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Inference of Inverted Exponentiated Lomax Distribution based on Step-Stress Partially Accelerated Life Test under Progressive Type II Censoring

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ABSTRACT

Due to the importance of the problem of testing the product units under stress higher than normal stress conditions, specially used in reliability analysis. In this paper, we discuss the problem of estimation with step stress partially accelerated life tests, the lifetime of testing items under use condition follows the inverted exponentiated Lomax distribution. The test is running under progressive Type-II censoring scheme, and the units drawn from the test were distributed as a binomial distribution. The model parameters and acceleration factor are estimated by maximum likelihood and Bayesian methods. The corresponding asymptotic confidence intervals as well as credible intervals are also constructed. Also, the theoretical results are assessed and compared through Monte Carlo simulation study. Two real data set are used to illustrate how the approaches will perform in practice. Finally, we reported some comments about numerical computation.

Keywords:
Partially accelerated life test, inverted exponentiated Lomax distribution, maximum likelihood estimation, Bayesian estimation, progressively Type II censoring

1. Introduction

The applications in numerous fields as econometrics, biological and engineering sciences, medical research, and life testing, need highlight the significance of inverted distributions. In 2009, Hassan and Mohamed presented a new three-parameter lifetime model, called the inverted exponentiated Lomax (*IEL*) distribution. The probability density function (pdf) and the survival function of *IEL* distribution are given, respectively by

$$f\ (t)=\alpha\theta\left(\lambda t^{-2}\right)\left(1+\lambda t^{-1}\right)^{-\alpha-1}\left\{1-\left(1+\lambda t^{-1}\right)^{-\alpha}\right\}^{\theta-1}; \qquad t>0 \quad ,\alpha,\theta,\lambda>0, \tag{1}$$

and

$$S(t)=\left[1-\left(1+\lambda t^{-1}\right)^{-\alpha}\right]^{\theta}, \tag{2}$$

where α and θ are two shape parameters, and λ is a scale parameter.

Testing units may be removed from reliability and life testing trials in a planned manner or in an unanticipated way, such as when an experimental unit breaks accidentally or quits the experiment. In order to save time and money, the removal is typically scheduled in advance. Utilizing the resources at hand effectively involves progressive censoring. To put it another way, if some of the experiment's surviving units are removed early, they can be put to use in other tests (Balakrishnan and Aggarwala 2000).
Lately, a progressively Type-II (PTII) censored sample has become popular enough to analyze highly reliable data because its flexibility and efficiency. In such censoring techniques, n independent and identical units are put to the test over a period of time, and m failures will be noted. The remaining $n-1$ surviving units have r_1 units randomly removed from them after the first failure occurs. All remaining surviving units $r_m = n - m - \sum_{k=1}^{m-1} r_k$ are all eliminated from the test at the conclusion of the mth failure, which causes the test to stop. In this scheme, $r_1, r_2, \ldots, r_m$ are all prefixed. However, these numbers might appear at random in some real-world circumstances. In some reliability trials, for instance, the researcher may determine it is unsuitable to continue testing any of the tested units even though they have not failed. In these situations, the removal pattern for each failure is random. Various authors have studied the statistical inference for different lifetime distributions under PTII censoring with random removals, such as; Yuen and Tse (1996); Tse et al. (2000); Wu et al. (2007); Abd Elghaly et al. (2007); Soliman et al. (2015), Nie and Gui (2019); Tashkandy et al.(2022) and Salem et al. (2023).

When the information of the failure is difficult to be obtained under standard environment, such as when dealing with a unit of high reliability, an accelerated lifetime method can be used to shorter the lifetime of the tested unit. As a result, partially accelerated life tests (PALTs) or accelerated life tests (ALTs) are frequently utilized in manufacturing industries since they significantly reduce test time and expense. This kind of experiments where testing is done under higher than normal stress levels are, in literature, known as ALT. However, in PALTs, both accelerated and usage conditions are used to test the goods. To assess the failure behavior of the items under typical use situations, data is gathered via tests conducted in ALTs or PALTs. The stress loading in ALT are applied in various ways and some of the widely used methods are constant-stress PALTs and step-stress PALTs (SS-PALTs). Nelson (1990) discussed the advantages and disadvantages of each of such methods. In SS-PALTs, a test item is run at normal (use) condition first, then at accelerated condition until the test is finished if it doesn't fail within a predetermined amount of time. In constant-stress PALTs each item is run at constant high stress until either failure occurs or the test is terminated. Specifically, SS-PALTs were studied under progressive censoring schemes by several authors; for example, see Ismail and Sarhan (2009); EL-Sagheer and Ahsanullah (2015); Soliman et al. (2017); Hassan et al. (2017); Mohie EL-Din et al. (2021); Yousef et al. (2022); Almarashi (2023); Alotaibi et al. (2022).
In this paper, the samples generated under progressive Type II censoring schemes with binomial removals and accelerated under SS-PALTs from *IEL* distribution. Yuen and Tse (1996) indicated that the number of patients drop out from a clinical test at each stage is random and cannot be pre-determined. So, in some reliability experiments, the pattern of removal of the tested units at each failure is random. Suppose that any test unit being removed from the life test is independent of the

others but with the same removal probability p. Then, we choose the binomial distribution as a removal distribution. The point and interval estimation of the model parameters are introduced with classical estimation method, maximum likelihood (*MLE*) and Bayes Method. The layout of the paper is as following. The paper is set up as follows. Section 2 provides a detailed description of the model and assumptions. To estimate the model parameters, the maximum likelihood method is described in Section 3, and approximate confidence interval estimations are obtained. In Section 4. The method for Bayesian estimates for the parameters of the *IEL* distribution under PTII censoring based on SS-PALT model is presented. The simulation technique is presented in Section 5 to illustrate theoretical results. Also, to illustrate the useful of the approaches in practice, we provided two real data sets in Section 6. Finally, some concluding remarks are summarized in Section 7.

2. Model Description

The model assumptions for SS-PALT procedure will be described as follows:

1. Under used condition, n identical and independent units are put on the test and the lifetime of each testing unit has the *IEL* distribution.
2. The test is terminated at the *mth* failure, where m is prefixed, $m \le n$ [i.e. m is a total number of units failed under SS-PALT $(m = m_1 + m_2)$].
3. Each of the n units is initially tested under typical operating conditions. It is put under accelerated condition (stress) if it does not fail or pass the test by a predetermined time τ.
4. At the k-th failure a random number of the surviving units, r_k, $k = 1,2,\ldots,m-1$ are randomly selected and removed from the test. Finally, at the m-th failure the remaining surviving units $R_m = n - m - \sum_{k=1}^{m-1} R_k$ are all removed from the test and the test is terminated.
5. Assume that each unit being eliminated from the test has the same removal probability p, despite being independent of the others. Following that, a binomial distribution is followed by the quantity of units deleted at each failure time. This means that $R_1 \sim bino(n-m, p)$ for $k = 2,3,\ldots,m-1$, $R_k \sim bino(n - m - \sum_{j=1}^{m} r_j, p)$ and $r_m = n - m - \sum_{k=1}^{m-1} r_k$.
6. The lifetime, say X , of a unit under SS-PALT can be rewritten as

$$X = \begin{cases} T & if \ \ T \le \tau \\ \tau + \dfrac{T-\tau}{\beta} & if \ \ T > \tau \end{cases} \tag{3}$$

