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Anisotropic pressure effect on strongly magnetized white dwarfs using quasi-local equation of state

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Abstract:

This work addresses the effect of anisotropic pressure on the strong magnetized white dwarf. A quasi-local equation of state is used to formulate the anisotropic factor inside the star. We use the assumption that the magnetic field is constant and strong enough to make degenerate electrons very energetic and occupy the first Landau level. Modified Lane-Emden equation is formulated and solved numerically. We interested in magnetized white dwarfs with mass range M=2.1-2.8 M_⊙ that was predicted as a progenitor for a peculiar type Ia-supernovae, which are characterized by low kinetic energy and over luminosities. In a good approximation, a mass-radius relation for this range of star masses was determined. It is found that the mass and radius of the stars increase due to anisotropic effect. The radius of the star decreases with increasing maximum energy of degenerate electrons and magnetic field intensity, which indicates that these quantities support occurrence of the explosion. The main disadvantage of used model is that the predicted internal magnetic field strength exceeds the maximum magnetic field needed for star stability, which refers that these stars are unstable and unbound.

Keywords: Magnetic white dwarfs, Supernovae, Anisotropic.

1. Introduction

It is known that white dwarf stars cannot have a mass exceeds Chandrasekhar limit 1.44 $M_{\odot}$. Sometimes white dwarf can increase its mass by accreting mass form a companion star. The increase in mass will raise the inward gravitational force which in turn makes the star contracts and heats up its core such that it can allow a nuclear fusion to start again. The star internal pressure and temperature increase quickly with time. Subsequently, within a few seconds, a substantial fraction of the white dwarf matter undergoes a runaway reaction which releases huge energy in highly shining explosion known as type Ia-supernova without leaving any remnant. All supernovae produced in the same mechanism provide information about the

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expansion of the universe and used as standard candles in measurement of cosmic distances [1,2].

Recent observations of some particular type Ia-supernovae such SN 2003fg, SN 2006gz, SN 2007if and SN 2009dc characterized by exceptionally high luminosity and lower kinetic energy [3, 4]. The kinetic energy comes from the difference between the binding energy of the white dwarf and the energy arising from the synthesis of elements in the explosion through the fusion. It is found that the light-curves of the previous supernovae are over-luminous and slow-rising, indicating that they cannot be calibrated as standard candles. This makes a lack of confidence in all type Ia-supernovae in measuring cosmic distances and expansion history of the universe. This phenomenon can be explained if we suggest that the progenitor of these supernovae has a highly super Chandrasekhar mass white dwarf because the larger mass implies a larger binding energy of the star and hence a low kinetic energy (or smaller velocity) and/or higher luminosity than that observed in a standard type Ia-supernova [3]. These observations indicate that the progenitor of type Ia-supernovae have masses up to $2.1 - 2.8\ M_{\odot}$ which is higher than the famous Chandrasekhar mass limit.

There is a lot of efforts to find mechanisms to explain the new super-Chandrasekhar mass of white dwarfs. For example, a rotating white dwarf accompanied by accretion from a companion star can explain super-Chandrasekhar limit [5]. Also, strong magnetic field can support that high mass limit. Das and Mukhopadhyay suggested that existence of super-strong central magnetic field about $8.8 \times 10^{17} G$ can support mass up to $2.58\ M_{\odot}$ [6]. Suh et al. [7] studied the influence of the weak magnetic field on the equation of state of a fully degenerate electron and obtained the mass radius relation for magnetic white dwarfs. They found that, at the same central density, the mass and radius of the white dwarf are increasing compared to non-magnetic white dwarfs. In Refs. [8–10] the authors obtained the equation of state for the degenerate electron of magnetic white dwarfs in a polytropic form by fitting the original equation of state of degenerate electron in strong magnetic field for one, two and three Landau levels. The mass radius relation is obtained by solving Lane-Emden equation, using fitting polytropic equations of states, and they proposed that the mass of strong magnetic white dwarfs lies in the range $2.3 - 2.6\ M_{\odot}$.

It is worth noting that, until now, strongly magnetized white dwarfs have not been observed. Perhaps the reason is that the surface magnetic field of a star can be screened by accretion process form the companion star as it happens in the type Ia-supernovae progenitors.

Where superficial condensed plasma can produce an opposite magnetic moment, which decreases the surface magnetic field strength of the star. Thus, it becomes more difficult to be observed [8].

On the other hand, it is believed that the stellar structure and its development depends on the idea that the star consists of a perfect fluid, and this requires its internal pressure be isotropic [11, 12]. In the recent decades, theoretical studies have shown that many systems particularly that have a high density do not follow this belief and shows anisotropic pressure beside its radial one. Generally, the pressure anisotropy in stellar structure is produced by many physical processes in high- and low-density regimes [13–15]. For low density objects anisotropy may be caused by slow rotation [15], anisotropic velocity distribution [15–17], mixture of two non-interacting fluids [18], viscosity [19, 20] or by repulsive force in low mass charged white dwarfs [21, 22]. Also, there are some static solutions of spherical symmetric Einstein-Vlasov equations that are anisotropic [23]. In highly dense systems, phase transitions by relativistic nuclear interactions can make nuclear matter be anisotropic, for more details see [24–28].

It was established that the magnetic field, in degenerate stars, can break the rotational symmetry $O(3)$ and produces anisotropic pressure [29]. The necessary criterion for the magnetic field, which cannot be neglected within the star, that contributes to pressure and energy with the same order as matter contribution has been calculated. It was found that the pressure is divided into two components, one in the direction of the field, i.e. parallel to the pressure, and the other perpendicular to the direction of the field, i.e. in the transverse direction of the pressure. The authors in [29] have indicated that these results are valid for relativistic systems composed of fermions in the presence of uniform and constant magnetic field. In addition to that, anisotropy can be induced into any isotropic compact stars naturally, Herrera [30] proved that shear in the stellar fluid flow, inhomogeneity in energy-density distribution and dissipative heat fluxes can initiate anisotropic effects into isotropic configuration. Therefore, magnetic white dwarfs fall under the influence of anisotropic pressure. In [31] the anisotropic effects were considered in highly magnetic white dwarfs. The strategy is that dealing with radial and anisotropic pressure on an equal footing enable them to recover the covariance form of Lane-Emden equations. This method enables them to obtain a new mass-radius relation for these stars.

