Assiut University Journal of Multidisciplinary Scientific Research (AUNJMSR)
Faculty of Science, Assiut University, Assiut, Egypt.
Printed ISSN 2812-5029
Online ISSN 2812-5037
Vol. 53(1): 74- 97 (2024)
https://aunj.journals.ekb.eg

Some Properties of Revolution Surfaces in Euclidean 3-space with Conformable Fractional Derivative

M. A. Soliman [1], H. N. Abd-Ellah [1] , M. A. Basuney [1]*

Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt

*Corresponding Author: mohamedbasuneytony@aun.edu.eg

ARTICLE INFO
Article History:
Received: 2023-08-30
Accepted: 2023-10-02
Online: 2023-12-28

Keywords:
Surfaces,
Revolution,
Conformable,
Fractional.

ABSTRACT

In this paper, our aim is to study some of the properties of revolution surfaces in Euclidean 3-space that were studied in [1], using new definition of fractional derivative called conformable fractional derivative [2].

INTRODUCTION

One of the earliest calculus to be developed is fractional calculus. It dates back to the 17^{th} century, when Newton and Leibniz created conventional calculus. Leibniz was aware that there are other ways to operate on functions except in an integer order when he began to develop the fundamental concepts of differentiation and integration. Leibniz himself questioned what would happen if the order of differentiation from integer values to fractions were expanded. Along with the integer order of differentiation, which is widely acknowledged today, Leibniz also indicated in a letter to L'Hopital that he had discussed the topic of differentiation [3].

Today, it is now known relatively natural to extend integer order differentiation to a fractional order of differentiation. Numerous books and articles on the topic of fractional derivative applications show that the issues addressed by Leibniz more than 300 years ago are still important (see [4, 5]).

Recent years have seen the emergence of fractional calculus as a branch of pure mathematics with numerous applications in physics and engineering [6, 7]. One can look at the specific characteristics of fractional derivatives, there are numerous definitions of fractional derivatives. Unlike traditional derivatives, which are defined locally, there are numerous definitions of fractional derivatives that are not local (see Refs. [8, 9]).

The conformable fractional derivative seems to be a natural extension of the usual derivative,and it coincides with the known fractional derivatives on polynomials. Clearly, if α is the order of the fractional derivative and it equals 1, then the definition coincides with the classical definition of first derivative. The conformable definition is the simplest and most natural and efficient definition of fractional derivativeof order $\alpha \in (0,1]$. We should remark that the definition include any α[2].

Newly, the authors in [2] and [10] define new well-behaved simple fractional derivatives called the conformable fractional derivative depending just on the basic limit definition of the derivative.

Khalil et al. [2] have introduced a new derivative called the conformable fractional derivative of f of order α, which is defined in Def. (2.8). They then defined the fractional derivative of higher order (i.e., of order $\alpha > 1$). They also defined the fractional integral of order $0 < \alpha \leq 1$ only. They then proved the product rule, and the fractional mean value theorem.
Katugampola introduced in [10] the new derivative which is defined in Def. (2.9). As a consequence of the above definitions, the authors in [2, 10] showed that the α derivatives obey the product rule and quotient rule and have results similar to Rolle's theorem and the mean value theorem of classical calculus.

Along with curves, surfaces of revolution were among the first topics covered in differential geometry. In both engineering and science, the use of surfaces of revolution is crucial. Because they occur frequently in nature, surfaces of revolution have long been recognized as both common and well known in geometric modeling. In addition, several items from daily life, including cans, furniture legs, and table glasses. They serve as illustrations of revolution surfaces [11, 12].

Weingarten first introduced W-surfaces, also known as Weingarten surfaces, in 1861 in relation to the challenge of identifying all surfaces that are isometric to a given surface of revolution. They have attracted geometers' interest over time. W-surface applications for computer-aided design and form analysis are shown in [13]. In Euclidean 3-space, a linear Weingarten surface, also known as a LW-surface, is a surface whose mean curvature H and Gaussian curvature K meet the relation a H + b K = c, where a, b, and c $\in$ R. Numerous geometers tried to find examples of LW-surfaces for a very long period; for instance, see [14]. The classification of LW-surfaces in the general case is almost completely open today. Along the history, they have been of interest for geometers, mainly when the surface is closed; see [15–20].

One of the most attractive geometric objects, and ones that have significant physical significance are harmonic surfaces. These are surfaces in space that minimize area, meaning that any sufficiently small section of the surface has the smallest area among all surfaces sharing the same boundary. In the real world, these surfaces can be seen spontaneously, such as in the appearance of a soap film spanned by a specific boundary curve, following physical laws. These surfaces are common in mathematics and physics. Any Riemannian manifold of at least three dimensions, or a manifold with a smooth field of inner products on their tangent spaces, is used to study them. Unique harmonic examples include holomorphic curves in complex Euclidean spaces of dimension greater than 1 [21].

The study of bi-conservative and bi-harmonic surfaces is nowadays a very active research area. Many enjoyable results on these types have been obtained in the last decade. In the last few years, from the theory of bi-harmonic submanifolds, arose the study of bi-conservative submanifolds that enforced itself as a very hopeful and interesting research topic [22].

The differential geometry of stability issues involving generic surfaces has recently piqued the curiosity of several geometers. As more researchers became involved and saw outcomes over the past two decades, this interest grew quickly. One may specifically cite works by [23-28]. The interaction of classical differential geometry with the calculus of stability is one of its most fascinating and significant features. The theory of harmonic surfaces, for instance, is where the roots of the calculus of stability can be found. Recent years have seen the careful study of a seemingly novel sort of stability problem proposed by the stability principles that give rise to the general theory of relativity's field equations. The earlier applications were the case [25, 29].

In this paper, we investigate some new results on the conformable fractional derivative, which has been recently proposed, and whose simple definition allows for many extensions of some properties in differential geometry of revolution surfaces, for which the applications are essential in the fractional differential models.

2. Basic concepts

2.1. Geometric preliminaries

In this section, we introduce some basic definitions and relations for our analysis for the surfaces in Euclidean 3-space.

Definition 2.1. [30, 31] We say M a revolution surface which is generated by a plane curve $r(u)$ when it is rotated around a straight line in the same plane. The parameterization of the plane curveis given by

$$r(u) = (\phi(u), \psi(u)). \tag{2.1}$$

Then the parameterization of revolution surface is given by

$$M: X(u,\upsilon) = (\phi(u)\cos\upsilon,\ \phi(u)\sin\upsilon,\ \psi(u)), 0 < \upsilon < 2\pi,\ \phi(u) > 0 \tag{2.2}$$

The coefficients of the first and second fundamental forms of the surface M are given by

$$\begin{aligned} &g_{11} = \psi'^2 + \phi'^2, \quad g_{22} = \phi^2, \quad g_{12} = 0, \\ &h_{11} = \phi'\psi'' - \psi'\phi'', \ h_{22} = \phi\ \psi', \quad h_{12} = 0. \end{aligned} \tag{2.3}$$

It is convenient to assume that the rotating curve is parameterized by arc length, that is, that

$$\phi'^2 + \psi'^2 = 1. \tag{2.4}$$

The Gaussian curvature G is given by

$$G = -\frac{\psi'(\psi'\phi'' - \phi'\psi'')}{\phi}, \tag{2.5}$$

and ϕ is always positive. It follows that the parabolic points are given by either $\psi' = 0$ (the tangent line to the generator curve is perpendicular to the axis of rotation) or $\phi'\psi'' - \psi'\phi'' = 0$ (the curvature of the generator curve is zero). A point which satisfies both conditions is a planar point, since these conditions imply that $h_{11} = h_{12} = h_{22}$.

It is convenient to put the Gaussian curvature in still another form. By differentiating (2.4) we obtain$\phi'\phi'' = -\psi'\psi''$. Thus,

$$G = -\frac{\psi'(\psi'\phi'' - \phi'\psi'')}{\phi} = -\frac{\psi'^2\phi'' + \phi'^2\phi''}{\phi} = -\frac{\phi''}{\phi}(\psi'^2 + \phi'^2) = -\frac{\phi''}{\phi} \tag{2.6}$$

The principal curvatures of a surface of revolution are given by

$$k_1 = \phi'\psi'' - \psi'\phi'',\ k_2 = \frac{\psi'}{\phi}, \tag{2.7}$$

hence, the mean curvature of such a surface is

$$H = \frac{\psi' + \phi(\phi'\psi'' - \psi'\phi'')}{2\phi} \tag{2.8}$$

Definition 2.2. [32] W-surfaces are the surfaces which satisfying $\eta(G,H)=0$, or, the corresponding Jacobian determinant is identically zero, i.e,

$$\eta(G,H)=\left|\frac{\partial(G,H)}{\partial(u,\upsilon)}\right|\equiv 0. \quad (2.9)$$

We can rewrite the condition (2.9) as follows

$$G_u H_\upsilon - G_\upsilon H_u = 0. \quad (2.10)$$

Definition 2.3. [33, 34] LW-surfaces are the surfaces which satisfying the linear equation

$$a\,G + b\,H = c,\ (a, b, c) \neq (0, 0, 0) \in R. \quad (2.11)$$

When the constant b=0, a LW-surface reduces to a surface with constant Gaussian curvature. And when the constant a =0, a LW-surface reduces to a surface with constant mean curvature. In such a sense, the LW-surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature.

