamma\hat{\Phi}_{n-2} = 0,$$

   we get that

   $$G_{\hat{\Phi}}(x) = \frac{e_1 + 2e_2 + (4+\gamma)e_3 + (e_0 + \gamma e_2 + 2\gamma e_3)x}{1 - 2x - \gamma x^2}.$$

2. The proof of the second item is similar to the first proof.

Thus, the proof is completed. □

Theorem 2.3. *The exponential generating functions of the pseudo-Fibonacci and pseudo-Lucas quaternions are, respectively,*

1. $E_{\hat{\Phi}}(x) = \dfrac{\hat{A}e^{Ax} - \hat{B}e^{Bx}}{A-B}$
2. $E_{\hat{\Psi}}(x) = \dfrac{\hat{A}e^{Ax} + \hat{B}e^{Bx}}{A+B}$

Proof. Let

$$E_{\hat{\Phi}}(x) = \sum_{n=0}^{\infty} \hat{\Phi}_n \frac{x^n}{n!} \quad \text{and} \quad E_{\hat{\Psi}}(x) = \sum_{n=0}^{\infty} \hat{\Psi}_n \frac{x^n}{n!}$$

be exponential generating functions of the pseudo-Fibonacci and pseudo-Lucas quaternions, respectively. Using the identity (11) and (12), we get

1.
$$\begin{aligned} E_{\hat{\Phi}}(x) = \sum_{n=0}^{\infty} \hat{\Phi}_n \frac{x^n}{n!} &= \frac{\hat{A}}{A-B} \sum_{n=0}^{\infty} \frac{(Ax)^n}{n!} - \frac{\hat{B}}{A-B} \sum_{n=0}^{\infty} \frac{(Bx)^n}{n!} \\ &= \frac{\hat{A}e^{Ax} - \hat{B}e^{Bx}}{A-B} \end{aligned}$$

2. The proof of the second item is similar to the first proof.

Thus, the proof is completed. □

Theorem 2.4. *Let $m, n \in \mathbb{Z}^+$. Then, the sums of the first n terms for the pseudo-Fibonacci and pseudo-Lucas quaternions are, respectively,*

1. $T_{\hat{\Phi}} = \dfrac{\hat{\Phi}_{n+2} - \hat{\Phi}_{n+1} + e_0 + 2e_1 + (4+\gamma)e_2 + (8+4\gamma)e_3}{\gamma+1}$
2. $T_{\hat{\Psi}} = \dfrac{\hat{\Phi}_{n+2} - \hat{\Phi}_{n+1} + e_0 + (2+\gamma)e_1 + (4+3\gamma)e_2 + (\gamma^2+8\gamma+8)e_3}{\gamma+1}$

Proof. Let

$$T_{\hat{\Phi}} = \sum_{k=1}^{n} \hat{\Phi}_k \quad \text{and} \quad T_{\hat{\Psi}} = \sum_{k=1}^{n} \hat{\Psi}_k$$

be sums of the first n terms for the pseudo-Fibonacci and pseudo-Lucas quaternions, respectively. Using identities (9) and (10), we have

1.
$$\begin{aligned} \hat{\Phi}_3 &= 2\hat{\Phi}_2 + \gamma\hat{\Phi}_1, \\ \hat{\Phi}_4 &= 2\hat{\Phi}_3 + \gamma\hat{\Phi}_2, \\ \hat{\Phi}_5 &= 2\hat{\Phi}_4 + \gamma\hat{\Phi}_3, \\ &\dots, \\ \hat{\Phi}_{n+1} &= 2\hat{\Phi}_n + \gamma\hat{\Phi}_{n-1}, \\ \hat{\Phi}_{n+2} &= 2\hat{\Phi}_{n+1} + \gamma\hat{\Phi}_n \end{aligned}$$

If we sum both of sides of the identities above, we obtain,

$$\hat{\Phi}_{n+2} + \hat{\Phi}_1 = \hat{\Phi}_{n+1} + (\gamma+1)\sum_{k=1}^{n} \hat{\Phi}_k.$$

2. The proof of the second item is similar to the first proof.

Thus, the proof is completed. □

Theorem 2.5. *Let $m, n \in \mathbb{Z}^+$. Then,*

1. $\sum_{n=1}^{m} \binom{n}{m} \hat{\Phi}_n = \hat{\Phi}_{2m}$
2. $\sum_{n=1}^{m} \binom{n}{m} \hat{\Psi}_n = \hat{\Psi}_{2m}$

Proof. 1. Applying identities (11), we obtain

$$\begin{aligned}\sum_{n=1}^{m} \binom{n}{m} \hat{\Phi}_n &= \frac{\hat{A}}{A-B}\sum_{n=1}^{m} \binom{n}{m} A^n 1^{m-n} - \frac{\hat{B}}{A-B}\sum_{n=1}^{m} \binom{n}{m} B^n 1^{m-n} \\ &= \frac{\hat{A}(A+1)^m - \hat{B}(B+1)^m}{A-B} \\ &= \frac{\hat{A}A^{2m} - \hat{B}B^{2m}}{A-B}\end{aligned}$$