In this paper the equation of state will be formulated using the hypothesis that the magnetic field inside the star is constant and very strong such that the degenerate electrons will be very energetic and occupy the first Landau level. A quasi-local equation of state will be used to formulate the anisotropic effect. Using these assumptions, a modified Lane-Emden equation is calculated. Results show that anisotropic effect leads to disappear the maximum mass limit found in [6] and a mass-radius relation is investigated. In section II equation of state for electrons in strong magnetic field is calculated in detail. The main features of the quasi-local equation of state for anisotropy and modified Lane-Emden equation is recorded in section III. In section IV a new mass-radius relation for the magnetic white dwarfs is achieved and dynamical stability of the star is examined.

2. Equation of state for electrons in strong magnetic field

We will suggest that the magnetic field will be very strong, such that it will prevent the electrons from moving, therefore, there will be no current, i.e. $\vec{J} = 0$ where $\vec{J}$ is the electrons current inside the star. Hence from steady state Maxwell equation $\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J}$ we can see that there is no spatial variation of the magnetic flux density $\vec{B}$ for $\vec{J} = 0$, which means that the magnetic field will be constant through the star [32]. It well known that the motion of the electron in magnetic field is quantized which known as Landau quantization [33]. If the electron moves perpendicularly to a constant uniform magnetic field such that $\hbar\omega_c \geq m_e c^2$ where $\hbar$ is the Planck constant, m_e is the mass of the electron, c is the speed of light, $\omega_c = \frac{eB}{m_e c}$ is the critical cyclotron frequency and e is the charge of the electron, the solution of Dirac equation is given by

$$\begin{aligned} E^2_{\upsilon,P_z} &= P_z^2 c^2 + m_e^2 c^4 + 2\upsilon eB\hbar \\ &= P_z^2 c^2 + m_e^2 c^4 (1 + 2\upsilon B_r) \qquad (1) \end{aligned}$$

$\upsilon = n_L + \frac{1}{2} + \sigma$, $n_L = 0, 1, 2, \ldots$ is the Landau level, $\sigma = \pm\frac{1}{2}$ is the electron spin, $B_r = \frac{B}{B_c}$ where $B_c = \frac{m_e^2 c^4}{e\hbar} = 4.414 \times 10^{13} G$ is the critical magnetic field strength and P_z is the electron momentum in z direction. In analogous to Eq. (1) Fermi energy of the electrons in υ-level can be defined as

$$E_F^2 = P_{F,\upsilon}^2 c^2 + m_e^2 c^4 (1 + 2\upsilon B_r) \qquad (2)$$

and the dimensionless Fermi energy can be written as

$$\epsilon_F = \frac{E_F}{m_e c^2} = [x_\upsilon^2 + 2\upsilon B_r + 1]^{\frac{1}{2}} \tag{3}$$

Where $x_\upsilon = \frac{P_{F,\upsilon}}{m_e c^2}$ is the dimensionless Fermi momentum and $p_{F,\upsilon}$ is the Fermi momentum of the electrons. We can determine the maximum number of occupied Landau level υ_m according to the maximum dimensionless Fermi energy, namely $\epsilon_{Fmax} = \frac{E_{Fmax}}{m_e c^2}$, from the condition $P_{F,\upsilon} \geq 0$ [6] such that

$$\upsilon_m \leq Integer\ \left(\frac{\epsilon_{Fmax}^2 - 1}{2B_r}\right) \tag{4}$$

On the other hand, the available density of states for electrons in magnetic field will also be modified. In this case, the electron density of state is given by [7]

$$\frac{2}{c\hbar^2} \sum_{\upsilon=0}^{\upsilon_m} g_\upsilon \int \frac{eB}{(2\pi)^2} dp_z \tag{5}$$

Where $g_\upsilon = (2 - \delta_{0\upsilon})$, $\delta_{0\upsilon}$ is Kronecker delta function and υ_m is the highest occupied landau level. Changing in density of state will modify the thermodynamical function of electrons. Then the electron number density in degenerate case will be given as

$$n_e = \frac{B_r}{2\pi^2\lambda^3} \sum_{\upsilon=0}^{\upsilon_m} g_\upsilon \int_0^{x_\upsilon} d\left(\frac{p_z}{m_e c}\right)$$

$$= \frac{B_r}{2\pi^2\lambda^3} \sum_{\upsilon=0}^{\upsilon_m} g_\upsilon x_\upsilon \tag{6}$$

Where $\lambda = \frac{\hbar}{m_e c}$ is the Compton wavelength of the electron. The mass density is related ρ to the number density through the relation

$$\rho = \mu_e m_\mu n_e \tag{7}$$

Where μ_e is the molecular weight per electron and m_μ is the unified atomic mass unit. Using the modified phase space Eq. (5), the energy density and the pressure of the degenerate electrons can be given as

$$\varepsilon_e = \frac{B_r}{2\pi^2\lambda^3} \sum_{\upsilon=0}^{\upsilon_m} g_\upsilon \int_0^{x_\upsilon} E_{\upsilon,p_z} d\left(\frac{p_z}{m_e c}\right)$$

$$= \frac{B_r}{2\pi^2\lambda^3} m_e c \sum_{\upsilon=0}^{\upsilon_m} g_\upsilon (1 + 2\upsilon B_r)\psi_+(z) \tag{8}$$

$$P_e = -\varepsilon_e + n_e E_F$$

$$= \frac{B_r}{2\pi^2\lambda^3} m_e c \sum_{\upsilon=0}^{\upsilon_m} g_\upsilon (1 + 2\upsilon B_r)\psi_-(z) \qquad (9)$$

Where

$$\psi_\pm = \frac{1}{2} z\sqrt{1+z^2} \pm \frac{1}{2} ln\left(z + \sqrt{1+z^2}\right) \qquad (10)$$

$$z = \frac{x_\upsilon}{\sqrt{1+2\upsilon B_c}} \qquad (11)$$

It is known that the strength of the magnetic field controls the separation gaps between energy levels and its number. Eq.(4) shows that the number of energy levels decrease and separation between energy levels increase with increasing magnetic field strength. We interested in strong magnetic field such that the degenerate electrons will be restricted to occupy only the first Landau level, where $\upsilon = 0$, while the higher levels will be empty [6]. In case of occupied first Landau level by degenerate electrons, the mass density can be calculated using Eqs. (6) and (7)

$$\rho = \frac{\mu_e m_\mu B_r}{2\pi^2\lambda^3} x_0 \qquad (12)$$

The pressure can be determined from Eq. (9)

$$P_e = \frac{m_e c^2 B_r}{4\pi^2\lambda^3}\left[x_0\sqrt{x_0^2+1} - ln\left(x_0 + \sqrt{x_0^2+1}\right)\right] \qquad (13)$$