Definition 2.4. [35] A surface M in Euclidean 3-space is bi-conservative if the mean curvature function H satisfies

$$A\,(\text{grad } H) = -H \text{ grad } H. \quad (2.12)$$

This condition can be split into two partial differential equations as follows

$$a_{11}H_u + a_{12}H_\upsilon + HH_u = 0 \quad (2.13)$$

$$a_{21}H_u + a_{22}H_\upsilon + HH_\upsilon = 0 \quad (2.14)$$

Where $A=(a_{ij}),\ i,j=1,2$ is given by

$$\left.\begin{aligned} a_{11} &= (h_{11}g_{22}-h_{12}g_{12})/g,\ \ a_{12} = (h_{12}g_{11}-h_{11}g_{12})/g \\ a_{21} &= (h_{12}g_{22}-h_{22}g_{12})/g,\ \ a_{22} = (h_{22}g_{11}-h_{12}g_{12})/g \end{aligned}\right\} \quad (2.15)$$

Definition 2.5. [21] A smooth surface in E^3 is a harmonic surface (minimal surface) if its mean curvature equals zero at every point, i. e, $k_1+k_2=0$, where k_1 and k_2 are the principal curvatures.

Definition 2.6. [36]. A surface M in Euclidean 3-space is said to be bi-harmonic if it satisfies the equation $\Delta^2 X = 0$, where $X = X(u,\upsilon)$ is the vector function representation of the surface M. According to the well-known Betrami's formula $\Delta X = -2\vec{H}$, the bi-harmonic condition in E^3 is also known as the equation

$$\Delta\vec{H} = 0, \quad \vec{H} = H\vec{N} \quad (2.16)$$

where Δ is the Laplacian operator (Laplacian-Beitrami operator) with respect to the first fundamental form of X and is given by

$$\Delta = -\frac{1}{\sqrt{g}}\sum_{i,j}\frac{\partial}{\partial u^i}\left[\sqrt{g}\,g^{ij}\frac{\partial}{\partial u^j}\right], \quad (2.17)$$

where, (g^{ij}) denotes the associated matrix with its inverse (g_{ij}). i.e, $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$ and u^i are the local coordinate on M.

Definition 2.7. [27]. The oriented compact immersion $X: M \to E^3$ is stable with respect to the integral $\int_M H^2 dA$ iff the following condition is valid

$$\Delta H + 2H(H^2 - G) = 0, \quad (2.18)$$

where dA is the volume element of M and Δ is given by (2.17).

2.2. Conformable fractional derivative

Definition 2.8. [2] Given a function f: $[0, \infty) \longrightarrow R$. Then the conformable fractional derivative of f of order $\boldsymbol{\alpha}$ is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \text{ for all } t > 0, \alpha \in (0,1).$$

We will sometimes, write $f^\alpha(t)$ for $T_\alpha(f)(t)$, to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable.

One can easily show that T_α satisfies all the properties in the following theorem.

Theorem 2.1. Let $\alpha \in (0,1\,]$ and f, g be α-differentiable at a point $t > 0$. Then

(1) $T_\alpha(a\,f + b\,g) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in R$.

(2) $T_\alpha(t^p) = p\,t^{p-\alpha}$, for all $a, b \in R$.

(3) $T_\alpha(c) = 0$, for all constant functions $f(t) = c$.

(4) $T_\alpha(fg) = f\,T_\alpha(g) + g\,T_\alpha(f)$.

(5) $T_\alpha\left(\frac{f}{g}\right) = \frac{g\,T_\alpha(f) - f\,T_\alpha(g)}{g^2}$.

(6) If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Conformable fractional derivative of certain functions

(1) $T_\alpha(t^p) = p\,t^{p-\alpha}$, for all $p \in R$.

(2) $T_\alpha(e^{cx}) = c\,x^{1-\alpha}\,e^{cx}, c \in R$.

(3) $T_\alpha(\sin bx) = b\,x^{1-\alpha}\cos bx, b \in R$.

(4) $T_\alpha(\cos bx) = -b\,x^{1-\alpha}\sin bx, b \in R$.

(5) $T_\alpha(\tan bx) = b\,x^{1-\alpha}\sec^2 bx, b \in R$.

(6) $T_\alpha(\cot bx) = -b\,x^{1-\alpha}\csc^2 bx, b \in R$.

(7) $T_\alpha(\sec bx) = b\,x^{1-\alpha}\sec bx \tan bx, b \in R$.

(8) $T_\alpha(\csc bx) = -bx^{1-\alpha}\csc bx \cot bx, b \in R$.

Definition 2.9. [10]. Let f : $[0, \infty) \longrightarrow R$ and $t > 0$. Then the fractional derivative of f of order α is defined by

$$D^\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t\,e^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}, \tag{2.19}$$

for $t > 0, \alpha \in (0,1)$. If f is α-differentiablein some (0, a), a> 0 and $\lim_{\varepsilon \to 0} D^\alpha(f)(t)$ exists, then define

$$D^\alpha(f)(0) = \lim_{t \to 0^+} D^\alpha(f)(t).$$

3. GENERAL PROPERTIES OF REVOLUTION SURFACE M USING CONFORMABLE FRACTIONAL CALCULUS

In this section, we shall describe and derive the fundamental quantities of revolution surface M (2.2) using conformable fractional calculus. Some properties of this surface are introduced. The general conditions for this surface to become type L/w-surface, bi-conservative, harmonic, bi-harmonic and stable are derived.
Using theorem (2.1), we have the following:

Corollary 3.1. The condition (2.4) will become

$$\phi'^2 + \psi'^2 = u^{2\alpha-2} \tag{3.1}$$

Proof

Since $\psi = \psi(u)$ and $\phi = \phi(u)$, then, $T_\alpha(\psi) = u^{1-\alpha}\psi'$, $T_\alpha(\phi) = u^{1-\alpha}\phi'$
$\Rightarrow (u^{1-\alpha}\psi')^2 + (u^{1-\alpha}\phi')^2 = 1$, then $\phi'^2 + \psi'^2 = u^{2\alpha-2}$.

By differentiating this condition using conformable fractional calculus, we get

$$\psi'\psi'' + \phi'\phi'' = (\alpha-1)u^{2\alpha-3} . \tag{3.2}$$

Using conformable fractional calculus, Eq. (3.1) and Eq. (3.2) we shall compute the propertiesof the surface (2.2) as follows:
The coefficients of the fractional first fundamental form are given by

$$g_{11}^{\alpha} = 1,\ g_{22}^{\alpha} = \phi^2 \upsilon^{2-2\alpha} \text{ and } g_{21}^{\alpha} = 0, \tag{3.3}$$

therefore, the fractional metric of the first fundamental form g of the surface (2.2) is given by

$$g^{\alpha} = \phi^2 \upsilon^{2-2\alpha}. \tag{3.4}$$

The fractional unit normal vector field N is given by

$$N^{\alpha} = u^{1-\alpha}(-\psi'\cos\upsilon, -\psi'\sin\upsilon,\ \ \phi') \tag{3.5}$$

Consequently, the coefficients of the fractional second fundamental form are written as follows:

$$h_{11}^{\alpha} = u^{3-3\alpha}(\phi'\psi'' - \psi'\phi''),\ h_{22}^{\alpha} = \phi\psi' u^{1-\alpha}\upsilon^{2-2\alpha} \text{ and } h_{12}^{\alpha} = 0, \tag{3.6}$$

hence the fractional metric of the second fundamental form h is given by

$$h^{\alpha} = \phi\psi' u^{4-4\alpha}\upsilon^{2-2\alpha}(\phi'\psi'' - \psi'\phi''). \tag{3.7}$$

Corollary 3.2. The fractional Gaussian and mean curvature G and H of surface (2.2) are given by

$$G^{\alpha} = \frac{u^{1-2\alpha}}{\phi}\big((\alpha-1)\phi' - u\phi''\big), \tag{3.8}$$

$$H^{\alpha} = \frac{1}{2\phi}\big(\psi' u^{1-\alpha} + \phi u^{3-3\alpha}(\phi'\psi'' - \psi'\phi'')\big). \tag{3.9}$$

And a_{ij}^{α} are given by

$$a_{11}^{\alpha} = u^{3-3\alpha}(\phi'\psi'' - \psi'\phi''),\ a_{22}^{\alpha} = \frac{\psi'}{\phi}u^{1-\alpha},\ a_{12}^{\alpha} = 0,\ a_{21}^{\alpha} = 0. \tag{3.10}$$

Lemma 3.1. If we put $\alpha = 1$ in Eqs. (3.8) and (3.9), we obtain the Gaussian and mean curvature in integer case Eqs. (2.6) and (2.8), respectively.

From (2.2) if this is taken as axis of z and u denotes perpendicular distance from it, the parametrization of the surface M is given by

$$M_0 : X(u,\upsilon) = (u\cos\upsilon, u\sin\upsilon, \psi(u)) \tag{3.11}$$

Corollary 3.3. The fractional Gaussian and mean curvature G and H of surface (3.11) are givenby

$$G^{\alpha} = (\alpha - 1)u^{-2\alpha}, \tag{3.12}$$

$$H^{\alpha} = \frac{1}{2}(u^{-\alpha}\psi' + u^{3-3\alpha}\psi''). \tag{3.13}$$

And a_{ij}^{α} are given by

$$a_{11}^{\alpha} = u^{3-3\alpha}\psi'', \ a_{22}^{\alpha} = \psi' u^{-\alpha}, \ a_{12}^{\alpha} = 0, \ a_{21}^{\alpha} = 0. \tag{3.14}$$

Corollary 3.4. The revolution surface M_0 is fractional LW-surface if the following equation is valid

$$2m_1(\alpha - 1)u^{-2\alpha} + m_2(u^{-\alpha}\psi' + u^{3-3\alpha}\psi'') - 2m_3 = 0, \tag{3.15}$$

where m_1, m_2 and m_3 are constants.