where, T is the lifetime of an item at normal condition, τ is the stress change time and β is the acceleration factor, $\beta > 1$. This model is called the tampered random variable model and proposed by DeGroot and Goel (1979). Thus, by the transformation-variable technique using the density in (1) and the model (3). Then, the pdf of *IEL* distribution under use and accelerated conditions in SSPALT is as follows

$$f(x) = \begin{cases} f_1(x) = \alpha\theta\left(\lambda x^{-2}\right)\left(1+\lambda x^{-1}\right)^{-(\alpha+1)}\left[1-\left(1+\lambda x^{-1}\right)^{-\alpha}\right]^{\theta-1}, & 0 < x \le \tau \\ f_2(x) = \alpha\theta\beta\lambda\left(\tau+\beta(x-\tau)\right)^{-2}\left[1+\lambda\left(\tau+\beta(x-\tau)\right)^{-1}\right]^{-(\alpha+1)}\left[1-\left[1+\lambda\left(\tau+\beta(x-\tau)\right)^{-1}\right]^{-\alpha}\right]^{\theta-1}, & x > \tau \end{cases} \tag{4}$$

3. Parameters Estimation

This section discusses maximum likelihood estimation (MLE) for the point and interval estimation of the model parameters. The observed data follow *IEL* distribution based on PTII censoring reported under SS-PALT model. Theoretical results are described as follows.

3.1. Maximum likelihood method

This section concerns with the *MLE*s of the unknown parameters θ, α , λ and β based on the progressively type II censored data with binomial removal.

Let $(x_k, r_k, u_{1k}, u_{2k})$, $k = 1,2,...,m$, denote the observation obtained from a progressively type-II censored sample under a step-stress PALT. Here $x_{(1)} \le x_{(2)} \le ... \le x_{(m)}$ and u_{1k}, u_{2k} are indicator functions: $u_{1k} \equiv I(x_k < \tau), u_{2k} \equiv I(x_k > \tau)$.

The conditional likelihood function of the observations $x = \{(x_k, r_k, u_{1k}, u_{2k}), k = 1,2, \ldots, m\}$ given the pre-determined number of removals $R = (R_1 = r_1, R_2 = r_2, \ldots R_{m-1} = r_{m-1})$, takes the following form

$$L_1(x_k, \alpha, \theta, \lambda, \beta, u_{1k}, u_{2k} | R = r) = \prod_{k=1}^{m} \left\{ f_1(x_k)\left(S_1(x_k)\right)^{r_k} \right\}^{u_{1k}} \left\{ f_2(x_k)\left(S_2(x_k)\right)^{r_k} \right\}^{u_{2k}}, \tag{5}$$

where,

$$S_1(x) = \left[1 - \left(1 + \lambda x^{-1}\right)^{-\alpha}\right]^{\theta} \text{ and } S_2(x) = \left[1 - \left[1 + \lambda\left(\tau + \beta(x - \tau)\right)^{-1}\right]^{-\alpha}\right]^{\theta}.$$

Inserting the probability density functions $f_1(x)$ and $f_2(x)$ defined in (3) and their corresponding survival functions in (5), then we have

$$L_1(x_k, \alpha, \theta, \lambda, \beta, u_{1k}, u_{2k} | R = r) =$$

$$\prod_{k=1}^{m} \left\{ \alpha\theta\left(\lambda x_k^{-2}\right)\left(1 + \lambda x_k^{-1}\right)^{-(\alpha+1)} \left[1 - \left(1 + \lambda x_k^{-1}\right)^{-\alpha}\right]^{\theta-1} (1 - (1 + \lambda x_k^{-1})^{-\alpha})^{\theta r_k} \right\}^{u_{1k}}$$

$$\left\{ \begin{matrix} \alpha\theta\beta\lambda\left(\tau + \beta(x_k - \tau)\right)^{-2} \left[1 + \lambda\left(\tau + \beta(x_k - \tau)\right)^{-1}\right]^{-(\alpha+1)} \\ \left[1 - \left[1 + \lambda\left(\tau + \beta(x_k - \tau)\right)^{-1}\right]^{-\alpha}\right]^{\theta-1} \left(1 - \left[1 + \lambda\left(\tau + \beta(x_k - \tau)\right)^{-1}\right]^{-\alpha}\right)^{\theta r_k} \end{matrix} \right\}^{u_{2k}}, \tag{6}$$

According to the following probability mass function, the number of units removed at each failure time is expected to follow a binomial distribution

$$P(R_1 = r_1) = \binom{n-m}{r_1} p^{r_1} (1-p)^{n-m-r_1}.$$

While, for $k = 2,3,...,m-1$

$$P(R_k = r_k | R_{k-1} = r_{k-1}, ..., R_1 = r_1) = \binom{n - m - \sum_{j=1}^{k-1} r_j}{r_k} p^{r_k} (1-p)^{n-m-\sum_{j=1}^{k} r_j}.$$

Moreover, suppose that R_k is independent of X_k for all *k*. Hence the likelihood function can be expressed as follows

$$L(x_k,\alpha,\theta,\lambda,\beta,p) = L_1(x_k,\alpha,\theta,\lambda,\beta,u_{1k},u_{2k}|R=r)P(R=r)\,, \qquad (7)$$

where

$$P(R=r)=P(R_1=r_1,R_2=r_2,...,R_{m-1}=r_{m-1})=P(R_{m-1}=r_{m-1}\big|R_{m-2}=r_{m-2},...,R_1=r_1)\times$$
$$P(R_{m-2}=r_{m-2}\big|R_{m-3}=r_{m-3},...,R_1=r_1)\times ...P(R_2=r_2\big|R_1=r_1)P(R_1=r_1).$$

That is,

$$P(R=r)=\frac{(n-m)!}{(n-m-\sum_{k=1}^{m-1}r_k)!\prod_{k=1}^{m-1}r_k}\,p^{\sum_{k=1}^{m-1}r_k}(1-p)^{(m-1)(n-m)-\sum_{k=1}^{m-1}(m-k)r_k},$$

The corresponding log-likelihood function, indicated by, $\ln L_1$, can be obtained from (6) as follows

$$\ln L_1 = m\ln\alpha + m\ln\theta + m\ln\lambda + m_2\ln\beta - 2\sum_{k=1}^{m_1}\ln x_k - (\alpha+1)\sum_{k=1}^{m_1}\ln D_k +$$
$$(\theta-1)\sum_{k=1}^{m_1}\ln(1-D_k^{-\alpha}) + \sum_{k=1}^{m_1}\theta r_k\ln(1-D_k^{-\alpha}) - 2\sum_{k=1}^{m_2}\ln W_k -$$
$$(\alpha+1)\sum_{k=1}^{m_2}\ln H_k + (\theta-1)\sum_{k=1}^{m_2}\ln(1-H_k^{-\alpha}) + \sum_{k=1}^{m_2}\theta r_k\ln(1-H_k^{-\alpha})\,,$$

where, $m=\sum_{k=1}^{m_1}u_{1k}+\sum_{k=1}^{m_2}u_{2k}$, $D_k=\left(1+\lambda x_k^{-1}\right)$, $W_k=\left(\tau+\beta\left(x_k-\tau\right)\right)$ and $H_k=\left(1+\lambda\left(\tau+\beta(x_k-\tau)\right)^{-1}\right)$.