Magnetic white dwarfs provide a very high-density medium with a high magnetic field strength, which stimulates the relativistic characteristics of degenerate electrons to appear and to have an effect, such that we can use the approximation $E_F \gg m_e c^2$ or $\epsilon_F \gg 1$. Using this approximation into Eq. (3) taking into account that there is only one occupied level by degenerate electrons, namely $\upsilon = 0$, one gets

$$x_0 = \sqrt{\epsilon_F^2 - 1} \approx \epsilon_F \qquad (14)$$

Using this approximation into Eqs. (12) and (13)

$$\rho = \frac{\mu_e m_\mu B_r}{2\pi^2\lambda^3}\sqrt{\epsilon_F^2 - 1}$$

$$\rho = \frac{\mu_e m_\mu B_r}{2\pi^2\lambda^3} \epsilon_F \qquad (15)$$

$$P_e = \frac{m_e c^2 B_r}{4\pi^2\lambda^3}\left[\epsilon_F^2 - \ln 2\epsilon_F\right] \qquad (16)$$

Logarithmic term in Eq. (16) can be neglected in the limit $\epsilon_F \gg 1$ and using Eq. (15) we can determine the equation of state for degenerate electrons

$$P_e = k_m \rho^2 \tag{17}$$

$$k_m = \frac{\pi^2 m_e c^2 \lambda^3}{\mu_e^2 m_\mu^2 B_r} \tag{18}$$

The central density can be calculated from Eq. (15) using maximum dimensionless Fermi energy ϵ_{Fmax} such that

$$\rho_c = \frac{\mu_e m_\mu B_r}{2\pi^2 \lambda^3} \sqrt{\epsilon_{Fmax}^2 - 1} \tag{19}$$

Only if the first level is filled with electrons $\upsilon = 0$, as we considered, then it should be $\upsilon_m = 1$, see [6]. Using this value in Eq.(4) to calculate the maximum dimensionless Fermi energy ϵ_{Fmax} which can be used to calculate the density at the center from Eq.(19)

$$\rho_c = \frac{\mu_e m_\mu}{\sqrt{2}\pi^2 \lambda^3} B_r^{\frac{3}{2}} \tag{20}$$

3. Anisotropic Lane-Emiden Equation

For non-rotating stars with spherical symmetry and uniform composition, the hydrodynamic equilibrium equation can be written as

$$\frac{dP_r}{dr} = -\rho \frac{d\phi}{dr} + \frac{2}{r}(P_\perp - P_r) \tag{21}$$

Where P_r is the radial pressure of the star, ρ is the matter density and ϕ is the Newtonian gravitational potential. $P_\perp$ is the tangential pressure and $P_\perp - P_r$ is the anisotropic factor, as called by Herrera and de Leon [34], measuring the anisotropic effect in the system. Eq. (21) is the non-relativistic limit of Tolman-Oppenheimer-Volkoff equation for anisotropic matter. The choice of the anisotropic factor should satisfy initial conditions of Eq.(21) that it should vanish at least as rapidly a $r \to 0$, or $\lim_{r\to 0} \frac{P_\perp - P_r}{r} = 0$. We can see that in the case $P_\perp > P_r$ the pressure gradient will increase, or the anisotropic force will be directed outward, so the mass of the star should increase. But for $P_\perp < P_r$ the pressure gradient will decrease, or the anisotropic force will be directed inward, and the mass of the star will decrease. Unfortunately, there is no evidence to suggest the real form of anisotropy in compact stars. Therefore, we follow some theoretical suggestions that meet the boundary conditions of the system. Bowers and Liang [35] suggested a form of anisotropic factor that dependence on pressure, density, and compactness of the star and that it satisfies the boundary conditions of the system. This model is studied in case of magnetic white dwarfs in [36]. Following [37] and [38] a quasi-local equation of state will be used to describe the anisotropy effect, such that

$$P_{\perp} - P_r = \alpha P_r \mu \tag{22}$$

where α is the anisotropy parameter which measures the anisotropy effect in the system and μ is the compactness defined as

$$\mu = \frac{2Gm(r)}{c^2 r} \tag{23}$$

The main feature of the quasi-local model is that it considers the quasi-local properties of the star configuration via compactness and local properties of the matter via pressure [38]. Also, we can notice that this model fulfills the boundary conditions of the system. The anisotropy factor vanishes at the center due to the compactness $\rightarrow r^2$, where the stellar fluid should be isotropic. Also at the surface of the star the anisotropic vanishes due the presence of the radial pressure which disappears there, Although there is no evidence that tangential pressure at the surface is equal to zero [39]. But we will accept this idea that adopted by this model. For more details and applications on this model see [40–43].

For spherical symmetry and uniform composition stars, Poisson equation can be written as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi\rho G \tag{24}$$

and polytropic equation of state (17) and mass equation may be written in the form

$$P_r = k\rho^{\gamma} \tag{25}$$

$$\frac{dm}{dr} = 4\pi\rho r^2 \tag{26}$$

$n = \frac{1}{\gamma - 1}$ is the polytropic index. Use Eqs. (22), (23) and (25) into (21)

$$\frac{d\phi}{dr} = -k\frac{1+n}{n}\rho^{\frac{1-n}{n}}\frac{d\rho}{dr} + \alpha\frac{4G}{c^2}\frac{mk\rho^{\frac{1}{n}}}{r^2} \tag{27}$$

Define the dimensionless density ω

$$\rho = \rho_c \omega^n \tag{28}$$

Where ρ_c is the cental density. Use Eqs. (27) and (28) into (24)

$$\frac{d}{dr}\left(r^2\frac{d\omega}{dr}\right) = -\frac{4\pi G}{k(1+n)}\rho^{\frac{n-1}{n}}r^2\omega^n + \alpha\frac{4G}{c^2(1+n)}\frac{d}{dr}(m\omega) \tag{29}$$