Theorem 3.1. The revolution surface M_0 is fractional bi-conservative if the following two equations are valid

$$a_{11}^{\alpha} H_u^{\alpha} + a_{12}^{\alpha} H_\upsilon^{\alpha} + H^{\alpha} H_u^{\alpha} = [u^{3-3\alpha}\psi'' + \frac{1}{2}(u^{-\alpha}\psi' + u^{3-3\alpha}\psi'')] \\ [u^{1-2\alpha}\psi'' - \alpha u^{-2\alpha}\psi' + u^{4-4\alpha}\psi''' + (3 - 3\alpha)u^{3-4\alpha}\psi''] = 0, \tag{3.16}$$

$$a_{21}^{\alpha} H_u^{\alpha} + a_{22}^{\alpha} H_\upsilon^{\alpha} + H^{\alpha} H_\upsilon^{\alpha} = 0, \tag{3.17}$$

where $H_u^{\alpha}, H_\upsilon^{\alpha}$ and a_{ij}^{α} are given by

$$\left.\begin{aligned} H_u^{\alpha} = \frac{1}{2}(u^{1-2\alpha}\psi'' - \alpha u^{-2\alpha}\psi' + u^{4-4\alpha}\psi''' + (3 - 3\alpha)u^{3-4\alpha}\psi''), \ H_\upsilon^{\alpha} = 0, \\ a_{11}^{\alpha} = u^{3-3\alpha}\psi'', \ a_{22}^{\alpha} = u^{-\alpha}\psi', \ a_{12}^{\alpha} = 0, \ a_{21}^{\alpha} = 0. \end{aligned}\right\} \tag{3.18}$$

Since $H_\upsilon^{\alpha} = a_{21}^{\alpha} = 0$, we have:

Remark 3.1. The condition (3.17) is vanished identically.

Corollary 3.5. The surface M_0 is fractional harmonic if the following equation is valid

$$u^{-\alpha}\psi' + u^{3-3\alpha}\psi'' = 0. \tag{3.19}$$

Theorem 3.2. The revolution surface M_0 is fractional bi-harmonic if the following conditions are valid

$$a_1^{\alpha} = w_0 \cos\upsilon = 0, \ a_2^{\alpha} = w_0 \sin\upsilon = 0, \tag{3.20}$$

$$a_3^{\alpha} = u^{3-4\alpha}\psi^{(3)} + (4 - 5\alpha)u^{2-4\alpha}\psi'' + (1 - 2\alpha)(2 - 3\alpha)u^{1-4\alpha}\psi' \\ + u^{6-6\alpha}\psi^{(4)} + (10 - 9\alpha)u^{5-6\alpha}\psi^{(3)} + 20(1 - \alpha)^2 u^{4-6\alpha}\psi'' = 0, \tag{3.21}$$

where w_0 is given by

$$w_0 = (1\text{-}2\alpha)(2\text{-}3\alpha)u^{1-4\alpha}\psi'^2 + (8 - 10\alpha)u^{2-4\alpha}\psi'\psi'' + 2u^{3-4\alpha}\psi''^2 \\ + 2u^{3-4\alpha}\psi'\psi^{(3)} + 20(1 - \alpha)^2 u^{4-6\alpha}\psi'\psi'' + (10 - 9\alpha)u^{5-6\alpha}\psi''^2 \\ + (10 - 9\alpha)u^{5-6\alpha}\psi'\psi^{(3)} + 3u^{6-6\alpha}\psi''\psi^{(3)} + u^{6-6\alpha}\psi'\psi^{(4)} - u^{-1-2\alpha}\psi'^2 - u^{2-4\alpha}\psi'\psi''. \tag{3.22}$$

Theorem 3.3. The revolution surface M_0 is fractional stable iff the following condition is valid

$$[u^{2-3\alpha}\psi^{(3)} + 2(1 - \alpha)u^{1-3\alpha}\psi'' - \alpha u^{1-3\alpha}\psi'' - \alpha(1 - 2\alpha)u^{-3\alpha}\psi' + u^{5-5\alpha}\psi^{(4)} \\ + (5 - 4\alpha)u^{4-5\alpha}\psi^{(3)} + 3(1 - \alpha)u^{4-5\alpha}\psi^{(3)} + 12(1 - \alpha)^2 u^{3-5\alpha}\psi''] \\ -2[u^{-\alpha}\psi' u^{3-3\alpha}\psi''][0.25(u^{-\alpha}\psi' u^{3-3\alpha}\psi'')^2 - (\alpha - 1)u^{-2\alpha}] = 0. \tag{3.23}$$

4. APPLICATIONS ON REVOLUTION SURFACE M_0

In this section, we studied possibility of obtaining the necessary conditions for revolution surfaces in special cases to become type fractional L/W-surfaces, bi-conservative, harmonic, bi-harmonic and stable in Euclidean 3-space E^3. The general conditions for these surfaces in the form nonlinear differential equations, so we were able to solve it using a new method. we get the theoretical or numerical solutions to those equations. Then, we translated these results into geometric shapes using computer-aided geometric design.

Here, we give the following cases:

4.1. Case 1. If we put $\psi(u) = u + 1$, we denote this surface by M_1.

So using Eqs. (3.15-3.17) and (3.19-3.23) , we have the following corollaries:

(i) M_1 is Lw-surface if the following equation is valid

$$2m_1(\alpha - 1)u^{-2\alpha} + m_2 u^{-\alpha} - 2m_3 = 0. \quad (4.1)$$

We shall take some different values of α in Eq. (4.1) as follows:

(a) At $\alpha = 1$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, we have

$$u = 3/8. \quad (4.2)$$

(b) At $\alpha = 0.75$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, we have

$$3u^{-3/4} - u^{-3/2} - 8 = 0. \quad (4.3)$$

(c) At $\alpha = 0.5$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, we have

$$3u^{-1/2} - 2u^{-1} - 8 = 0. \quad (4.4)$$

(d) At $\alpha = 0.1$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, we have

$$15u^{-1/10} - 18u^{-1/5} - 40 = 0. \quad (4.5)$$

The solutions of Eqs. (4.3 – 4.5) are complex. So, we have the following:

Corollary 4.1. The surface M_1 is Lw-surface in integer alpha case and not Lw-surface in the fractional cases.

The Eqs. (4.2 – 4.5) are illustrated in (Fig. 1).

(ii) M_1 is bi-conservative if the following equation is valid

$$\alpha u^{-3\alpha} = 0. \quad (4.6)$$

(iii) M_1 is harmonic if the following equation is valid

$$u^{-\alpha} = 0. \quad (4.7)$$

Putting $\alpha = 1, 0.75, 0.5, 0.1$ in conditions (4.6) and (4.7) one can see that these conditions are not satisfied, so we have the following:

Corollary 4.2. The surface M_1 is neither bi-conservative nor harmonic.

(iv) M_1 is bi-harmonic if the following equations are valid

$$a_1 = w_1 \cos \upsilon = 0, \quad a_2 = w_1 \sin \upsilon = 0, \tag{4.8}$$

$$a_3 = (1\text{-}2\alpha)(2\text{-}3\alpha)u^{1-4\alpha} = 0, \tag{4.9}$$

where w_1 is given by

$$w_1 = (1\text{-}2\alpha)(2\text{-}3\alpha)u^{1-4\alpha} - u^{-1-2\alpha}. \tag{4.10}$$

We shall take some different values of α in Eqs. (4.8) and (4.9) as follows:

(a) At $\alpha = 1$, we have a_1 and a_2 are vanished identically and a_3 is not valid, then the surface is not bi-harmonic.

(b) At $\alpha = 0.75$, we have

$$a_1 = (u^{-2} - 8u^{-5/2}) \cos \upsilon = 0, \quad a_2 = (u^{-2} - 8u^{-5/2}) \sin \upsilon = 0, \quad a_3 = 1 \neq 0. \tag{4.11}$$

Then the surface is not bi-harmonic in this case because $a_3 \neq 0$.

(c) At $\alpha = 0.5$, we have

$$a_1 = \cos \upsilon = 0, \quad a_2 = \sin \upsilon = 0, \quad a_3 = 0. \tag{4.12}$$

Then the surface is not bi-harmonic in this case because in generally $\cos v \neq 0$ and $\sin v \neq 0$ in the main surface (3.11).

(d) At $\alpha = 0.1$, we have

$$a_1 = (34u^{3/5} - 25u^{-6/5}) \cos \upsilon = 0, \quad a_2 = (34u^{3/5} - 25u^{-6/5}) \sin \upsilon = 0, \quad a_3 = u^{3/5} = 0. \tag{4.13}$$

Then the surface is not bi-harmonic in this case because $a_3 = u^{3/5} = 0$ but $u \neq 0$ in the main surface (3.11).

Corollary 4.3. The surface M_1 is not bi-harmonic in integer or fractional alpha cases.