2. The proof of the second item is similar to the first proof.

Thus, the proof is completed. □

Theorem 2.6. *(The Honsberger-like identity). Let $\hat{\Phi}_n$ and $\hat{\Psi}_n$ be the $n-$th the pseudo-Fibonacci and pseudo-Lucas quaternions, respectively. For $n, m \in \mathbb{Z}^+$, we have*

1. $\hat{\Phi}_n \hat{\Phi}_{m+1} + \gamma \hat{\Phi}_{n-1} \hat{\Phi}_m = \dfrac{\hat{A}^2 A^{n+m} - \hat{B}^2 B^{n+m}}{2\sqrt{1+\gamma}}$
2. $\hat{\Psi}_n \hat{\Psi}_{m+1} + \gamma \hat{\Psi}_{n-1} \hat{\Psi}_m = \sqrt{1+\gamma}\left(\dfrac{\hat{A}^2 A^{n+m} - \hat{B}^2 B^{n+m}}{2}\right)$

Proof. 1. Applying identities (11), we get

$$\begin{aligned}\hat{\Phi}_n \hat{\Phi}_{m+1} + \gamma \hat{\Phi}_{n-1} \hat{\Phi}_m &= \left(\frac{\hat{A}A^n - \hat{B}B^n}{A-B}\right)\left(\frac{\hat{A}A^{m+1} - \hat{B}B^{m+1}}{A-B}\right) \\ &\quad + \gamma\left(\frac{\hat{A}A^{n-1} - \hat{B}B^{n-1}}{A-B}\right)\left(\frac{\hat{A}A^m - \hat{B}B^m}{A-B}\right) \\ &= \frac{\hat{A}^2 A^{n+m} A - \hat{A}\hat{B}A^n B^{m+1} - \hat{B}\hat{A}B^n A^{m+1} + \hat{B}^2 B^{n+m} B}{(A-B)^2} \\ &\quad + \gamma\left(\frac{\hat{A}^2 A^{n+m} B - \hat{A}\hat{B}A^n B^{m+1} - \hat{B}\hat{A}B^n A^{m+1} + \hat{B}^2 B^{n+m} A}{-\gamma(A-B)^2}\right) \\ &= \frac{\hat{A}^2 A^{n+m}(A-B) - \hat{B}^2 B^{n+m}(A-B)}{(A-B)^2} \\ &= \frac{\hat{A}^2 A^{n+m} - \hat{B}^2 B^{n+m}}{2\sqrt{1+\gamma}}\end{aligned}$$

2. The proof of the second item is similar to the first proof.

Thus, the proof is completed. □

Proposition 1. *For $n, m \in \mathbb{Z}^+$, The following identities are valid:*

1. $\hat{\Phi}^2_{m+1} + \gamma\hat{\Phi}^2_m = \dfrac{\hat{A}^2 A^{2m+1} - \hat{B}^2 B^{2m+1}}{2\sqrt{1+\gamma}}$
2. $\hat{\Psi}^2_{m+1} + \gamma\hat{\Psi}^2_m = \sqrt{1+\gamma}\left(\dfrac{\hat{A}^2 A^{2m+1} - \hat{B}^2 B^{2m+1}}{2}\right)$

Proof. If $m+1$ is written instead of n in the Theorem 2.6, the identities are proven. □

Theorem 2.7. *(The d'Ocagne-like identity). Let $\hat{\Phi}_n$ and $\hat{\Psi}_n$ be the $n-th$ the pseudo-Fibonacci and pseudo-Lucas quaternions, respectively. For $m \geq n+1$, we have*

1. $\hat{\Phi}_n\hat{\Phi}_{m+1} - \hat{\Phi}_{n+1}\hat{\Phi}_m = (-\gamma)^m \left(\dfrac{\hat{A}\hat{B}A^{n-m} - \hat{B}\hat{A}B^{n-m}}{2\sqrt{1+\gamma}}\right)$
2. $\hat{\Psi}_n\hat{\Psi}_{m+1} - \hat{\Psi}_{n+1}\hat{\Psi}_m = (-\gamma)^m \sqrt{1+\gamma}\left(\dfrac{\hat{B}\hat{A}B^{n-m} - \hat{A}\hat{B}A^{n-m}}{2}\right)$