Define the dimensionless coordinate ξ

$$r = r_c \xi \tag{30}$$

$$r_c = \frac{k(1+n)}{4\pi G}\rho_c^{\frac{1-n}{n}} \tag{31}$$

Use Eqs. (30) and (31) into Eqs.(29) and (26), one gets

$$\frac{d}{d\xi}\left(\xi^2\frac{d\omega}{d\xi}\right) = -\xi^2\omega^n + \alpha\frac{4G}{(1+n)c^2 r_c}\frac{d}{d\xi}(m\omega) \tag{32}$$

$$\frac{dm}{d\xi} = 4\pi\rho_c r_c^3 \xi^2 \omega^n \tag{33}$$

Use Eq . (33) into (32)

$$\frac{d}{d\xi}\left(\xi^2\frac{d\omega}{d\xi}\right) = -\frac{1}{4\pi\rho_c r_c^3}\frac{dm}{d\xi} + \alpha\frac{4G}{(1+n)c^2 r_c}\frac{d}{d\xi}(m\omega) \tag{34}$$

Integrate Eq. (34) form the center of the star where $\xi = 0$ to the star surface where $\xi = \xi_R$ taking into account the following boundary conditions

$$m(\xi_R) = M, \quad m(0) = 0, \quad \omega(\xi_R) = 0, \quad \omega(0) = 0 \tag{35}$$

where $\omega(\xi_R) = 0$ guarantees that the stellar density is vanished at the surface of the star and ξ_R is the dimensionless total star radius. We can obtain the total mass of the star as

$$M = 4\pi\rho_c r_c^3 \xi_R^3 \left(\frac{d\omega}{d\xi}\right)_{\xi_R} \tag{36}$$

Also, from Eq. (34) we can see that

$$\xi^2\frac{d\omega}{d\xi} + \left[\frac{1}{4\pi\rho_c r_c^3} - \frac{4G\alpha\omega}{(1+n)c^2 r_c}\right]m = const \tag{37}$$

Using the boundary conditions $\xi = 0$ and $m(0) = 0$, we find that $const = 0$, hence we can obtain the mass as a function of ξ

$$m(\xi) = -\left[\frac{1}{4\pi\rho_c r_c^3} - \frac{4G\alpha\omega}{(1+n)c^2 r_c}\right]^{-1}\xi^2\frac{d\omega}{d\xi} \tag{38}$$

It is clear that Eq.(38) goes to Eq.(36) at $\xi = \xi_R$ where $m(\xi_R) = M$ and $\omega(\xi_R) = 0$. now removing the mass from Eq .(32) using Eqs. (33) and (38), one finds

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\omega}{d\xi}\right) = -\omega^n + \alpha\beta\left[\omega^{n+1} - (1-\alpha\beta\omega)^{-1}\left(\frac{d\omega}{d\xi}\right)^2\right] \tag{39}$$

$$\beta = \frac{16\pi G}{(1+n)c^2}\rho_c r_c^2 \tag{40}$$

Eq. (39) is the Modified Lane-Emden equation for polytropic index n. In our case $= 1$, use Eqs. (20) and (31) into Eqs. (39) and (40), one gets

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\omega}{d\xi}\right) = -\omega + \alpha\beta\left[\omega^2 - (1-\alpha\beta\omega)^{-1}\left(\frac{d\omega}{d\xi}\right)^2\right] \quad (41)$$

$$\beta = \frac{2\sqrt{2}m_e}{\mu_e m_\mu} B_r^{\frac{1}{2}} \quad (42)$$

And from Eq. (30) we can calculate the radius of the star

$$R = r_c\xi_r \quad (43)$$

Eq.(41) can be solved numerically with the boundary conditions $\omega(0) = 1$ and $\left(\frac{d\omega}{d\xi}\right)_{\xi_R} = 0$. In turn the values of ξ_R and $\left(\frac{d\omega}{d\xi}\right)_{\xi_R}$, which can be determined at $\omega(\xi_R) = 0$, can be used to calculate the mass and radius of the star from Eqs.(36) and (43).

4. Results and discussion

From Eq. (41) we can recover the original case when $= 0$. In this case the solution is $\omega_{\alpha=0} = \frac{\sin\xi}{\xi}$ with $\xi_r = \pi$. Using this solution into Eqs. (36) and (43) to calculate the mass and radius of the star

$$M_{\alpha=0} = \frac{1}{\mu_e^2 m_\mu^2}\left(\frac{c\pi\hbar}{G}\right)^{\frac{3}{2}} = 2.58\ M_\odot \quad (44)$$

$$R_{\alpha=0} = \frac{1}{\mu_e m_\mu m_e}\left(\frac{\pi^3\hbar^3}{2cGB_r}\right)^{\frac{1}{2}} \quad (45)$$

Eq.(44) shows that the mass of the star depends on the fundamental constants and does not depend on the central density of the star or the magnetic field strength of the star. But the advantage of this formula is that the mass of the star $2.58\ M_\odot$ lies in the range of progenitor masses that predicted by the observations of type Ia-supernovae. While the radius of the star depends on the magnetic field strength of the star which depends on the maximum energy of degenerate electrons ϵ_{Fmax} . Hence the star with calculated mass in Eq.(44) can take any value of the radius depending on the magnetic field strength.

The solution of Eq. (41) diverges for the values ≥ 1 . In this work we interested in stars with masses in the range $= 2.1 - 2.8\ M_\odot$. Hence the singularity will not present any problem because it falls outside the scope of our solutions where the maximum mass limit $2.8\ M_\odot$ can be calculated at $= 0.098$. Use Eqs. (20) and (31) into Eq. (36), one gets

$$M = -\frac{\pi^{\frac{1}{2}}}{\mu_e^2 m_\mu^2}\left(\frac{c\hbar}{G}\right)^{\frac{3}{2}} \xi_R^2 \left(\frac{d\omega}{d\xi}\right)_{\xi_R} \tag{46}$$

Fig. (1) shows the numerical solution of Eq. (41) using values $\alpha\beta = -0.287$ and 0.089 which correspond to masses $M = 2.1$ and $2.8\ M_\odot$, respectively. The star mass, Eq. (46), as a function of $\alpha\beta$ is shown in Fig. (2). Obviously, the used model addresses the problem of having a single mass value, as shown earlier in [6]. Hence our results can cover the uncertainty in the progenitor mass for the type Ia-supernova between 2.1 and 2.8 $M_\odot$ which is predicted by recent observed data of that explosions.