The Eqs. (4.11, 4.12) are illustrated in (Fig. 2) and the Eq. (4.13) is illustrated in (Fig. 3).

(v) M_1 is stable iff the following equation is valid

$$\left(\frac{5}{4} - \frac{\alpha}{2} - \alpha^2\right)u^{-3\alpha} = 0 \tag{4.14}$$

Also, putting $\alpha = 1, 0.75, 0.5, 0.1$ in condition (4.14), one can see that this condition is not satisfied, so we have the following

Corollary 4.4. The surface M_1 is unstable in integer and fractional alpha cases.

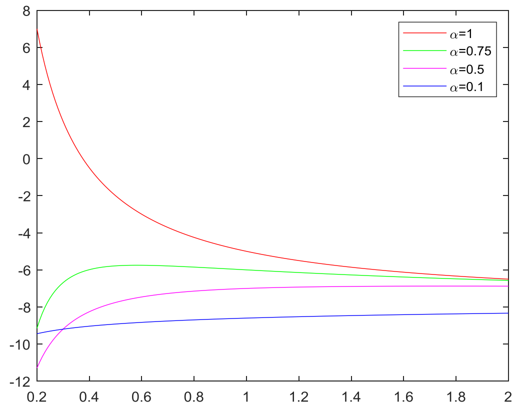

Figure 1: Lw-surface of M_1

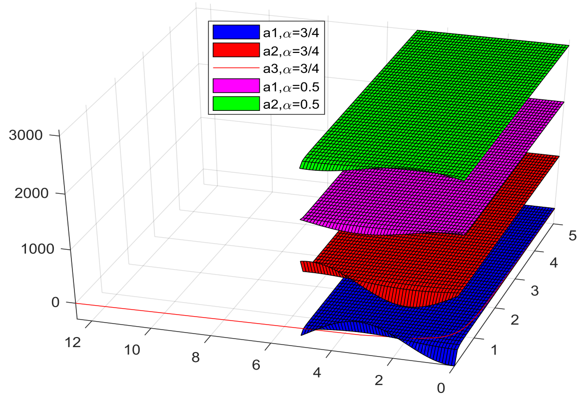

Figure 2: Bi-harmonic of $M_1, \alpha = 0.75, 0.5$

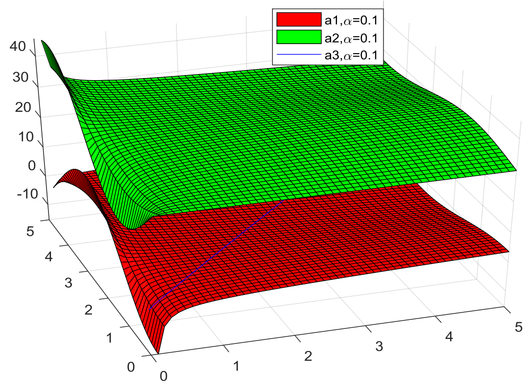

Figure 3: Bi-harmonic of $M_1, \alpha = 0.1$

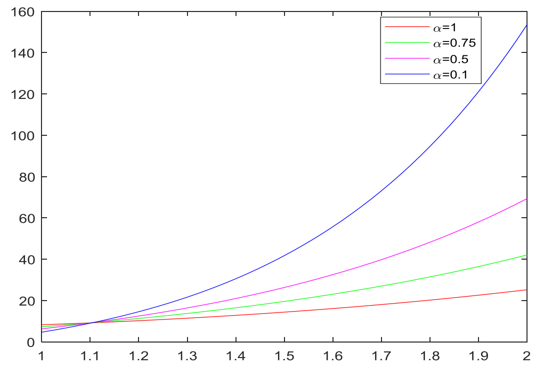

Figure 4: Lw-surface of M_2

4.2. Case 2. If we put $\psi(u) = e^u$, we denote this surface by M_2 .
So using Eqs. (3.15-3.17) and (3.19-3.23) , we have the following corollaries:
(i) M_2 is Lw-surface if the following equation is valid

$$2m_1(\alpha-1)u^{-2\alpha} + m_2 e^u(u^{-\alpha} + u^{3-3\alpha}) - 2m_3 = 0. \tag{4.15}$$

We shall take some different values of α in Eq. (4.15) as follows:

(a) At $\alpha = 1, m_1 = 2, m_2 = 3, m_3 = 4$, we have

$$3e^u(u^{-1} + 1) - 8 = 0. \tag{4.16}$$

The Eq. (4.16) has not explicit or numerical solution.
(b) At $\alpha = 0.75, m_1 = 2, m_2 = 3, m_3 = 4$, we have

$$3e^u(u^{-3/4} + u^{3/4} - u^{-3/2}) - 8 = 0. \tag{4.17}$$

The numerical solution of Eq. (4.17) is u = 2296/4953.

(c)At $\alpha = 0.5, m_1 = 2, m_2 = 3, m_3 = 4$, we have

$$3e^u\,(u^{-1/2} + u^{3/2}) - 2u^{-1} - 8 = 0. \tag{4.18}$$

The numerical solution of Eq. (4.18) is u = −733/847+438/127i.
(d) At $\alpha = 0.1, m_1 = 2, m_2 = 3, m_3 = 4$, we have .

$$3e^u\,(u^{-1/10} + u^{27/10}) - 3.6u^{-1/5} - 8 = 0. \tag{4.19}$$

The numerical solution of Eq. (4.19) is u = 848/995.
From foregoing results, we have the following:

Corollary 4.5. The surface M_2 is lw-surface when $\alpha = 0.75,\ 0.1$ while it is not lw-surface when $\alpha = 1,\ 0.5$.

The Eqs. (4.16 – 4.19) are illustrated in (Fig. ٤).
(ii) M_2 is bi-conservative if the following equation is valid

$$[3u^{3-3\alpha} + u^{-\alpha}][u^{1-2\alpha} - \alpha u^{-\alpha} + u^{4-4\alpha} + 3(1-\alpha)u^{3-4\alpha}] = 0. \tag{4.20}$$

we shall take some different values of α in Eq. (4.20) as follows:

(a) At $\alpha = 1$, we have

$$3u^3 + 4u^2 - 2u - 1 = 0. \tag{4.21}$$

The solution of Eq. (4.21) is $u_1 = -\frac{1}{3},\ u_{2,3} = \frac{1}{2}(-1 \pm \sqrt{5})$.

(b) At $\alpha = 0.75$, we have

$$[u^{-3/4} + 3u^{3/4}][u + u^{-1/2} - 0.75u^{-3/2} + 0.75] = 0. \tag{4.22}$$

The numerical solution of Eq. (4.22) is $u = -\frac{3267}{2521} + \frac{730}{761}i$.
(c) At $\alpha = 0.5$, we have

$$[u^{-1/2} + 3u^{3/2}][1.5u - 0.5u^{-1} + u^2 + 1] = 0. \tag{4.23}$$

The numerical solution of Eq. (4.23) is u = 707/2229.
(d) At $\alpha = 0.1$, we have

$$[u^{-1/10} + 3u^{27/10}][u^{4/5} - 0.1u^{-1/5} + 2.7u^{13/5} + u^{18/5}] = 0. \tag{4.24}$$

The numerical solution of Eq. (4.24) is u = 381/3967.
Thus, we have the following

Corollary 4.6. The surface M_2 is bi-conservative when $\alpha = 1,\ 0.5,\ 0.1$ while it is not bi-conservative when $\alpha = 0.75$.

The Eqs. (4.21 – 4.24) are illustrated in (Fig. 5).
(iii) M_2 is harmonic if the following equation is valid

$$(u^{-\alpha} + u^{3-3\alpha}) = 0. \tag{4.25}$$

We shall take some different values of α in Eq. (4.25) as follows:
(a) At $\alpha = 1$, we have

$$u = -1. \tag{4.26}$$

(b) At $\alpha = 0.75$, we have

$$u^{-3/4} + u^{3/4} = 0. \quad (4.27)$$

The solution of Eq. (4.27) is $u_{1,2} = \frac{1}{4}(1 \pm \sqrt{3}i)^2$.

(c) At α = 0.5, we have

$$u^{-1/2} + u^{3/2} = 0. \quad (4.28)$$

The solution of Eq. (4.28) is $u_{1,2} = \pm i$.

(d) At α = 0.1, we have

$$u^{-1/10} + u^{27/10} = 0. \quad (4.29)$$

All roots of Eq. (4.29) are complex.

Thus, we have the following:

Corollary 4.7. The surface M_2 is harmonic in the integer alpha case while it is not harmonic in fractional alpha cases

The Eqs. (4.26 – 4.29) are illustrated in (Fig. 6)

(iv) M_2 is bi-harmonic if the following equations are valid

$$a_1 = w_2 \cos \upsilon = 0, \quad a_2 = w_2 \sin \upsilon = 0, \quad (4.30)$$

$$a_3 = u^{3-4\alpha} + (4-5\alpha)u^{2-4\alpha} + (1-2\alpha)(2-3\alpha)u^{1-4\alpha} + u^{6-6\alpha} + (10-9\alpha)u^{5-6\alpha} + 20(1-\alpha)^2 u^{4-6\alpha} = 0, \quad (4.31)$$

where w_2 is given by

$$w_2 = (1-2\alpha)(2-3\alpha)u^{1-4\alpha} + (7-10\alpha)u^{2-4\alpha} + 4u^{3-4\alpha} + 20(1-\alpha)^2 u^{4-6\alpha} + 2(10-9\alpha)u^{5-6\alpha} + 4u^{6-6\alpha} - u^{-1-2\alpha}. \quad (4.32)$$

We shall take some different values of α in Eqs. (4.30) and (4.31) as the following

(a) At α =1, we have

$$\left.\begin{aligned} a_1 &= (4u^2 + 6u - 3)\cos\upsilon, \\ a_2 &= (4u^2 + 6u - 3)\sin\upsilon, \\ a_3 &= u^3 + 2u^2 - u + 1 = 0. \end{aligned}\right\} \quad (4.33)$$

The solution of Eq. (4.33) is $u_1 = -\frac{1}{3}, u_{2,3} = \frac{2}{1 \pm \sqrt{5}}$.