Since $P(R=r)$ does not depend on the parameters θ,α, λ and β, the maximum likelihood estimators $\hat{\theta},\hat{\alpha},\hat{\lambda}$ and $\hat{\beta}$ can be produced by directly maximising $\ln L_1$. Therefore, by solving the following non-linear equations, the maximum probability estimates of θ, α, λ and β may be found.

$$\frac{\partial \ln L_1}{\partial\theta} = \frac{m}{\theta} + \sum_{k=1}^{m_1}\ln(1-D_k^{-\alpha}) + \sum_{k=1}^{m_2}\ln(1-H_k^{-\alpha}) + \sum_{k=1}^{m_1}r_k\ln(1-D_k^{-\alpha}) + \sum_{k=1}^{m_2}r_k\ln(1-H_k^{-\alpha}) = 0,$$

$$\frac{\partial \ln L_1}{\partial\alpha} = \frac{m}{\alpha} - \sum_{k=1}^{m_1}\ln D_k + (\theta-1)\sum_{k=1}^{m_1}\frac{\ln D_k}{(D_k^{\alpha}-1)} + \sum_{k=1}^{m_1}\frac{\theta r_k\ln D_k}{(D_k^{\alpha}-1)} - \sum_{k=1}^{m_2}\ln H_k +$$
$$(\theta-1)\sum_{k=1}^{m_2}\frac{\ln H_k}{(H_k^{\alpha}-1)} + \sum_{k=1}^{m_2}\frac{\theta r_k\ln H_k}{(H_k^{\alpha}-1)} = 0,$$

$$\frac{\partial \ln L_1}{\partial \lambda}=\frac{m}{\lambda}-(\alpha+1)\sum_{k=1}^{m_1}\frac{1}{x_k D_k}+(\theta-1)\sum_{k=1}^{m_1}\frac{\alpha}{x_k D_k(D_k^{\alpha}-1)}+\sum_{k=1}^{m_1}\frac{\theta\alpha r_k}{x_k D_k(D_k^{\alpha}-1)}$$
$$-(\alpha+1)\sum_{k=1}^{m_2}\frac{1}{W_k H_k}+(\theta-1)\sum_{k=1}^{m_2}\frac{\alpha}{W_k H_k(H_k^{\alpha}-1)}+\sum_{k=1}^{m_2}\frac{\theta\alpha r_k}{W_k H_k(H_k^{\alpha}-1)}=0,$$

$$\frac{\partial \ln L_1}{\partial \beta}=\frac{m_2}{\beta}-2\sum_{k=1}^{m_2}\frac{(x_k-\tau)}{W_k}+(\alpha+1)\sum_{k=1}^{m_2}\frac{\lambda\left(x_k-\tau\right)}{H_k W_k^{\,2}}-(\theta-1)\sum_{k=1}^{m_2}\frac{\alpha\lambda\left(x_k-\tau\right)}{H_k W_k^{\,2}(H_k^{\,\alpha}-1)}$$
$$-\sum_{k=1}^{m_2}\frac{\alpha\lambda\theta r_k\left(x_k-\tau\right)}{H_k W_k^{\,2}(H_k^{\,\alpha}-1)}=0.$$

Since it is obvious that there is no closed form solution for the aforementioned non-linear equations, the MLEs of the unknown parameters can be determined using an iterative technique. Similarly, the MLE of can be obtained immediately by maximizing (6) because $L_1(x_k,\alpha,\theta,\lambda,\beta,u_{1k},u_{2k}|R=r)$ does not require the binomial parameter p. Consequently, the following equation must be solved to determine the MLE of p:

$$\frac{\partial \ln L}{\partial p}=\frac{\sum_{k=1}^{m-1} r_k}{p}-\frac{(m-1)(n-m)-\sum_{k=1}^{m-1}(m-k)r_k}{1-p}=0.$$

Hence,

$$\hat{p}=\frac{\sum_{k=1}^{m-1} r_k}{(m-1)(n-m)-\sum_{k=1}^{m-1}(m-k)r_k}.$$

3.2. Asymptotic confidence interval

The most common method to set confidence bounds for the parameters is to use the asymptotic normal distribution of the *MLE*s. By computationally inverting the asymptotic Fisher-information matrix F, it is possible to approximate the asymptotic variances and covariance matrix of the MLE of the parameters. It is made up of the negative second and mixed derivatives of the likelihood function's natural logarithm as determined by the MLE. It is possible to write the asymptotic Fisher information matrix F as follows:

$$F=\begin{bmatrix}\frac{-\partial^2 \ln L_1}{\partial\theta^2} & \frac{-\partial^2 \ln L_1}{\partial\theta\partial\alpha} & \frac{-\partial^2 \ln L_1}{\partial\theta\partial\lambda} & \frac{-\partial^2 \ln L_1}{\partial\theta\partial\beta}\\ \frac{-\partial^2 \ln L_1}{\partial\alpha\partial\theta} & \frac{-\partial^2 \ln L_1}{\partial\alpha^2} & \frac{-\partial^2 \ln L_1}{\partial\alpha\partial\lambda} & \frac{-\partial^2 \ln L_1}{\partial\alpha\partial\beta}\\ \frac{-\partial^2 \ln L_1}{\partial\lambda\partial\theta} & \frac{-\partial^2 \ln L_1}{\partial\lambda\partial\alpha} & \frac{-\partial^2 \ln L_1}{\partial\lambda^2} & \frac{-\partial^2 \ln L_1}{\partial\lambda\partial\beta}\\ \frac{-\partial^2 \ln L_1}{\partial\beta\partial\theta} & \frac{-\partial^2 \ln L_1}{\partial\beta\partial\alpha} & \frac{-\partial^2 \ln L_1}{\partial\beta\partial\lambda} & \frac{-\partial^2 \ln L_1}{\partial\beta^2}\end{bmatrix}\downarrow(\hat{\theta},\hat{\alpha},\hat{\lambda},\hat{\beta})$$

The log-likelihood function's second and mixed partial derivatives with regard to θ, α , λ and β are acquired as shown here

$$\frac{\partial^2 \ln L_1}{\partial \theta^2} = \frac{-m}{\theta^2},$$

$$\frac{\partial^2 \ln L_1}{\partial \alpha^2} = \frac{-m}{\alpha^2} - (\theta-1)\sum_{k=1}^{m_1} \frac{D_k^\alpha (\ln D_k)^2}{(D_k^\alpha - 1)^2} - \sum_{k=1}^{m_1} \frac{\theta r_k D_k^\alpha (\ln D_k)^2}{(D_k^\alpha - 1)^2} -$$
$$(\theta-1)\sum_{k=1}^{m_2} \frac{H_k^\alpha (\ln H_k)^2}{(H_k^\alpha - 1)^2} - \sum_{k=1}^{m_2} \frac{\theta r_k H_k^\alpha (\ln H_k)^2}{(H_k^\alpha - 1)^2},$$

$$\frac{\partial^2 lnL_1}{\partial\alpha\partial\theta} = \sum_{k=1}^{m_1} \frac{lnD_k}{(D_k^\alpha - 1)} + \sum_{k=1}^{m_2} \frac{lnH_k}{(H_k^\alpha - 1)} + \sum_{k=1}^{m_1} \frac{r_k lnD_k}{(D_k^\alpha - 1)} + \sum_{k=1}^{m_2} \frac{r_k lnH_k}{(H_k^\alpha - 1)},$$