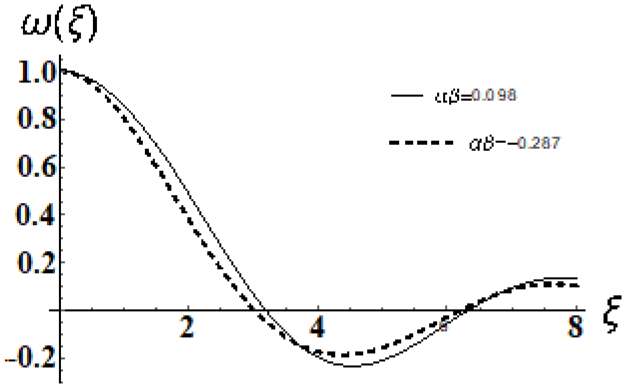

FIG. 1: Numerical solution of modified Lane-Emden equation with quasi-local anisotropic factor

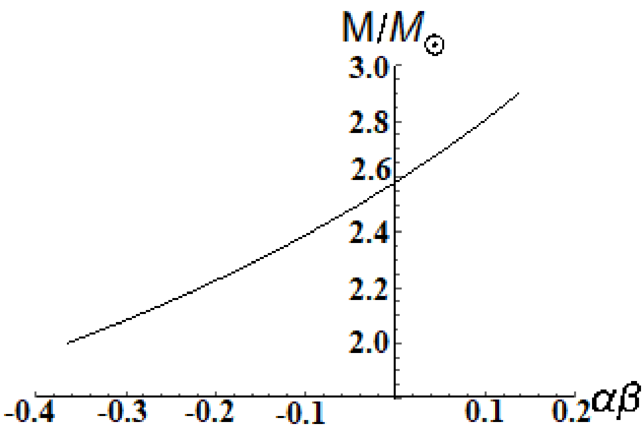

FIG. 2: Masses of highly magnetized white dwarf as function of αβ for one energy level

From Fig. (2) we can see that at $\alpha\beta = 0$ the mass of the star will be $M = 2.8\ M_\odot$ where $P_\perp - P_r = 0$. The positive value of the quantity $\alpha\beta$ appears when $P_\perp > P_r$. In this case the quantity $\frac{P_\perp - P_r}{r}$ represents a force directed outward and increases the internal pressure, hence it leads the system to increase its mass over $2.58\ M_\odot$ to achieve stability. And vice versa, the negative value of the quantity $\alpha\beta$ appears when $P_\perp < P_r$. The force in this case directed inward which decreases the internal pressure, and it leads the system to decrease its mass less 2.58 $M_{\odot}$ to achieve stability.

To calculate the radius of a star, we must choose values for the parameter ϵ_{Fmax}, which is completely arbitrary, but it must fulfill the condition $\epsilon_{Fmax}^2 \gg 1$. We will follow [44–46] in choosing values of the parameter, namely $\epsilon_{Fmax} = 20, 100$ and 200. The central density and magnetic field strength corresponding to these values are listed in table 1

Table 1: The central density and magnetic field strength with energy parameter

ϵ_{Fmax}	$\rho_c(gm/cm)$	$B(G)$
20	1.2×10^{10}	8.8×10^{15}
100	1.5×10^{12}	2.2×10^{17}
200	1.2×10^{13}	8.8×10^{17}

Fig. (3) shows the mass radius relation for three categories of magnetic white dwarfs which corresponding to above mentioned energy values. Mass-radius relation at $\epsilon_{Fmax} = 200$ is shown in more detail in Fig. (4). We can see that the radius of the magnetic star increases with mass in each category, the increase seems almost linear. The increase in radius with mass in each category due to the presence of the anisotropy in the system, represented by the anisotropy parameter α. It seems that the anisotropic pressure increases the total pressure inside the star which leads to an increase in the mass and radius of the star to fulfill the requisite condition for stability.

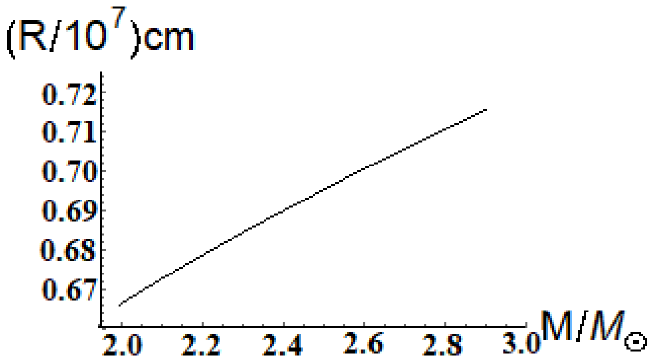

FIG. 3: Mass-radius relation for highly magnetized white dwarfs for one energy level

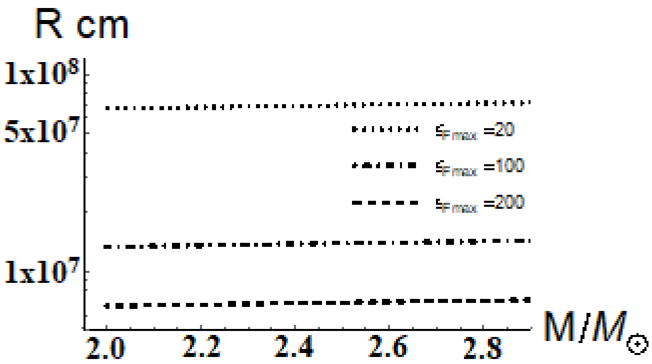

FIG. 4: mass-radius relation of magnetic white dwarfs at different energies

On the other hand, as the system energy (we mean maximum energy of degenerate electrons that occupy lowest Landau level) increases, the radius of the star decreases at the same mass despite the increase in central density of the star as shown in above listed values. Perhaps this result indicates that choosing large values of system energy, and hence a large values of internal magnetic field strength of the star, is more sense because the increase in system energy leads to smaller radius, which increases the probability occurrence of the supernova explosion. We can touch this result by dividing Eq. (36) by the square of Eq. (43), one finds that

$$\frac{M}{R^2} = -\frac{2m_e^2 c^{\frac{5}{2}}}{(\pi G\hbar^3)^{\frac{1}{2}}} B_r \left(\frac{d\omega}{d\xi}\right)_{\xi_R} \qquad (47)$$

The numerical value of the derivative $\left(\frac{d\omega}{d\xi}\right)_{\xi_R}$ correspond to mass range $M = 2.1 - 2.8\ M_\odot$ lies in between -0.280 and -0.330, so we can use its average value $\left(\frac{d\omega}{d\xi}\right)_{\xi_R} = -0.305$, then Eq.(47) can be rewritten as

$$\frac{M}{M_\odot} = \frac{B_r}{8.2\times10^{-5}} \left(\frac{R}{R_\odot}\right)^2 \qquad (48)$$

Eq. (48) shows that the mass-radius relation of magnetic white dwarfs, in a good approximation, depends only on the magnetic field, the effect of the anisotropy is given up by fixing the derivative at -0.305, which point out that its small impact. It is worth noting that the maximum deviation in radius between above equation and the exact result that shown in Fig. (3), for masses $2.1\ M_\odot$ and $2.8\ M_\odot$ is around 4% and we should stress that this equation valid only for these mass range of magnetic white dwarfs. Eq. (48) points out that the effect of anisotropy on the mass-radius relation of the star so small. This result harmonizes with the comment about the used model in [37] that the compactness is much smaller than unity in non-relativistic regime, so the anisotropy not expected to play important part.