Proof. 1. Applying identities (11), we get

$$\begin{aligned}
\hat{\Phi}_n\hat{\Phi}_{m+1} - \hat{\Phi}_{n+1}\hat{\Phi}_m &= \left(\frac{\hat{A}A^n - \hat{B}B^n}{A-B}\right)\left(\frac{\hat{A}A^{m+1} - \hat{B}B^{m+1}}{A-B}\right) \\
&- \left(\frac{\hat{A}A^{n+1} - \hat{B}B^{n+1}}{A-B}\right)\left(\frac{\hat{A}A^m - \hat{B}B^m}{A-B}\right) \\
&= \frac{\hat{A}^2A^{n+m}A - \hat{A}\hat{B}A^nB^{m+1} - \hat{B}\hat{A}B^nA^{m+1} + \hat{B}^2B^{n+m}B}{(A-B)^2} \\
&- \left(\frac{\hat{A}^2A^{n+m}A - \hat{A}\hat{B}A^{n+1}B^m - \hat{B}\hat{A}B^{n+1}A^m + \hat{B}^2B^{n+m}B}{(A-B)^2}\right) \\
&= \frac{\hat{A}\hat{B}A^nB^m(A-B) - \hat{B}\hat{A}A^mB^n(A-B)}{(A-B)^2} \\
&= \frac{\hat{A}\hat{B}A^nB^m - \hat{B}\hat{A}A^mB^n}{A-B} \\
&= (-\gamma)^m\left(\frac{\hat{A}\hat{B}A^{n-m} - \hat{B}\hat{A}B^{n-m}}{A-B}\right)
\end{aligned}$$

2. The proof of the second item is similar to the first proof.

Thus, the proof is completed. □

Theorem 2.8. *(The Catalan-like identity). Let $\hat{\Phi}_n$ and $\hat{\Psi}_n$ be the $n-th$ the pseudo-Fibonacci and pseudo-Lucas quaternions, respectively. For $n \geq m \geq 0$, we have*

1. $\hat{\Phi}_{n-m}\hat{\Phi}_{n+m} - \hat{\Phi}^2_n = (-\gamma)^{n-m}\left(\dfrac{\hat{A}\hat{B}B^m - \hat{B}\hat{A}A^m}{2\sqrt{1+\gamma}}\right)\Phi_m$

2. $\hat{\Psi}_{n-m}\hat{\Psi}_{n+m}-\hat{\Psi}_n^2=(-\gamma)^{n-m}\left(\frac{\hat{B}\hat{A}A^m-\hat{A}\hat{B}B^m}{2}\right)\left(\frac{A^m-B^m}{2}\right)$

Proof. 1. Applying identities (11), we get

$$\begin{aligned}\hat{\Phi}_{n-m}\hat{\Phi}_{n+m}-\hat{\Phi}_n^2 &= \left(\frac{\hat{A}A^{n-m}-\hat{B}B^{n-m}}{A-B}\right)\left(\frac{\hat{A}A^{n+m}-\hat{B}B^{n+m}}{A-B}\right)\\ &-\left(\frac{\hat{A}A^n-\hat{B}B^n}{A-B}\right)^2\\ &=\frac{\hat{A}^2A^{2n}-\hat{A}\hat{B}A^{n-m}B^{n+m}-\hat{B}\hat{A}B^{n-m}A^{n+m}+\hat{B}^2B^{2n}}{(A-B)^2}\\ &-\left(\frac{\hat{A}^2A^{2n}-\hat{A}\hat{B}A^nB^n-\hat{B}\hat{A}B^nA^n+\hat{B}^2B^{2n}}{(A-B)^2}\right)\\ &=\frac{(-\gamma)^{n-m}(\hat{A}\hat{B}B^m-\hat{B}\hat{A}A^m)(A^m-B^m)}{(A-B)^2}\\ &=(-\gamma)^{n-m}\left(\frac{\hat{A}\hat{B}B^m-\hat{B}\hat{A}A^m}{2\sqrt{1+\gamma}}\right)\Phi_m\end{aligned}$$

2. The proof of the second item is similar to the first proof.

Thus, the proof is completed. □

Theorem 2.9. *(The Cassini-like identity). Let $\hat{\Phi}_n$ and $\hat{\Psi}_n$ be the $n-th$ the pseudo-Fibonacci and pseudo-Lucas quaternions, respectively. For $n\geq 1$, we have*

1. $\hat{\Phi}_{n-1}\hat{\Phi}_{n+1}-\hat{\Phi}_n^2=(-\gamma)^{n-1}\left(\frac{\hat{A}\hat{B}B-\hat{B}\hat{A}A}{2\sqrt{1+\gamma}}\right)$
2. $\hat{\Psi}_{n-1}\hat{\Psi}_{n+1}-\hat{\Psi}_n^2=(-\gamma)^{n-1}\sqrt{1+\gamma}\left(\frac{\hat{B}\hat{A}A-\hat{A}\hat{B}B}{2}\right)$

Proof. If 1 is written instead of m in the Theorem 2.8, the identities are proven. □

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