$$\frac{\partial^2 lnL_1}{\partial\lambda\partial\theta} = \sum_{k=1}^{m_1} \frac{\alpha}{x_k D_k (D_k^\alpha - 1)} + \sum_{k=1}^{m_2} \frac{\alpha}{W_k H_k (H_k^\alpha - 1)} + \sum_{k=1}^{m_1} \frac{\alpha r_k}{x_k D_k (D_k^\alpha - 1)} + \sum_{k=1}^{m_2} \frac{\alpha r_k}{W_k H_k (H_k^\alpha - 1)}$$

$$\boldsymbol{\frac{\partial^2 lnL_1}{\partial\beta\partial\theta} = \sum_{k=1}^{m_2} \frac{-\alpha\lambda(x_k-\tau)}{W_k^2 H_k (H_k^\alpha - 1)} - \sum_{k=1}^{m_2} \frac{\alpha\lambda r_k (x_k-\tau)}{W_k^2 H_k (H_k^\alpha - 1)}},$$

$$\frac{\partial^2 lnL_1}{\partial\lambda\partial\alpha} = -\sum_{k=1}^{m_1} \frac{1}{x_k D_k} - \sum_{k=1}^{m_2} \frac{1}{W_k H_k} + (\theta-1)\sum_{k=1}^{m_1} \frac{(D_k^\alpha - 1) - \alpha D_k^\alpha lnD_k}{x_k D_k (D_k^\alpha - 1)^2} +$$
$$+ \sum_{k=1}^{m_1} \frac{\theta r_k [(D_k^\alpha - 1) - \alpha D_k^\alpha lnD_k]}{x_k D_k (D_k^\alpha - 1)^2} + (\theta-1)\sum_{k=1}^{m_2} \frac{(H_k^\alpha - 1) - \alpha H_k^\alpha lnH_k}{W_k H_k (H_k^\alpha - 1)^2}$$
$$+ \sum_{k=1}^{m_2} \frac{\theta r_k [(H_k^\alpha - 1) - \alpha H_k^\alpha lnH_k]}{W_k H_k (H_k^\alpha - 1)^2},$$

$$\frac{\partial^2 lnL_1}{\partial\beta\partial\lambda} = -(\alpha+1)\sum_{k=1}^{m_2} \left\{ \frac{-(x_k-\tau)}{H_k W_k^2} + \frac{\lambda(x_k-\tau)}{H_k^2 W_k^3} \right\} +$$
$$(\theta-1)\sum_{k=1}^{m_2} \alpha \left\{ \frac{-(x_k-\tau)}{W_k^2 H_k (H_k^\alpha - 1)} + \frac{\lambda(x_k-\tau)}{H_k^2 W_k^3 (H_k^\alpha - 1)} + \frac{\alpha\lambda(x_k-\tau) H_k^\alpha}{H_k^2 W_k^3 (H_k^\alpha - 1)^2} \right\}$$
$$+ \sum_{k=1}^{m_2} \theta\alpha r_k \left\{ \frac{-(x_k-\tau)}{W_k^2 H_k (H_k^\alpha - 1)} + \frac{\lambda(x_k-\tau)}{H_k^2 W_k^3 (H_k^\alpha - 1)} + \frac{\alpha\lambda(x_k-\tau) H_k^\alpha}{H_k^2 W_k^3 (H_k^\alpha - 1)^2} \right\},$$

$$\frac{\partial^2 lnL_1}{\partial\beta\partial\alpha} = \sum_{k=1}^{m_2}\frac{\lambda(x_k-\tau)}{H_kW_k^2} + (\theta-1)\sum_{k=1}^{m_2}\frac{[-\lambda(x_k-\tau)(H_k^\alpha-1)+\alpha\lambda(x_k-\tau)H_k^\alpha lnH_k]}{H_kW_k^2(H_k^\alpha-1)^2}$$
$$+\sum_{k=1}^{m_2}\frac{\theta r_k\lambda[-(x_k-\tau)(H_k^\alpha-1)+\alpha(x_k-\tau)H_i^\alpha lnH_k]}{H_kW_k^2(H_k^\alpha-1)^2},$$

$$\frac{\partial^2 lnL_1}{\partial\lambda^2} = \frac{-m}{\lambda^2} + (\alpha+1)\sum_{k=1}^{m_1}\frac{1}{{x_k}^2{D_k}^2} - (\theta-1)\sum_{k=1}^{m_1}\left\{\frac{\alpha}{{x_k}^2{D_k}^2(D_k^\alpha-1)} + \frac{\alpha^2D_k^{\alpha-1}}{{x_k}^2D_k(D_k^\alpha-1)^2}\right\} -$$
$$\sum_{k=1}^{m_1}\left\{\frac{\theta\alpha r_k}{{x_k}^2{D_k}^2(D_k^\alpha-1)} + \frac{\theta\alpha^2 r_kD_k^{\alpha-1}}{{x_k}^2D_k(D_k^\alpha-1)^2}\right\} + (\alpha+1)\sum_{k=1}^{m_2}\frac{1}{W_k^2H_k^2} -$$
$$(\theta-1)\sum_{k=1}^{m_2}\left\{\frac{\alpha}{W_k^2H_k^2(H_k^\alpha-1)} + \frac{\alpha^2H_k^{\alpha-1}}{W_k^2H_k(H_k^\alpha-1)^2}\right\} - \sum_{k=1}^{m_2}\left\{\frac{\theta\alpha r_k}{W_k^2H_k^2(H_k^\alpha-1)} + \frac{\theta\alpha^2 r_kH_k^{\alpha-1}}{W_k^2H_k(H_k^\alpha-1)^2}\right\},$$

Under the normality property of MLEs of the parameters α,θ and β. The $100(1-\gamma)\%$ approximate confidence interval (ACI) for α,θ and β can be, respectively, easily constructed as

$$\hat{\alpha} \pm z_{\gamma/2}\sigma_{\hat{\alpha}},\ \hat{\theta} \pm z_{\gamma/2}\sigma_{\hat{\theta}},\ \hat{\beta} \pm z_{\gamma/2}\sigma_{\hat{\beta}},$$

where, $z_{\frac{\gamma}{2}}$ is the $(\gamma/2)$ 100% lower percentile of standard normal distribution, γ is the specified significance level and $\sigma(.)$ is the standard deviation for the maximum likelihood estimates.