Now let us suggest how does the increase in the magnetic field affect the supernova mechanism. Increasing the mass of white dwarfs by mass accretion from the accompanied star increases also the gravitational inward force which leads to contract the radius of the star. Due to the conservation property of the total magnetic flux inside the star, the internal magnetic field strength increases with star contraction. According to Eq. (48), the magnetic field increases the contraction rate of the star. Hence growing the mass of the star, by matter accretion, and decreasing the radius of the star, by gravitational contraction and magnetic field, heats up the star core and allows to the nuclear fusion to start again. With time, the rate of nuclear fusion increases and internal pressure increases. Later, within a few seconds, the pressure overcomes the gravitational collapse, and a large portion of the white dwarf substance undergoes a runaway reaction that releases enormous energy in a very bright burst known as type Ia-supernova without leaving any residue.

Now we will use the virial theorem to check the value of the magnetic field within the star and its stability. According to [47, 48, 49] the necessary condition for stability is that the magnetic energy W_B of the star should equal to the gravitational potential energy W_B of the star, which are defined as follow, respectively,

$$W_B = \frac{B^2 R^2}{6} \tag{49}$$

$$|W_G| = \frac{3}{5-n}\frac{GM^2}{R} = \frac{3}{4}\frac{GM^2}{R} \tag{50}$$

where $n = 1$. The maximum possible magnetic field B_{max} that needed for stability can be terminated from the stability condition $W_B = |W_G|$

$$B_{max} = \sqrt{\frac{9G}{2}}\frac{M}{R^2} \tag{51}$$

Use Eq. (48) into (51)

$$B_{max} = \sqrt{\frac{9G}{2}}\frac{B_r}{8.2\times10^{-5}} \quad \frac{M_\odot}{R_\odot^2} \tag{52}$$

The equilibrium takes place when the internal magnetic field $B = B_c B_r$ is equal to the maximum field for stability B_{max} , the ratio between these fields

$$\frac{B}{B_{max}} = 8.2 \times 10^{-5} \sqrt{\frac{2}{9G}} B_c \frac{R_\odot^2}{M_\odot} = 16 \tag{53}$$

It means that the internal magnetic field of the star exceeds the maximum magnetic field needed for stability about 16 times. Of course, this result will reflect on the stability condition which needs $\frac{W_B}{|W_G|} \approx 1$, to calculate this value

$$\frac{W_B}{|W_G|} = \frac{2}{9G} B^2 \frac{R^4}{M^2} \tag{54}$$

use $\frac{R^2}{M}$ from Eq. (51) into (54), one gets

$$\frac{W_B}{|W_G|} = \frac{B^2}{B_{max}^2} = 256 \tag{55}$$

Thus, the ratio is larger than unity, so that the strongly magnetized white dwarfs are unstable and unbound according to used model. On the other hand, Chandrasekhar and Fermi [47] proved that the star can abandon the spherical symmetry due to its internal magnetic field and become oblate by contracting along the field direction. The star proceeds to contract along the direction of the magnetic field and turns into an oblate spheroidal shape until the ratio $\frac{\epsilon}{R}$ becomes

$$\frac{\epsilon}{R} = -\frac{35}{24} \frac{B^2 R^4}{G M^2} \tag{56}$$

where ϵ is the eccentricity. Use Eq. (51) into (56), one gets

$$\frac{\epsilon}{R} = -6.56 \frac{B^2}{B_{max}^2} = -1679 \tag{57}$$

This result point out that the star becomes flat at the poles. This means that the star will deviate strongly from spherical symmetry to have a spheroidal shape due to presence of its ultra-magnetic field. According to used model these stars are unstable and unbound. Perhaps using a constant magnetic field strength in calculations, where the surface magnetic field of the star is several order magnitudes lower less than the central one, and lack of consideration for breaking the spherical symmetry, due to magnetic field, are reasons that cause appearance of instability in the star.

5. Conclusion

This paper is devoted to the study of a model of spherically symmetric anisotropic fluid distribution, aiming to explain some super luminous type Ia-supernovae recently reported from astrophysical observations. The anisotropy in such stars is produced by breaking rotational symmetry by magnetic field or by some other physical processes such as shear in stellar fluid flow, inhomogeneity in energy-density distribution or dissipative heat fluxes. Quasi-local equation of state is used to describe the anisotropic behavior in such compact stars. Unfortunately, there is no evidence to suggest the real form of anisotropy in compact stars. Therefore, we follow some theoretical suggestions that meet the boundary conditions of the system and the evidence for its validity is the comparisons with the experimental results. This model was chosen here only, thinking that it is closer to the reality without evidence. Considers the quasi-local properties of the star configuration via compactness and local properties of the matter as pressure, as well as it fulfills the necessary boundary conditions for the system. Following [6], we suggest that the magnetic field is very strong and has a constant strength through the star such that the degenerate electrons will be very energetically and restricted to occupy the first Landau level only. In this case the equation of state takes a polytropic form with polytropic index $n = 1$. A modified Lane-Emden equation is formulated and solved numerically. A new mass-radius relation was formulated and used through virial theorem to examine the stability of the star.

It is found that for fixed system energy (maximum energy of degenerate electrons that occupy lowest Landau level), or equivalently for fixed magnetic field strength, the radius of the star increases with mass slowly. Also, as the energy of the system increases, the radius of the star decreases for the same star mass. This indicates that choosing a large value of the energy of the system is closer to reality due to that it decreases the star radius, which increases the probability of supernova explosion. The calculations point out that the effect of the anisotropy is weak, due to that the compactness is much less than unity in non-relativistic regime. At least a relationship was found between radius and mass of the star that was not present in the original work. We used a virial theorem to study the stability conditions of the star. Calculations show that the predicted magnetic field of the star is exceeds the maximum magnetic field of the star that needed for stability configuration. The strong internal magnetic field makes these stars unstable and unbound. As well as it makes them deviate from spherical symmetry to oblate spheroidal shape.