(b) At α =0.75, we have

$$\left.\begin{aligned} a_1 &= \left(u^{-2} - 4u^{-1} + 10u^{-1/2} + 52u^{1/2} + 32u^{3/2} - 8u^{-5/2} + 32\right)\cos\upsilon = 0, \\ a_2 &= \left(u^{-2} - 4u^{-1} + 10u^{-1/2} + 52u^{1/2} + 32u^{3/2} - 8u^{-5/2} + 32\right)\sin\upsilon = 0, \\ a_3 &= 2u^{-1} + u^{-2} + 10u^{-1/2} + 26u^{1/2} + 8u^{3/2} + 8 = 0. \end{aligned}\right\} \quad (4.34)$$

The numerical solution of Eq. (4.34) is $u = -\frac{531}{386} + \frac{805}{1038}i$.

(c) At $\alpha = 0.5$, we have

$$\left.\begin{aligned} a_1 &= (4u^5 + 11u^4 + 9u^3 + 2u^2 - 1)\cos v = 0, \\ a_2 &= (4u^5 + 11u^4 + 9u^3 + 2u^2 - 1)\sin v = 0, \\ a_3 &= 2u^3 + 11u^2 + 12u + 3 = 0. \end{aligned}\right\} \quad (4.35)$$

The real numerical solution of Eqs. (4.35) is $u = 1641/3221$, and the other four roots are complex.

(d) At α =0.1, we have

$$\left.\begin{aligned} a_1 &= \left(34u^{\frac{3}{5}} - 25u^{-\frac{6}{5}} + 150u^{\frac{8}{5}} + 100u^{\frac{13}{5}} + 405u^{\frac{17}{5}} + 455u^{\frac{22}{5}} + 100u^{\frac{27}{5}}\right)\cos v = 0, \\ a_2 &= \left(34u^{\frac{3}{5}} - 25u^{-\frac{6}{5}} + 150u^{\frac{8}{5}} + 100u^{\frac{13}{5}} + 405u^{\frac{17}{5}} + 455u^{\frac{22}{5}} + 100u^{\frac{27}{5}}\right)\sin v = 0, \\ a_3 &= 68u^{3/5} + 175u^{8/5} + 50u^{13/5} + 810u^{17/5} + 455u^{22/5} + 50u^{27/5} = 0. \end{aligned}\right\} \quad (4.36)$$

The numerical solution of Eqs. (4.36) is u = 950/1791.
From the previous results, we have the following:

Corollary 4.8. The surface M_2 is bi-harmonic when α = 1, 0.5, 0.1 and not bi-harmonic when α= 0.75.

The Eqs. (4.33, 4.34) are illustrated in (Fig. 7), and the Eqs. (4.35, 4.36) are illustrated in (Fig. 8).

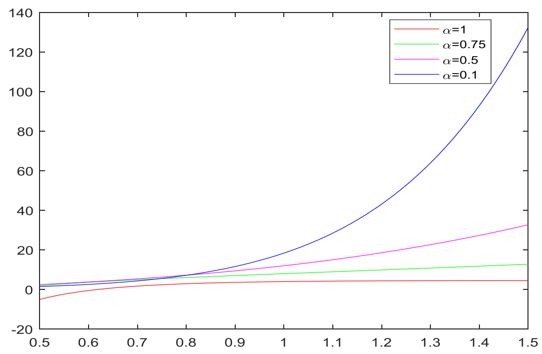

Figure 5: Bi-conservative of M_2

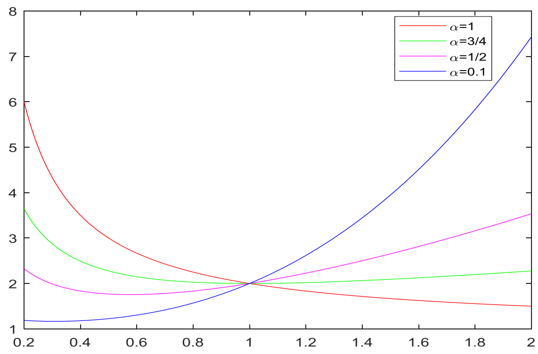

Figure 6: Harmonic of M_2

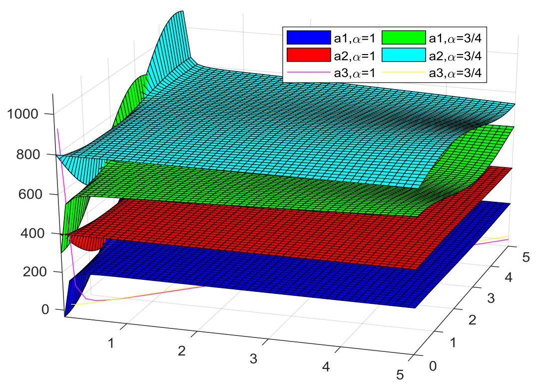

Figure 7: Bi-harmonic of $M_2, \alpha = 1, 0.75$

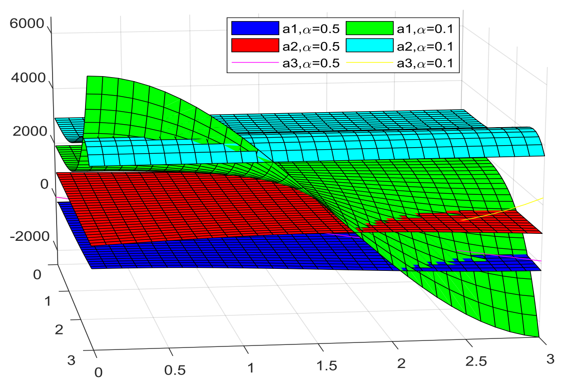

Figure 8: Bi-harmonic of $M_2, \alpha = 0.5, 0.1$

(v) M_2 is stable iff the following equation is valid

$$[u^{2-3\alpha} + 2(1-\alpha)u^{1-3\alpha} - \alpha u^{1-3\alpha} - \alpha(1-2\alpha)u^{-3\alpha} + u^{5-5\alpha} + (5-4\alpha)u^{4-5\alpha} + 3(1-\alpha)u^{4-5\alpha} + 12(1-\alpha)^2u^{3-5\alpha}] - 2[u^{-\alpha} + u^{3-3\alpha}][0.25e^{2u}(u^{-\alpha} + u^{3-3\alpha})^2 - (\alpha-1)u^{-2\alpha}] = 0. \quad (4.37)$$

We shall take some different values of α in Eq. (4.37) as follows:

(a) At $\alpha = 1$, we have

$$(u+1)^3e^{2u} - 2u^3 - 4u^2 + 2u - 2 = 0. \quad (4.38)$$

The numerical solution of Eq. (4.38) is u = 456/3685.
(b) At $\alpha = 0.75$, we have

$$2u^{\frac{3}{2}}(4u^2 + 11u + 4u^{\frac{1}{2}} + 3) - 2u - 4e^{2u} - u^{\frac{3}{2}}(12e^{2u} + 4) - 12u^3e^{2u} - 4u^{9/2}e^{2u} - 1 = 0. \quad (4.39)$$

The numerical solution of Eq. (4.39) is u = −2310/841+779/1168 i.

(c) At $\alpha = 0.5$, we have

$$2u^4 + 9u^3 + 8u^2 + u - (u^2+1)(e^{2u} + 2u^2e^{2u} + u^4e^{2u} + 2) = 0. \quad (4.40)$$

The numerical solution of Eq. (4.40) is u = −1031/273.

(d) At $\alpha = 0.1$, we have

$$\left[25u^{\frac{1}{5}}e^{2u}\left(u^{-\frac{1}{10}} + u^{\frac{27}{10}}\right)^2 + 90\right]\left[1 + u^{\frac{28}{10}}\right] - \left[486u^{\frac{14}{5}}365u^{\frac{19}{5}} + 50u^{\frac{24}{5}} + 85u + 50u^2\right] + 4 = 0. \quad (4.41)$$

The Eq. (4.41) has no exact or numerical solution.

Thus, we have the following:

Corollary 4.9. The surface M_2 is stable when α = 1, 0.5 while it is unstable when α= 0.75, 0.1.

The Eqs. (4.38 – 4.41) are illustrated in (Fig. 9).