4. Bayesian Estimation

This section explains the method for Bayesian estimates for the parameters θ,α, λ and β of the *IEL* distribution under PTII censoring based on SS-PALT model. These estimations are based on the square error (SE) loss function, which will be discussed in this section. We suppose the parameter prior distributions to be gamma priors. The following is an example of the joint gamma prior density of θ,α, λ and β:

$$\pi(\theta,\alpha,\beta,\lambda) \propto \theta^{q_1-1}e^{-w_1\theta}\alpha^{q_2-1}e^{-w_2\alpha}\beta^{q_3-1}e^{-w_3\beta}\lambda^{q_4-1}e^{-w_4\lambda};\ q_j,w_j>0, j=1,2,3,4. \quad (8)$$

In order to extract the hyper-parameters of the informative priors, the estimates and their variances were equated with the inverse of the Fisher information matrix of alpha and beta to produce the ML estimator for θ,α, λ and β, which it denoted as elective hyper-parameters and this contributed by Dey et al. (2016). Using (6) and (8), the joint posterior of the *IEL* under PTII censoring based on SS-PALT model with parametersθ,α, λ and β is derived as

$$L_1(x_i,\alpha,\theta,\lambda,\beta,u_{1i},u_{2i}|R=r)$$
$$=\theta^{m+q_1-1}e^{-w_1\theta}\alpha^{m+q_2-1}e^{-w_2\alpha}\beta^{q_3-1}e^{-w_3\beta}\lambda^{q_4-1}e^{-w_4\lambda}\prod_{i=1}^{m}\{(\lambda {x_i}^{-2})(1$$
$$+\lambda{x_i}^{-1})^{-(\alpha+1)}[1-(1+\lambda{x_i}^{-1})^{-\alpha}]^{\theta-1}(1-(1+\lambda{x_i}^{-1})^{-\alpha})^{\theta r_i}\}^{u_{1i}}\{\beta\lambda(\tau+\beta(x_i$$
$$-\tau))^{-2}[1+\lambda(\tau+\beta(x_i-\tau))^{-1}]^{-(\alpha+1)}[1-[1+\lambda(\tau+\beta(x_i-\tau))^{-1}]^{-\alpha}]^{\theta-1}(1$$
$$-[1+\lambda(\tau+\beta(x_i-\tau))^{-1}]^{-\alpha})^{\theta r_i}\}^{u_{2i}}, \quad (9)$$

The SE loss function (SELF), which is defined as follows:
$\boldsymbol{L(\tilde{\alpha},\alpha)=(\tilde{\alpha}-\alpha)^2, L(\tilde{\beta},\beta)=\left(\tilde{\beta}-\beta\right)^2, L(\tilde{\lambda},\lambda)=\left(\tilde{\lambda}-\lambda\right)^2}$ and $\boldsymbol{L\left(\tilde{\theta},\theta\right)=\left(\tilde{\theta}-\theta\right)^2}$, which it is used to evaluate the Bayesian estimates of $\boldsymbol{\theta}$, $\boldsymbol{\alpha}$, λ and $\boldsymbol{\beta}$.

In this section, we are used the Markov chain Monte Carlo (MCMC) method to simulate the posterior distribution as well as the estimation problem. The MCMC technique can be used to obtain the Bayesian estimators. Useful MCMC subclasses include Gibbs sampling and the more versatile Metropolis inside Gibbs sampling. The Metropolis-Hastings (MH) algorithm and Gibbs sampling are the two most popular MCMC techniques. Using the MH inside Gibbs sampling steps, we generate random samples from conditional posterior densities of θ,$\boldsymbol{\alpha}$, λ and $\boldsymbol{\beta}$.
The MH method has the following steps for taking a sample from the posterior density:
Step 1. Set the initial value of $\underline{\Phi}$ as $\underline{\Phi}^{(0)}=\left(\tilde{\theta},\tilde{\alpha},\tilde{\lambda},\tilde{\beta}\right)$.
Step 2. For $r=1,2,\dots,M$ the following steps are repeated:

- Set $\underline{\Phi}=\underline{\Phi}^{(i-1)}$.
- Generate a new candidate parameter value $\underline{\acute{\Phi}}$ from $N\left(\underline{\Phi},S_{\underline{\Phi}}\right)$, where $S_{\underline{\Phi}}$ is variance of vector parameters.
- Calculate $\xi=\pi^*\left(\underline{\acute{\Phi}}|x\right)/\pi^*\left(\underline{\Phi}|x\right)$, where $\pi^*\left(\underline{\acute{\Phi}}|x\right)$ is the posterior density.
- Generate a sample u from the uniform distribution $U(0,1)$.
- Accept or reject the new candidate $\underline{\acute{\Phi}}$

$$\begin{cases} If \quad u\le\ \xi \ \ set\ \underline{\Phi}^{(i)}=\underline{\acute{\Phi}} \\ otherwise\ set\ \underline{\Phi}^{(i)}=\underline{\acute{\Phi}}\,. \end{cases}$$

Through SELF, the Bayesian estimators are obtained. The method suggested by Chen and Shao (1999) is used to provide the 95% two-sided greatest density region credible interval for the unknown parameters or any function of them. For more recently papers, see Tolba (2022), Salem et al. (2023), and Hamdy et al. (2023).

5. Simulation Study

We will use a simulation study to estimate the unknown parameters of the *IEL* distribution under PTII censoring based on the SS-PALT model in order to assess the effectiveness of the proposed approach. For different sample combinations, the *MLE* and Bayesian estimation methods' respective MSE and bias, as well as CIs with their width, are determined.
Sample combinations:

- The sample size n has been taken as 50, 100, and 200.
- Censored sample size m has been taken as 35, and 45 when *n* = 50, *m* = 60, and 80 when *n* =100, and *m* = 150, and 180 when *n* =200.
- Different values of parameters as actual values have been chosen as:
  - Case I: $\alpha=0.5,\theta=1.2,\lambda=0.7,\beta=2$.
  - Case II: $\alpha=2,\theta=1.2,\lambda=0.7,\beta=2$.
  - Case III: $\alpha=1.5,\theta=2.5,\lambda=0.5,\beta=3$.
- Determine the value of binomial parameter p as 0.4 and 0.7.

The simulation procedure for estimation is as follows:

I. Create an n-piece random sample using the uniform distribution $\boldsymbol{U}(\mathbf{0},\mathbf{1})$.

II. Generate a random number of R from Binomial as follows:
- ❖ $\boldsymbol{R_1}$ is produced randomly using the binomial $(\boldsymbol{n}-\boldsymbol{m},\boldsymbol{p})$;
- ❖ $\boldsymbol{R_k}$ is produced randomly using the binomial $\left(\boldsymbol{n}-\boldsymbol{m}-\sum_{j=1}^{k-1}\boldsymbol{R_j}\,,\boldsymbol{p}\right),\boldsymbol{k}=\mathbf{2},\dots,\boldsymbol{m}-\mathbf{1}$;
- ❖ $\boldsymbol{R_m}$ is produced as $(\boldsymbol{n}-\boldsymbol{m}-\sum_{j=1}^{m-1}\boldsymbol{R_j})$;

III. By algorithm in Balakrishnan and Sandhu (1995), we can generate PTII censoring sample.

IV. Generate *IEL* sample based on PTII censoring under SS-PALT.

V. To obtain the MLEs of the model parameters, the Newton-Raphson method is employed to simultaneously solve the nonlinear equations.

VI. To obtain the Bayesian of the model parameters, the Metropolis-Hastings (M-H) algorithm is employed.

VII. When evaluating the performance of estimators, their bias has been taken into account. Also, mean square error (MSE), width of CI (WCI) and coverage probability (CP) are evaluated.

VIII. We replicate the process 1000 times for each setting to get average estimations.

Additionally, the approximate confidence intervals (ACIs) and greatest posterior density (HPD) intervals, two distinct confidence interval-estimation techniques, are built. The average WCI for MLE as approximate WCI can be denoted as (WACI) while for Bayesian as credible WCI can be denoted as (WCCI) of the acquired estimators. The MLEs are evaluated numerically using the 'maxLik' package by Henningsen and Toomet (2011), which implements the Newton-Raphson technique, and the Bayes estimation are assessed using the 'CODA' package by Plummer et al. (2006), which generates the MCMC variates.