The effects of general relativity in the maximum mass of the star are small, around 2% [50- 53], so we do not expect these effects to change the stability state of the star. May be the reason of the instability is using the assumption of spherical symmetry, where the results shows that the star is deformed by its internal magnetic field. Also using the assumption of the constancy of the internal magnetic field of the star does not describe the true state of the star as the surface magnetic field is less than the field at the center and therefore there is a gradient of the field inside the star [54, 55]. So, we must solve the structure equations along with Maxwell equations assuming a varying magnetic field profile.

References

[1] M.M. Philips, "The absolute magnitudes of Type IA supernovae." The Astrophysical Journal 413: (1993) L105-L108.

[2] G. Goldhader, D.E. Groom, A. Kim, "Timescale stretch parameterization of type Ia supernova B-band light curves." The Astrophysical Journal 558.1: (2001) 359.

[3] D. A. Howell, M. Sullivan and P. E. Nugent, "The type Ia supernova SNLS-03D3bb from a super-Chandrasekhar-mass white dwarf star." Nature 443.7109: (2006)308-311.

[4] R. A. Scalzo, G. Aldering and P. Antilogus, "Nearby supernova factory observations of SN 2007if: First total mass measurement of a super-Chandrasekhar-mass progenitor." The Astrophysical Journal 713. 2: (2010) 1073.

[5] I. Hachisu, M. Kato, H. Saio, H. Nomoto, "A single degenerate progenitor model for type Ia supernovae highly exceeding the Chandrasekhar mass limit." The Astrophysical Journal 744. 1: (2011) 69.

[6] U. Das, B. Mukhopadhyay, "New mass limit for white dwarfs: super-Chandrasekhar type Ia supernova as a new standard candle." Physical review letters 110. 7: (2013) 071102.

[7] I. S. Suh and G.J. Mathews, "Mass-radius relation for magnetic white dwarfs." The Astrophysical Journal 530.2: (2000) 949.

[8] U. Das, B. Mukhopadhyay, "Strongly magnetized cold degenerate electron gas: Mass-radius relation of the magnetized white dwarf." Physical Review D 86. 4: (2012) 042001.

[9] U. Das, B. Mukhopadhyay, "Violation of Chandrasekhar mass limit: The exciting potential of strongly magnetized white dwarfs." International Journal of Modern Physics D 21. 11: (2012) 1242001.

[10] A. Kundu, B. Mukhopadhyay, "Mass of highly magnetized white dwarfs exceeding the Chandrasekhar limit: An analytical view." Modern Physics Letters A 27. 15: (2012) 1250084.

[11] R. Kippenhahn, A. Weigert, "Book Review: Stellar structure and evolution/Springer-Verlag, 1990." Space Science Reviews 58: (1991) 190.

[12] M. Ruderman,"Pulsars: structure and dynamics." Annual Review of Astronomy and Astrophysics 10.1: (1972) 427-476.

[13] V. Canuto, "Equation of state at ultrahigh densities." Annual Review of Astronomy and Astrophysics 12. 1: (1974) 167-214.

[14] K. Dev, M. Gleiser, "Anisotropic stars: exact solutions." General relativity and gravitation 34: (2002) 1793-1818.

[15] L. Herrera, N.O. Santos, "Local anisotropy in self-gravitating systems." Physics Reports 2862: (1997) 53-130

[16] J.H. Jeans, "The motions of stars in a Kapteyn universe." Monthly Notices of the Royal Astronomical Society 82: (1922) 122-132.

[17] J. Binney, "Dynamics of elliptical galaxies and other spheroidal components." Annual Review of Astronomy and Astrophysics 20. 1: (1982) 399-429.

[18] P. Alencar, P. Letelier, "Anisotropic fluids with multifluid components." Physical Review D 34.2: (1986) 343.

[19] W. Barreto, S. Rojas, "An equation of state for radiating dissipative spheres in general relativity." Astrophysics and space science 193: (1992) 201-215.

[20] W. Barreto, "Exploding radiating viscous spheres in general relativity." Astrophysics and space science 201: (1993) 191-201.

[21] F. Shojai, M.R. Fazel, A. Estepanian, M. Kohandel, "On the Newtonian anisotropic configurations." The European Physical Journal C 75: (2015) 1-9.

[22] H. Liu, X. Zhang and D. Wen, "Remarks about the tensor mode detection by the BICEP2 collaboration and the super-planckian excursions of the inflaton field." Physical Review D 89. 10: (2014) 101301.

[23] H. Andreasson, "The Einstein-Vlasov system/kinetic theory." Living Reviews in Relativity 14: (2011) 1-55.

[24] O. H. Silva, F. B. C. Macedo, E. Berti and C. B. L Crispino, "Slowly rotating anisotropic neutron stars in general relativity and scalar–tensor theory." Classical and Quantum Gravity 32. 14: (2015) 145008.

[25] M. Ruderman, "Pulsars: structure and dynamics." Annual Review of Astronomy and Astrophysics 10. 1: (1972) 427-476.

[26] V. Canuto and S. Chitre, "Crystallization of dense neutron matter." Physical Review D 9.6: (1974) 1587.

[27] R. Sawyer, "Condensed $\pi-$ phase in neutron-star matter." Physical Review Letters 29. 6: (1972) 382.

[28] B. Carter and D. Langlois, "Relativistic models for superconducting-superfluid mixtures." Nuclear Physics B 531.1-3: (1998) 478-504.

[29] E. J. Ferrer, V. de la Incera, J. P. Keith, I. Portillo and P. L. Springsteen, "Equation of state of a dense and magnetized fermion system." Physical Review C 82. 6: (2010) 065802.

[30] L. Herrera, "Stability of the isotropic pressure condition." Physical Review D 101.10: (2020) 104024.

[31] M. Moussa, "Mass–radius relation for strongly magnetized white dwarfs with anisotropy." Annals of Physics 385: (2017) 347-357.

[32] U. Das and B. Mukhopadhyay, "Revisiting some physics issues related to the new mass limit for magnetized white dwarfs." Modern Physics Letters A 29. 07: (2014) 1450035.