4.3. Case 3. If we put $\psi(u) = \cos u$, we denote this surface by M_3.

So using Eqs. (3.15-3.17) and (3.19-3.23), we have the following corollaries:
(i) M_3 is Lw-surface if the following equation is valid

$$2m_1(\alpha-1)u^{-2\alpha} - m_2\,(u^{-\alpha}\sin u + u^{3-3\alpha}\cos u) - 2m_3 = 0. \quad (4.42)$$

We shall take some different values of α in Eq. (4.42) as follows:
(a) At $\alpha = 1$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, we have

$$3u\cos u + 3\sin u + 8u = 0. \quad (4.43)$$

The solution of Eq. (4.43) is u = 0.
(b) At $\alpha = 0.75$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, we have

$$3u^{3/4}\cos u + 3u^{-3/4}\sin u + u^{-3/2} + 8 = 0. \quad (4.44)$$

The numerical solution of Eq. (4.44) is $u = -\frac{3416}{25} + \frac{152}{4681} i$.

(c) At $\alpha = 0.5$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, we have

$$3u^{3/2}\cos u + 3u^{-1/2}\sin u + 2u^{-1} + 8 = 0. \quad (4.45)$$

The numerical solution of Eq. (4.45) is $u = 20979/212$.

(d) At $\alpha = 0.1$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, we have

$$3u^{27/10}\cos u + 3u^{-1/10}\sin u + 3.6u^{-1/5} + 8 = 0. \quad (4.46)$$

The numerical solution of Eq. (4.46) is $u = -\frac{18449}{81} + \frac{1}{1194494} i$.

Thus, we have the following:

Corollary 4.10. The surface M_3 is lw-surface when $\alpha = 0.5$ while it is not lw-surface when $\alpha = 1, 0.75, 0.1$.

The Eqs. (4.43 – 4.46) are illustrated in (Fig. 10).

(ii) M_3 is bi-conservative if the following equation is valid

$$[3u^{3-3\alpha}\cos u + u^{-\alpha}\sin u][(\alpha u^{-2\alpha} + u^{4-4\alpha})\sin u - \{u^{1-2\alpha} + 3(1-\alpha)u^{3-4\alpha}\}\cos u] = 0. \quad (4.47)$$

We shall take some different values of α in Eq. (4.47) as follows:

(a) At $\alpha = 1$, we have

$$[3\cos u + u^{-1}\sin u][u^{-1}\cos u - (u^{-2} + 1)\sin u] = 0. \quad (4.48)$$

The numerical solution of Eq. (4.48) is $u = -28809/131$.

(b) At $\alpha = 0.75$, we have

$$[(u + 0.75u^{-3/2})\sin u - (u^{-1/2} + 0.75)\cos u][3u^{3/4}\cos u + u^{-3/4}\sin u] = 0. \quad (4.49)$$

The numerical solution of Eq. (4.49) is $u = -\frac{10336}{47} - \frac{11}{35875} i$.

(c) At $\alpha = 0.5$, we have

$$[(1.5u+1)\cos u - (0.5u^{-1} + u^2)\sin u][3u^{3/2}\cos u + u^{-1/2}\sin u] = 0. \quad (4.50)$$

The numerical solution of Eq. (4.50) is $u = -18826/85$.

(d) At $\alpha = 0.1$, we have

$$[3u^{27/10}\cos u + u^{-1/10}\sin u][(u^{4/5} + 2.7u^{13/5})\cos u - (0.1u^{-1/5} + u^{18/5})\sin u] = 0. \quad (4.51)$$

The numerical solution of Eq. (4.51) is $u = 22365/26$.

Thus, we have the following:

Corollary 4.11. The surface M_3 is bi-conservative when $\alpha = 1, 0.5, 0.1$while it is not bi-conservative when $\alpha = 0.75$.

The Eqs. (4.48 – 4.51) are illustrated in (Fig. 11).

(iii) M_3 is harmonic if the following equation is valid

$$u^{-\alpha}\sin u + u^{3-3\alpha}\cos u = 0. \quad (4.52)$$

We shall take some different values of α in Eq. (4.52) as follows:

(a) At $\alpha = 1$, we have

$$\cos u + u^{-1}\sin u = 0. \quad (4.53)$$

The numerical solution of Eq. (4.53) is $u = -28699/126$.

(b) At $\alpha = 0.75$, we have

$$u^{3/4}\cos u + u^{-3/4}\sin u = 0. \quad (4.54)$$

The numerical solution of Eq. (4.54) is $u = -18449/81+17/58436i$.

(c) At $\alpha = 0.5$, we have

$$u^{3/2}\cos u + u^{-1/2}\sin u = 0. \quad (4.55)$$

The numerical solution of Eq. (4.55) is $u = -18449/81$.

(d) At $\alpha = 0.1$, we have

$$u^{27/10}\cos u + u^{-1/10}\sin u = 0. \quad (4.56)$$

The numerical solution of Eq. (4.56) is $u = 22365/226$.

Thus, we have the following:

Corollary 4.12. The surface M_3 is harmonic when $\alpha = 1,\ 0.5,\ 0.1$ while it is not harmonic when $\alpha = 0.75$

The Eqs. (4.53 – 4.56) are illustrated in (Fig. 12).

(iv) M_3 is bi-harmonic if the following equations are valid

$$a_1 = w_3\cos\upsilon = 0, \quad a_2 = w_3\sin\upsilon = 0, \quad (4.57)$$

$$a_3 = [u^{6-6\alpha} - (4-5\alpha)u^{2-4\alpha} - 20(1-\alpha)^2u^{4-6\alpha}]\cos u$$
$$+ [u^{3-4\alpha} + (10\text{-}9\alpha)u^{5-6\alpha} - (1-2\alpha)(2-3\alpha)u^{1-4\alpha})]\sin u=0, \quad (4.58)$$

where w_3 is given by

$$\begin{aligned} w_3 = {} & [2u^{3-4\alpha} + (10-9\alpha)u^{5-6\alpha}]\cos^2 u + [(7-10\alpha)u^{2-4\alpha} - 4u^{6-6\alpha} \\ & + 20(1-\alpha)^2u^{4-6\alpha}]\sin u\cos u + [(1-2\alpha)(2-3\alpha)u^{1-4\alpha} - u^{-1-2\alpha} \\ & - 2u^{3-4\alpha} - (10-9\alpha)u^{5-6\alpha}]\sin^2 u. \end{aligned} \quad (4.59)$$

We shall take some different values of α in Eqs. (4.57) and (4.58) as follows:

(a) At $\alpha = 1$, we have

$$\left.\begin{aligned} a_1 &= [3u^{-1}(\cos^2 u - \sin^2 u) - (3u^{-2}+4)\sin u\cos u]\cos\upsilon = 0, \\ a_2 &= [3u^{-1}(\cos^2 u - \sin^2 u) - (3u^{-2}+4)\sin u\cos u]\sin\upsilon = 0, \\ a_3 &= (2u^2-1)\sin u + u(u^2+1)\cos u = 0. \end{aligned}\right\} \quad (4.60)$$

The numerical solution of Eqs. (4.60) is $u = -39789/170$.

(b) At $\alpha = 0.75$, we have

$$\left.\begin{aligned} a_1 = &\ [-(26u^{\frac{1}{2}}+16)\cos^2 u + \left(4u^{-1} - 10u^{-\frac{1}{2}} + 32u^{\frac{3}{2}}\right)\sin u \cos u \\ &+(26u^{\frac{1}{2}} - u^{-2} + 8u^{-\frac{5}{2}} + 16)\sin^2 u]\cos \upsilon = 0, \\ a_2 = &\ [-(26u^{\frac{1}{2}}+16)\cos^2 u + \left(4u^{-1} - 10u^{-\frac{1}{2}} + 32u^{\frac{3}{2}}\right)\sin u \cos u \\ &+(26u^{\frac{1}{2}} - u^{-2} + 8u^{-\frac{5}{2}} + 16)\sin^2 u]\sin \upsilon = 0, \\ a_3 = &\left(26u^{\frac{1}{2}} - u^{-2} + 8\right)\sin u - \left(2u^{-1} + 10u^{-\frac{1}{2}} - 8u^{\frac{3}{2}}\right)\cos u = 0. \end{aligned}\right\} \quad (4.61)$$

The numerical solution of Eqs. (4.61) is u =−9929/43+3/31579 i.

(c) At α = 0.5, we have

$$\left.\begin{aligned} a_1 = &\ [(11u^2+4u)\cos^2 u - 2(4u^3 - 5u - 2)\sin u \cos u \\ &-(4u + 2u^{-2} + 11u^2)\sin^2 u\,]\cos \upsilon = 0, \\ a_2 = &\ [(11u^2+4u)\cos^2 u - 2(4u^3 - 5u - 2)\sin u \cos u \\ &-(4u + 2u^{-2} + 11u^2)\sin^2 u\,]\sin \upsilon = 0, \\ a_3 = &\ (11u^2 + 2u)\sin u + (2u^3 - 10u - 3)\cos u = 0. \end{aligned}\right\} \quad (4.62)$$

The numerical solution of Eqs. (4.62) is u = $-2491/11$.
(d) At α = 0.1, we have

$$\left.\begin{aligned} a_1 = &\ [(500u^{13/5} + 2275u^{22/5})\cos^2 u + 10(150u^{8/5} + 405u^{17/5} \\ &-100u^{\frac{27}{10}})\sin u \cos u + (340u^{3/5} - 250u^{-6/5} - 500u^{13/5} \\ &-2275u^{\frac{22}{5}})\sin^2 u]\cos \upsilon = 0, \\ a_2 = &\ [(500u^{13/5} + 2275u^{22/5})\cos^2 u + 10(150u^{8/5} + 405u^{17/5} \\ &-100u^{\frac{27}{10}})\sin u \cos u + (340u^{3/5} - 250u^{-6/5} - 500u^{13/5} \\ &-2275u^{\frac{22}{5}})\sin^2 u]\sin \upsilon = 0, \\ a_3 = &\left[250u^{\frac{18}{5}} - 340u^{\frac{3}{5}} + 2275u^{\frac{22}{5}}\right]\sin u \\ &-[875u^{\frac{8}{5}} + 4050u^{\frac{17}{5}} - 250u^{27/5}]\cos u = 0. \end{aligned}\right\} \quad (4.63)$$

The numerical solution of Eqs. (4.63) is u = 3611/36.