Results and discussions: From Tables 1-3, some comments can be made as:

- In terms of bias, MSE, WCI, and CP, the given MLE and Bayes estimation of the unknown *IEL* parameters under PTII censoring based on the SS-PALT model are quite satisfactory.
- The collected estimates improve considerably more in terms of their bias, MSE, and WCI when m as expected, increases.
- Frequentist estimates are outperformed by Bayesian estimates based on gamma conjugate prior functions because they take prior knowledge into account.
- The value of the removal probability $\boldsymbol{p}$ has an impact on how m affects the precision of the MLE of the parameters. For constant n and m, the MSE of the parameter estimations occasionally falls as $\boldsymbol{p}$ rises.
- Bayesian estimation is better than MLE in this model and under the presented scheme.

Table 1: Bias, MSE, WCI and CP for parameters of *IEL distribution* under PTII censoring based on SS-PALT model: $\alpha = 0.5, \theta = 1.2, \lambda = 0.7, \beta = 2$

P			0.4								0.7							
$\tau = 0.15$			MLE				Bayesian				MLE				Bayesian			
n	m		Bias	MSE	WACI	CP	Bias	MSE	WCCI	CP	Bias	MSE	WACI	CP	Bias	MSE	WCCI	CP
50	35	α	0.1019	0.0376	0.6466	94.9%	0.0057	0.0012	0.2354	98.5%	0.1015	0.0395	0.6703	96.3%	0.0051	0.0013	0.2345	99.0%
		θ	0.9286	1.4581	3.0275	97.6%	0.3674	0.2493	1.1086	90.2%	0.9170	1.4483	3.0566	97.0%	0.3662	0.2587	1.1042	90.1%
		λ	0.0434	0.3239	2.2255	96.7%	-0.0865	0.0552	0.7123	92.2%	0.0141	0.2629	2.0102	95.6%	-0.0845	0.0560	0.7224	92.2%
		β	0.2490	1.0902	3.9770	95.7%	0.0473	0.0242	0.7332	96.8%	0.1881	1.0235	3.8985	95.7%	0.0379	0.0213	0.7102	96.0%
	45	α	0.0549	0.0172	0.4674	96.3%	-0.0214	0.0011	0.1746	99.1%	0.0502	0.0166	0.4645	96.0%	-0.0237	0.0013	0.1705	98.9%
		θ	0.9086	1.3775	2.9140	96.8%	0.1079	0.0314	0.3857	84.3%	0.8877	1.3236	2.8702	96.6%	0.0996	0.0317	0.3814	81.1%
		λ	0.0357	0.2520	1.9638	95.5%	-0.0856	0.0287	0.3719	79.7%	0.0136	0.2485	1.9810	95.4%	-0.0929	0.0302	0.3665	78.0%
		β	0.1397	0.7864	3.4346	95.9%	0.0159	0.0148	0.3301	83.6%	0.2095	0.8747	3.5749	95.5%	0.0170	0.0142	0.3306	84.3%
100	60	α	0.0892	0.0233	0.4862	95.9%	0.0051	0.0007	0.1755	98.9%	0.0761	0.0198	0.4639	96.3%	0.0025	0.0006	0.1743	98.9%
		θ	0.8203	1.1718	2.7700	96.9%	0.4128	0.2608	1.0145	91.9%	0.7506	1.0353	2.6942	96.9%	0.4077	0.2501	1.0002	92.1%
		λ	-0.0708	0.1881	1.6781	95.7%	-0.0881	0.0407	0.6330	92.0%	-0.0950	0.1867	1.6532	95.5%	-0.0924	0.0357	0.6232	93.5%
		β	0.0704	0.5702	2.9487	95.6%	0.0305	0.0079	0.5788	99.2%	0.0884	0.6029	3.0254	95.8%	0.0264	0.0084	0.5852	98.6%
	80	α	0.0417	0.0095	0.3448	96.1%	-0.0167	0.0006	0.1367	99.7%	0.0350	0.0080	0.3239	95.5%	-0.0181	0.0005	0.1373	99.4%
		θ	0.8300	1.0815	2.4573	95.5%	0.1487	0.0404	0.3980	86.5%	0.6833	1.0086	2.4566	96.4%	0.1509	0.0405	0.4029	86.4%
		λ	0.0200	0.1623	1.6172	94.8%	-0.1263	0.0309	0.3566	86.1%	0.0325	0.1625	1.4957	94.3%	-0.1301	0.0327	0.3606	85.8%
		β	0.1315	0.5216	2.7852	95.6%	0.0192	0.0067	0.3063	98.7%	0.1356	0.5141	2.7613	95.7%	0.0146	0.0079	0.3016	90.4%
200	150	α	0.0240	0.0041	0.2328	95.3%	-0.0022	0.0001	0.1082	99.8%	0.0282	0.0045	0.2391	95.4%	-0.0024	0.0001	0.1096	99.0%
		θ	0.7269	0.7454	1.8268	96.2%	0.4648	0.2665	0.8703	95.7%	0.7245	0.7639	1.9172	96.6%	0.4641	0.2624	0.8585	95.5%
		λ	-0.0458	0.1229	1.3632	97.0%	-0.1185	0.0302	0.5309	95.3%	-0.0587	0.1283	1.3859	96.0%	-0.1154	0.0276	0.5224	96.1%
		β	0.1238	0.2692	1.9760	95.8%	0.0180	0.0019	0.4036	99.9%	0.0709	0.2480	1.9333	95.1%	0.0166	0.0017	0.3945	99.9%
	180	α	0.0140	0.0029	0.2055	94.5%	-0.0022	0.0001	0.0994	100.0%	0.0156	0.0030	0.2065	95.1%	-0.0011	0.0001	0.1001	99.9%
		θ	0.6756	0.7169	1.7376	96.0%	0.2215	0.0644	0.4207	91.1%	0.7223	0.7174	1.7352	96.6%	0.2208	0.0629	0.4174	90.1%
		λ	-0.0207	0.0918	1.1856	97.0%	-0.1072	0.0296	0.3403	91.8%	-0.0458	0.0934	1.1852	97.2%	-0.1074	0.0240	0.3384	92.1%
		β	0.0944	0.2361	1.8692	95.8%	0.0121	0.0012	0.2593	96.6%	0.1081	0.2427	1.8851	95.8%	0.0134	0.0014	0.2624	96.1%

Table 2: Bias, MSE, WCI and CP for parameters of *IEL distribution* under PTII censoring based on SS-PALT model: $\alpha = 2, \theta = 1.2, \lambda = 0.7, \beta = 2$