[33] L.D. Landau, E.M. Lifshitz, (2013). Quantum mechanics: non-relativistic theory (Vol. 3). Elsevier.

[34] L. Herrera and J. Ponce de Leon, "Isotropic and anisotropic charged spheres admitting a one-parameter group of conformal motions." Journal of mathematical physics 26.9: (1985) 2302-2307.

[35] L. Herrera and W. Barreto, "Newtonian polytropes for anisotropic matter: General framework and applications." Physical Review D 87. 8: (2013) 087303.

[36] M. Moussa, "Mass–radius relation and dynamical stability of strongly magnetized white dwarfs in anisotropic configuration using Bowers and Liang model." Annals of Physics 420: (2020) 168263.

[37] D. Horvat, S. Ilijic, and A. Marunovic, "Classical Quantum Gravity 28: (2011) 025009.

[38] V. Folomeev and V. Dzhunushaliev, "Magnetic fields in anisotropic relativistic stars." Physical Review D 91 .4: (2015) 044040.

[39] H. Abreu, H. Hernandez, L.A. Nunez, "Sound speeds, cracking and the stability of self-gravitating anisotropic compact objects." Classical and Quantum Gravity 24. 18: (2007) 4631.

[40] C. Cattoen, T. Faber and M. Visser, "Gravastars must have anisotropic pressures." Classical and Quantum Gravity 22. 20: (2005) 4189.

[41] D. Horvat, S. Ilijic and A. Marunovic,"Radial pulsations and stability of anisotropic stars with a quasi-local equation of state." Classical and Quantum Gravity 28. 2: (2010) 025009.

[42] D. D. Doneva and S. S. Yazadjiev, "Nonradial oscillations of anisotropic neutron stars in the Cowling approximation." Physical Review D 85.12: (2012) 124023.

[43]] J. D.V. Arbail, M. Malheiro, "Radial stability of anisotropic strange quark stars." Journal of Cosmology and Astroparticle Physics 2016. 11: (2016) 012.

[44] U. Das, B. Mukhopadhyay, "Strongly magnetized cold degenerate electron gas: Mass-radius relation of the magnetized white dwarf." Physical Review D 86.4: (2012) 042001.

[45] A. Kundu and B. Mukhopadhyay, "Mass of highly magnetized white dwarfs exceeding the Chandrasekhar limit: An analytical view." Modern Physics Letters A 27.15: (2012) 1250084.

[46] U. Das and B. Mukhopadhyay, "Violation of Chandrasekhar mass limit: The exciting potential of strongly magnetized white dwarfs." International Journal of Modern Physics D 21. 11 : (2012) 1242001.

[47] S. Chandrasekhar and E. Fermi, "Magnetic fields in spiral arms." Astrophysical Journal 118: (1953) 113-115.

[48] S. L. Shapiro and S .A. Teukolsky," Black holes, white dwarfs, and neutron stars: The physics of compact objects". John Wiley & Sons, (2008).

[49] F. Hardy, P. Dufour and S. Jordan, "Spectrophotometric analysis of magnetic white dwarf–I. Hydrogen-rich compositions." Monthly Notices of the Royal Astronomical Society 520.4 (2023): 6111-6134.

[50] U. Das and B. Mukhopadhyay, "Revised cosmological parameters after BICEP 2 and BOSS." Journal of Cosmology and Astroparticle Physics 2015. 02: (2015) 016.

[51] P. Bera and D. Bhattacharya, "Mass–radius relation of strongly magnetized white dwarfs: dependence on field geometry, GR effects and electrostatic corrections to the EOS." Monthly Notices of the Royal Astronomical Society 456.3: (2016) 3375-3385.

[52] I. Ablimit, P. Podsiadlowski, R. Di Stefano, S. Rappaport, J. Wicker, "White Dwarf—Red Giant Star Binaries as Type Ia Supernova Progenitors: With and without Magnetic Confinement." The Astrophysical Journal Letters 941.2 (2022): L33.

[53] L. L. Amorim, S. O. Kepler, Baybars Külebi, S. Jordan, A. D. Romero, "Catalog of magnetic white dwarfs with hydrogen dominated atmospheres." The Astrophysical Journal 944.1 (2023): 56.

[54] U. Das and B. Mukhopadhyay, "Revisiting some physics issues related to the new mass limit for magnetized white dwarfs." Modern Physics Letters A 29.07 : (2014) 1450035.

[55] D. Deb, B. Mukhopadhyay and F. Weber, "Anisotropic Magnetized White Dwarfs: Unifying Under-and Overluminous Peculiar and Standard Type Ia Supernovae." The Astrophysical Journal 926.1 (2022): 66.

الملخص العربى

تأثير الضغط متباين الخواص على الأقزام البيضاء شديدة المغنطة باستخدام معادلة الحالة شبه المحلية

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يتناول هذا العمل تأثير الضغط متباين الخواص على القزم الأبيض شديد المغنطة. تُستخدم معادلة الحالة شبه المحلية لصياغة عامل متباين الخواص داخل النجم. نستخدم افتراض أن المجال المغناطيسي ثابت وقوي بما يكفي لجعل الإلكترونات المنحلة نشطة للغاية وتحتل المستوى الأول من مستويات لانداو. تمت صياغة معادلة لان-امدن المعدلة وحلها عدديًا. نحن مهتمون بالأقزام البيضاء الممغنطة ذات النطاق الكتلي $M = 2.1 - 28\ M_{\odot}$ والتي تم التنبؤ بها كأسلاف لمستعرات عظمى غريبة من النوع Ia، والتي تتميز بانخفاض الطاقة الحركية والإضاءة الزائدة. في تقدير تقريبي جيد، تم تحديد علاقة الكتلة بنصف القطر لهذا النطاق من النجوم. لقد وجد أن كتلة النجوم ونصف قطرها تزداد بسبب تأثير متباين الخواص. يتناقص نصف قطر النجم مع زيادة الطاقة القصوى للإلكترونات المنحلة وشدة المجال المغناطيسي ، مما يشير إلى أن هذه الكميات تدعم حدوث الانفجار. العيب الرئيسي للنموذج المستخدم هو أن شدة المجال المغناطيسي الداخلية المتوقعة تتجاوز المجال المغناطيسي الأقصى اللازم لاستقرار النجوم ، مما يشير إلى أن هذه النجوم غير مستقرة وغير مرتبطة.