From the previous results, we have the following:
Corollary 4.13. The surface M_3 is bi-harmonic when α = 1, 0.5, 0.1 and not bi-harmonic when α = 0.75.

The Eqs. (4.60, 4.61) are illustrated in (Fig. 13) and the Eqs. (4.62, 4.63) are illustrated in (Fig. 14).

(v) M_3 is stable iff the following equation is valid

$$u^{2-3\alpha}\sin u - 2(1-\alpha)u^{1-3\alpha}\cos u + \alpha u^{1-3\alpha}\cos u + \alpha(1-2\alpha)u^{-3\alpha}\sin u + u^{5-5\alpha}\cos u$$
$$+(5-4\alpha)u^{4-5\alpha}\sin u + 3(1-\alpha)u^{4-5\alpha}\sin u - 12(1-\alpha)^2 u^{3-5\alpha}\cos u$$

$$-2[-u^{-\alpha}\sin u-u^{3-3\alpha}\cos u][0.25(-u^{-\alpha}\sin u-u^{3-3\alpha}\cos u)^2-(\alpha-1)u^{-2\alpha}]=0. \quad (4.64)$$

We shall take some different values of α in Eq. (4.64) as follows:

(a) At α = 1, we have

$$2u^3\cos u+2u\cos u+4u^2\sin u-2\sin u+u^3(\cos u+u^{-1}\sin u)^3=0. \quad (4.65)$$

The numerical solution of Eq. (4.65) is u = −7061/31.

(b) At α = 0.75, we have

$$4u^{9/2}\cos^3 u+8u^{7/2}\cos u+22u^{5/2}\sin u+10u^{3/2}\cos u-12u^{3/2}\cos^3 u$$
$$+8u^2\sin u+12u^3\cos^2 u\sin u+2u\cos u+5\sin u-4\cos^2 u\sin u=0. \quad (4.66)$$

The numerical solution of Eq. (4.66) is u = 0.

(c) At α = 0.5, we have

$$[u^{3/2}\cos u+u^{-1/2}\sin u][2u^{-1}+(u^{3/2}\cos u+u^{-1/2}\sin u)^2]+2u^{5/2}\cos u+9u^{3/2}\sin u$$

$$+2u^{1/2}\sin u-6u^{1/2}\cos u-u^{-1/2}\cos u=0 \quad (4.67)$$

The numerical solution of Eq. (4.67) is u = −22094/97.

(d) At α = 0.1, we have

$$[u^{27/10}\cos u+u^{-1/10}\sin u][90u^{-1/5}+25(u^{27/10}\cos u+u^{-1/10}\sin u)^2]$$
$$+50u^{9/2}\cos u+365u^{7/2}\sin u+4u^{-3/10}\sin u+50u^{17/10}\sin u-486u^{5/2}\cos u-85u^{7/10}\cos u=0. \quad (4.68)$$

The numerical solution of Eq. (4.68) is u = −18449/81-10/16929 i.

Thus, we have the following:

Corollary 4.14. The surface M_3 is stable when α = 1, 0.5 while it is unstable when α = 0.75, 0.1.

The Eqs. (4.65 – 4.68) are illustrated in (Fig. 15).

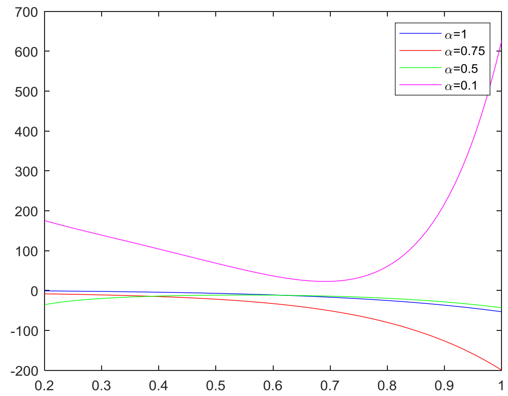

Figure 9: Stability of M_2

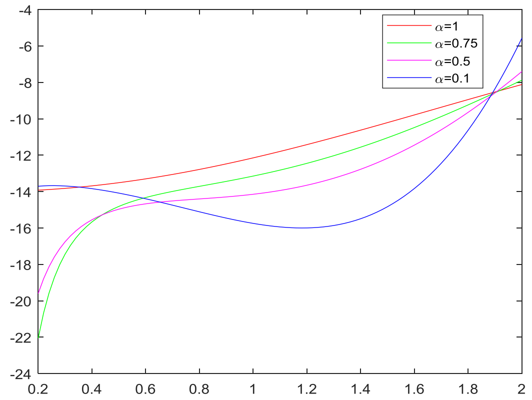

Figure 10: Lw-surface of M_3

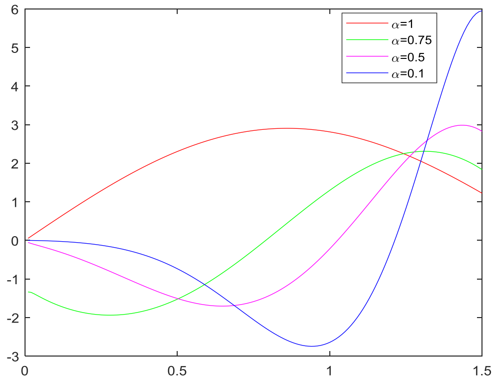

Figure 11: Bi-conservative of M_3

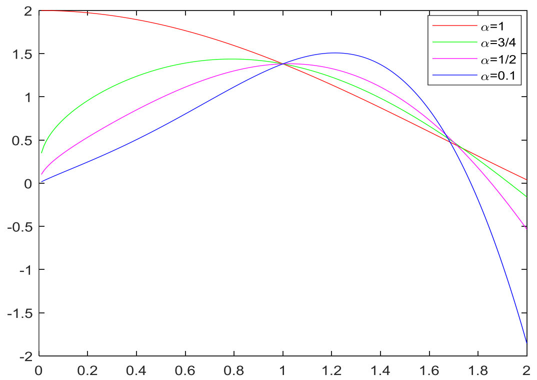

Figure 12: Harmonic of M_3

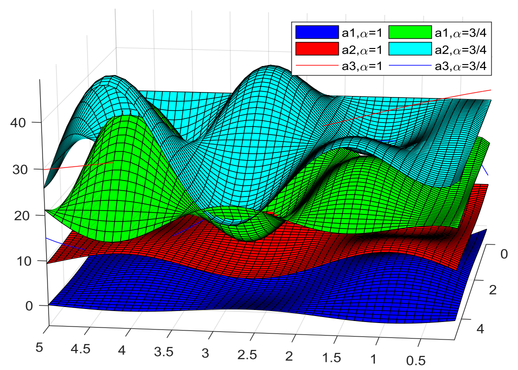

Figure 13: Bi-harmonic of $M_3, \alpha = 1, 0.75$

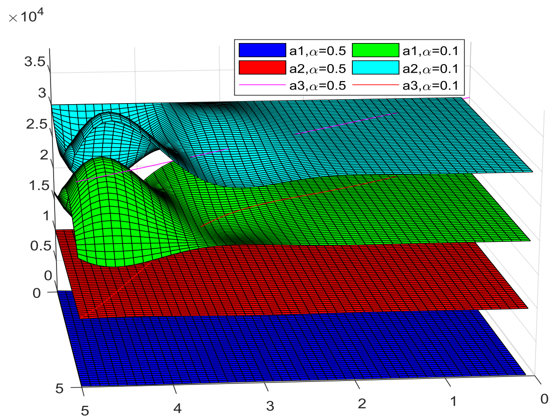

Figure 14: Bi-harmonic of $M_3, \alpha = 0.5, 0.1$

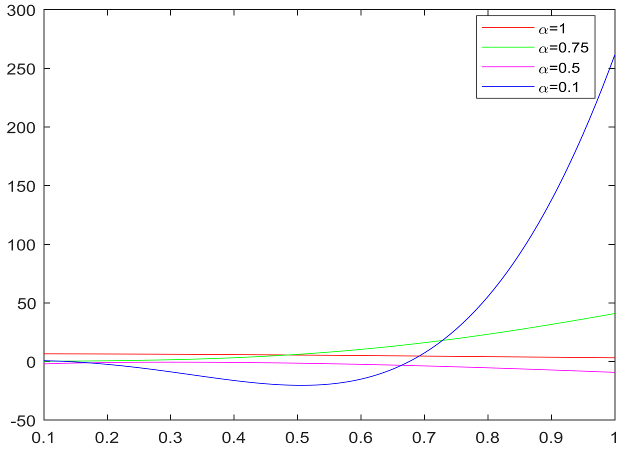

Figure 15: Stability of M_3

The previous results can be summarized in the following table:

Property	**Case 1:** $\psi(u) = u + 1$	**Case 2:** $\psi(u) = e^u$	**Case 3:** $\psi(u) = \cos u$
	α of M_1	α of M_2	α of M_3
Lw-surface	1	0.75, 0.1	0.5
Not Lw-surface	0.75, 0.5, 0.1	1, 0.5	1, 0.75, 0.1
Bi-conservative	No results	1, 0.5, 0.1	1, 0.5, 0.1
Not bi-conservative	1, 0.75, 0.5, 0.1	0.75	0.75
Harmonic	No results	1	1, 0.5, 0.1
Not harmonic	1, 0.75, 0.5, 0.1	0.75, 0.5, 0.1	0.75
Bi- harmonic	No results	1, 0.5, 0.1	1, 0.5, 0.1
Not bi- harmonic	1, 0.75, 0.5, 0.1	0.75	0.75
Stable	No results	1, 0.5	1, 0.5
Unstable	1, 0.75, 0.5, 0.1	0.75, 0.1	0.75, 0.1

5. CONCLUSION

In this study, we have introduced some properties of revolution surfaces using a conformable fractional derivative, which is a natural extension of the usual derivative and its definition is the simplest and most natural definition of fractional derivative.