P			0.4								0.7							
$\tau = 0.5$			MLE				Bayesian				MLE				Bayesian			
n	m		Bias	MSE	LACI	CP	Bias	MSE	LCCI	CP	Bias	MSE	LACI	CP	Bias	MSE	LCCI	CP
50	35	α	0.7430	1.7166	4.2324	97.3%	0.0274	0.1003	1.0489	91.8%	0.7766	1.8164	4.3200	97.5%	0.0207	0.1029	1.0304	90.4%
		θ	1.5464	4.6409	5.8823	95.2%	0.3884	0.2664	1.1405	91.8%	1.5316	4.2786	5.4525	95.4%	0.3875	0.2776	1.1384	91.7%
		λ	0.2158	0.8964	3.6156	94.1%	-0.0332	0.0254	0.6357	95.2%	0.1698	0.7393	3.3057	94.2%	-0.0399	0.0237	0.6336	95.9%
		β	0.2062	1.4947	4.7262	95.1%	0.0675	0.0283	0.6658	95.6%	0.1147	1.2262	4.3196	96.2%	0.0493	0.0160	0.6460	97.2%
	45	α	0.4166	1.0437	3.6585	97.1%	-0.0489	0.0246	0.3768	79.4%	0.3762	0.9910	3.6147	96.1%	-0.0489	0.0247	0.3659	77.8%
		θ	1.4286	3.8405	5.2612	95.5%	0.0880	0.0276	0.3765	81.9%	1.4467	3.8070	5.1346	94.9%	0.0994	0.0322	0.3833	80.7%
		λ	0.2813	0.8669	3.4811	93.9%	-0.1020	0.0227	0.3346	85.7%	0.3183	0.6895	3.4945	93.8%	-0.0989	0.0218	0.3306	85.9%
		β	0.1299	0.9406	3.7694	95.6%	0.0230	0.0113	0.3189	89.5%	0.1391	0.9770	3.8381	95.0%	0.0248	0.0117	0.3175	87.0%
100	60	α	0.6502	1.2778	3.6266	96.2%	0.0393	0.0619	0.9217	94.5%	0.6661	1.3105	3.6513	96.4%	0.0258	0.0523	0.9251	95.6%
		θ	1.2715	2.8173	4.2973	94.7%	0.4720	0.3190	1.0829	93.6%	1.2286	2.5789	4.0561	94.8%	0.4322	0.2666	1.0351	95.3%
		λ	0.0851	0.4073	2.4806	94.0%	-0.0303	0.0152	0.5576	97.3%	0.0539	0.3852	2.4249	94.5%	-0.0453	0.0138	0.5378	97.3%
		β	0.0644	0.6047	3.0392	96.3%	0.0439	0.0080	0.5255	98.5%	0.0584	0.6833	3.2339	96.0%	0.0430	0.0076	0.5207	98.4%
	80	α	0.3572	0.6795	2.9137	95.4%	-0.0562	0.0229	0.3607	79.5%	0.3881	0.7420	3.0161	96.6%	-0.0485	0.0237	0.3623	78.3%
		θ	1.2345	2.5641	3.9998	94.8%	0.1433	0.0390	0.3974	86.2%	1.1703	2.3103	3.8042	95.2%	0.1391	0.0369	0.3879	86.2%
		λ	0.1321	0.4024	2.4335	94.7%	-0.0411	0.0149	0.3014	92.0%	0.1102	0.3891	2.4079	94.8%	-0.0411	0.0120	0.3019	90.8%
		β	0.0231	0.4978	2.7657	95.6%	0.0251	0.0065	0.2871	92.8%	0.0369	0.5626	2.9382	96.2%	0.0273	0.0070	0.2891	93.9%
200	150	α	0.2824	0.3886	2.1796	95.3%	-0.0118	0.0153	0.7018	98.7%	0.3064	0.4666	2.3945	95.2%	-0.0073	0.0197	0.7037	97.7%
		θ	0.9863	1.3619	2.4464	95.0%	0.4535	0.2518	0.8410	95.4%	0.9910	1.3941	2.5178	94.8%	0.4630	0.2660	0.8456	94.7%
		λ	0.0200	0.1432	1.4821	95.5%	-0.0653	0.0092	0.4205	98.6%	0.0276	0.1624	1.5770	94.4%	-0.0647	0.0085	0.4166	98.8%
		β	-0.0048	0.2398	1.9207	95.6%	0.0234	0.0016	0.3384	99.5%	-0.0180	0.2327	1.8906	95.5%	0.0215	0.0014	0.3351	99.9%
	180	α	0.1517	0.2741	1.9652	96.2%	-0.0105	0.0146	0.3329	84.6%	0.1547	0.2810	1.9884	95.5%	-0.0377	0.0151	0.3300	85.7%
		θ	0.9558	1.2557	2.2942	96.2%	0.2035	0.0540	0.4012	93.4%	0.9949	1.3557	2.3725	95.1%	0.2100	0.0568	0.4098	92.9%
		λ	0.0656	0.1421	1.4557	95.7%	-0.0513	0.0082	0.2555	94.6%	0.0756	0.1631	1.5561	95.5%	-0.0613	0.0072	0.2528	94.4%
		β	0.0550	0.2395	1.9073	95.5%	0.0207	0.0013	0.2393	97.4%	0.0011	0.2155	1.8208	94.8%	0.0126	0.0013	0.2389	97.8%

Table 3: Bias, MSE, WCI and CP for parameters of *IEL distribution* under PTII censoring based on SS-PALT model: $\alpha = 1.5, \theta = 2.5, \lambda = 0.5, \beta = 3$