In section 2, we have defined the revolution surfaces and some properties which are related to our work such that, w-surface, lw-surface, bi-conservative, harmonic, bi-harmonic and stability of the surfaces. Also, we define the conformable fractional derivative of any function f and have introduced the rules of differentiation related to it.

In section 3, we have studied some properties of revolution surface M_0 using conformable fractional derivative.

In section 4 we presented three applications (examples) on the main surface M_0 by replacing $\psi(u)$ by special functions and we have plotted the results using Matlab program v.18, and we have summarized all applications results in previous table, which makes it easier for the reader to understand the applications results easily. The reader should note that the results when alpha equals one are the same as the results in Ref. [1].

The table shows the following results: lw-surface is intrinsic property to the surface M_1 at $\alpha = 1$ only, while for the surface M_2 it is intrinsic property at α =0.75, 0.1 Also, for the surface M_3 lw-surface is intrinsic property at $\alpha = 0.5$ only, and the all other cases of α the surfaces cannot be have intrinsic property. Similarly, with respect to the bi-conservative and bi-harmonic properties, they are intrinsic properties for the surfaces M_2 and M_3 at $\alpha = 1, 0.5, 0.1$ and they are not intrinsic property at another alpha cases and for M_1. Regarding to the harmonic property, it is intrinsic property for the surface M_2 at $\alpha = 1$ only, and intrinsic property for M_3 at $\alpha = 1$, 0.5, 0.1 only but it is not intrinsic property for M_1 at all alpha cases. Finally, the stability property, it is intrinsic property for the surfaces M_2 and M_3 at $\alpha = 1, 0.5$ only, and it is not intrinsic property for the surface M_1 at all alpha cases.

REFERENCES

[1] M. A. Soliman, H. N. Abd-Ellah, and M. A. Basuney, "Some new results of revolution surfaces in Euclidean 3-space E^3," Assiut University Journal of Multidisciplinary scientific Research, Vol. 52, pp. 322–340, (2023).

[2] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," Journal of computational and applied mathematics, vol. 264, pp. 65–70, (2014).

[3] G. Baumann and F. Stenger, "Fractional calculus and sinc methods," Fractional calculus and applied analysis, vol. 14, no. 4, pp. 568–622, (2011).

[4] J. Sabatier, O. P. Agrawal, and J. T. Machado, Advances in fractional calculus, vol. 4. Springer, (2007).

[5] S. Das, Functional Fractional Calculus for System Identification and Controls, Springer, (2008).

[6] V. E. Tarasov, Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, fields and media. Springer Science & Business Media, (2010).

[7] V. E. Tarasov, "Fractional vector calculus and fractional maxwell's equations," Annals of Physics, vol. 323, no. 11, pp. 2756–2778, (2008).

[8] I. Podlubny, "Fractional differential equations (an introduction to fractional derivatives, fractional differential equations, some methods of their solution and some of their applications)," Math. Sci. Eng, vol. 198, p. 340, (1999).

[9] S. G. Samko, A. A. Kilbas, O. I. Marichev, et al., Fractional Integrals and Derivatives, vol. 1. Gordon and breach science publishers, Yverdon Yverdon-les-Bains, Switzerland, (1993).

[10] U. N. Katugampola, "A New Fractional Derivative with Classical Properties," arXiv preprint arXiv:1410.6535, 2014.

[11] A. Gray, E. Abbena, and S. Salamon, "Modern geometry of curves and surfaces with mathematica," Studies in Advanced Mathematics, vol. 47, (2006).

[12] J. Vršek and M. Lávička, "Determining surfaces of revolution from their implicit equations," Journal of computational and applied mathematics, vol. 290, pp. 125–135, (2015).

[13] B. van Brunt and K. Grant, "Potential applications of Weingarten surfaces in CAGD, part I: Weingarten surfaces and surface shape investigation," Computer aided geometric design, vol. 13, no. 6, pp. 569–582, (1996).

[14] L. Rafael, "On linear Weingarten surfaces," International Journal of Mathematics, vol. 19, 08 (2006).

[15] B. S.-S. CHERN, "Some new characterizations of the Euclidean sphere," Selected Papers: Vol. 2, p. 123, (1978).

[16] S.-s. Chern, “On special W-surfaces,” Proceedings of the American Mathematical Society, vol. 6, no. 5, pp. 783–786, (1955).

[17] H. Hopf, “Über flächen mit einer relation zwischen den hauptkrümmungen. meinem lehrer erhard schmidt in verehrung und freundschaft zum 75. geburtstag gewidmet.,” Mathematische Nachrichten, vol. 4, no. 1-6, pp. 232–249, (1950).

[18] R. Osserman, “Differential geometry in the large. by heinz hopf,” The American Mathematical Monthly, vol. 93, no. 1, pp. 71–74, (1986).

[19] W. Kühnel and M. Steller, “On closed Weingarten surfaces,” Monatshefte für Mathematik, vol. 146, no. 2, pp. 113–126, (2005).

[20] K. Voss, “Über geschlossene weingartensche flächen ,” Mathematische Annalen, vol. 138, no. 1, pp. 42–54, (1959).

[21] F. Forstneric, “Minimal Surfaces for Undergraduates,” arXiv preprint arXiv:2101.01375, (2021).

[22] H. N. Abd-Ellah and A. K. Omran, “Study on BCN and BAN ruled surfaces in E^3,” Korean Journal of Mathematics, vol. 25, no. 4, pp. 513–535, (2017).

[23] A. Ros and E. Vergasta, “Stability for hypersurfaces of constant mean curvature with free boundary,” Geometriae Dedicata, vol. 56, no. 1, pp. 19–33, (1995).

[24] M. Sakaki, “Stability of surfaces with constant mean curvature in 3-dimensional Riemannian manifolds,” Osaka Journal of Mathematics, vol. 27, no. 4, pp. 893–898, (1990).

[25] B.-Y. Chen, “On a variational problem on hypersurfaces,” Journal of the London Mathematical Society, vol. 2, no. 2, pp. 321–325, (1973).

[26] M. A. Soliman, S. Hassan, and E. Abd ElMonem, “Examples of surfaces gained by variation of pedal surfaces in E^{n+1},” J. Egyptian Math. Soc, vol. 18, pp. 91–105, (2010).

[27] N. H. A. All and H. N. Abd-Ellah, “Stability of closed hyperruled surfaces,” Chaos, Solitons & Fractals, vol. 13, no. 5, pp. 1077–1092, (2002).

[28] N. A. All and H. N. Abd-Ellah, “The tangential variation on hyperruled surfaces,” Applied mathematics and computation, vol. 149, no. 2, pp. 475–492, (2004).

[29] M. A. Soliman, H. N. Abd-Ellah, S. Hassan, and S. Saleh, “Finiteness property of deformed revolution surfaces in E^3 (part I),” Electronic Journal of Mathematical Analysis and Applications, vol. 4, no. 1, pp. 211–225, (2016

[30] R. M. P. Appl, “Finite type surfaces of revolution,” Riv. Mat. Pura Appl, vol. 12, pp. 75–87, (1992).

[31] H. Pottmann and J. Wallner, Computational Line Geometry. Springer, (2001).

[32] H. N. Abd-Ellah, “Translation L/W-surfaces in Euclidean 3-space ,” Journal of the Egyptian Mathematical Society, vol. 23, no. 3, pp. 513–517, (2015).

[33] F. Dillen and W. K hnel, “Ruled Weingarten surfaces in minkowski 3-space,” manuscripta mathematica, vol. 98, no. 3, pp. 307–320, (1999).

[34] J. Suk Ro and D. Won Yoon, “Tubes of Weingarten types in a Euclidean 3-space, j,” Chungcheong Mathematical Society, vol. 22, no. 3, pp. 359–366, (2009).

[35] Y. Fu and L. Li, “A class of Weingarten surfaces in Euclidean 3-space,” in Abstract and Applied Analysis, vol. 2013, Hindawi, (2013).

[36] B.-Y. Chen, “Some open problems and conjectures on submanifolds of finite type,” Soochow J. math, vol. 17, no. 2, pp. 169–188, (1991]