P			0.4								0.7							
$\tau = 0.5$			MLE				Bayesian				MLE				Bayesian			
n	m		Bias	MSE	LACI	CP	Bias	MSE	LCCI	CP	Bias	MSE	LACI	CP	Bias	MSE	LCCI	CP
50	35	α	0.5022	0.7381	2.7339	95.9%	0.0642	0.0448	0.8185	95.1%	0.4728	0.7000	2.7071	96.2%	0.0448	0.0373	0.8134	96.0%
		θ	0.8419	2.1522	4.7119	96.0%	0.1216	0.2440	1.2759	83.0%	0.7357	2.0198	4.7690	95.7%	0.1391	0.2558	1.3026	81.8%
		λ	0.0057	0.1456	1.4963	95.5%	-0.0059	0.0161	0.5144	95.3%	0.0071	0.1638	1.5872	95.4%	0.0007	0.0150	0.5190	95.2%
		β	0.1715	1.4077	4.6044	95.9%	0.0451	0.0543	0.9546	94.4%	0.2611	1.6538	4.9386	94.5%	0.0367	0.0559	0.9713	94.5%
	45	α	0.2891	0.4291	2.3053	95.0%	-0.0159	0.0163	0.3406	82.4%	0.2982	0.4406	2.3256	95.9%	-0.0131	0.0161	0.3376	81.8%
		θ	0.5943	1.2264	3.9947	95.2%	0.0164	0.0292	0.3939	75.6%	0.4899	1.2599	3.2484	95.4%	0.0138	0.0286	0.3874	74.1%
		λ	0.0375	0.1287	2.0573	94.3%	-0.0295	0.0078	0.2743	91.7%	0.0941	0.1251	1.3177	93.8%	-0.0337	0.0080	0.2729	91.9%
		β	0.1708	1.3461	4.1650	95.5%	0.0022	0.0210	0.3621	78.5%	0.2589	1.6380	4.9157	94.9%	0.0044	0.0211	0.3638	79.5%
100	60	α	0.4350	0.4983	2.1804	96.2%	0.0623	0.0268	0.7002	97.3%	0.4088	0.4569	2.1115	96.1%	0.0612	0.0276	0.7028	96.5%
		θ	0.5117	1.0853	3.5591	94.6%	0.1819	0.2022	1.2268	86.7%	0.5364	1.1335	3.6069	95.5%	0.1859	0.2230	1.2325	85.0%
		λ	-0.0633	0.0902	1.1515	94.9%	-0.0187	0.0091	0.4268	97.1%	-0.0516	0.0925	1.1753	95.0%	-0.0215	0.0078	0.4302	98.2%
		β	0.1347	0.8579	3.5941	94.6%	0.0291	0.0260	0.8398	97.9%	0.0996	0.8787	3.6555	94.7%	0.0349	0.0258	0.8294	97.6%
	80	α	0.2454	0.2513	1.7144	95.4%	-0.0043	0.0127	0.3182	86.5%	0.2177	0.2307	1.6793	95.2%	-0.0107	0.0112	0.3145	86.7%
		θ	0.4790	0.9818	3.3602	95.1%	0.0277	0.0287	0.3912	76.0%	0.5063	1.1258	3.5096	95.7%	0.0127	0.0286	0.3906	75.3%
		λ	-0.0098	0.0896	1.0213	94.7%	-0.0412	0.0059	0.2430	94.1%	0.0013	0.0905	1.0212	95.2%	-0.0396	0.0060	0.2458	94.1%
		β	0.1392	0.7534	3.3602	94.5%	0.0013	0.0164	0.3477	82.6%	0.1388	0.7043	3.2460	94.8%	0.0095	0.0176	0.3535	81.1%
200	150	α	0.1784	0.1277	1.2145	95.8%	0.0210	0.0055	0.4829	99.3%	0.1623	0.1230	1.2195	96.0%	0.0119	0.0045	0.4756	99.7%
		θ	0.4841	0.8568	3.0942	94.8%	0.2281	0.1532	1.0878	90.8%	0.4474	0.7873	3.0051	95.8%	0.2026	0.1415	1.0761	91.3%
		λ	-0.0350	0.0522	0.8854	95.4%	-0.0321	0.0040	0.3163	99.0%	-0.0315	0.0536	0.8992	96.0%	-0.0336	0.0042	0.3170	99.0%
		β	0.0905	0.3877	2.4160	95.5%	0.0219	0.0063	0.6123	99.5%	0.1555	0.4744	2.6317	95.6%	0.0256	0.0067	0.6226	99.6%
	180	α	0.0912	0.0748	1.0116	96.1%	-0.0078	0.0043	0.2781	92.9%	0.0932	0.0830	1.0693	95.5%	-0.0104	0.0041	0.2794	91.6%
		θ	0.5246	0.7404	2.6749	95.3%	0.0476	0.0258	0.3747	76.9%	0.5243	0.6840	2.9482	96.3%	0.0421	0.0252	0.3751	78.7%
		λ	0.0014	0.0423	0.8071	95.7%	-0.0510	0.0039	0.2009	96.6%	0.0033	0.0473	0.8528	95.6%	-0.0315	0.0041	0.2008	95.7%
		β	0.1214	0.3892	2.4001	94.4%	0.0039	0.0051	0.3143	88.4%	0.1227	0.3760	2.3564	95.3%	0.0095	0.0060	0.3133	89.1%

6. Application

This section examines two genuine data sets, the first of which was researched and presented by Hassan and Mohamed (2019) and the second of which was researched and presented by Klakattawi et al. (2022). The *IEL* better distribution, however (see Hassan and Mohamed (2019)), is consistent with the smaller values of the earlier measures.

Data Set 1: The first data set shows how many million rotations a set of 23 ball bearings could withstand before failing, according to Lawless (1982). According to the findings of Hassan and Mohamed (2019), the *IEL* model is appropriate for this data set based on the chosen criteria. The goodness of fit measurements for the *IEL* model is the smallest. The data can be displayed as follows: 0.1788, 0.2892, 0.33, 0.4152, 0.4212, 0.4560, 0.4848, 0.5184, 0.5196, 0.5412, 0.5556, 0.6780, 0.6864, 0.6864, 0.6888, 0.8412, 0.9312, 0.9864, 1.0512, 1.0584, 1.2792, 1.2804, 1.7340.

The Kolmogorov-Smirnov (K-S) statistic (with associated P-value) is obtained to demonstrate the reliability of the *IEL* model. First, MLEs for parameters, respectively, are 5.0620(4.4929), 12.1849(11.3937) and 0.5025(0.4896), with their standard errors (SEs), and the K-S(P-value) is 0.1079(0.9514). The *IEL* distribution fits the failure data, according to this finding. The p-values, Figure 1, the K-S distances, and the model's reasonable agreement with the data are all indicators of this.

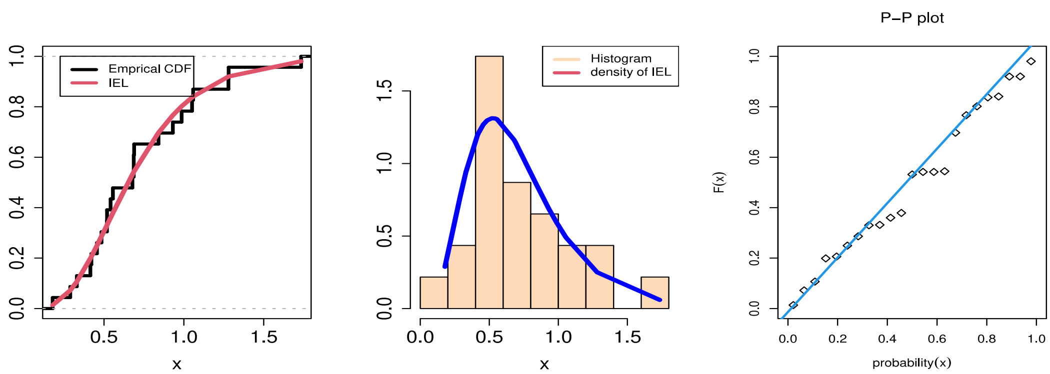

Figure 1: Estimated CDF and PDF with data also P-P plot of *IEL* distribution: data I

Table 4: *MLE* and Bayesian by estimates values and SE with different accretion time: Data I

		MLE		Bayesian	
tau		estimates	SE	estimates	SE
0.5	α	4.9921	5.3041	5.7638	2.6159
	θ	15.0241	32.6351	15.1050	2.0881
	λ	0.5501	1.1639	0.6706	0.4307
	β	0.9130	0.6329	0.9849	0.2893
	S	0.6877		0.8935	
	H	1.8104		0.7349	
0.7	α	3.6436	1.3995	3.6678	1.1600
	θ	345.2720	1051.8409	349.3677	68.9581
	λ	2.7907	4.4575	3.6948	2.1062
	β	0.4295	0.1938	0.4419	0.1678
	S	0.3718		0.6609	
	H	1.7706		0.8068	

For α, θ, λ and β in Figures 2 and 3, the parameter distributions from Data Set I were trace plotted to correspond to the posterior density of the MCMC outputs at $\tau = 0.5$ and $\tau = 0.7$, respectively. Figures 4 and 5 also shows, for α, θ, λ and β, the marginal posterior density estimates of the parameters of the *IEL* distribution under PTII censoring based on the SS-PALT model, together with their histograms based on 10,000 chain values, where $\tau = 0.5$ and $\tau = 0.7$, respectively. Based on the SS-PALT model for data I, Figures 6 and 7 were obtained to determine whether the estimators are maximal or not for parameters of the *IEL* distribution under PTII censoring.

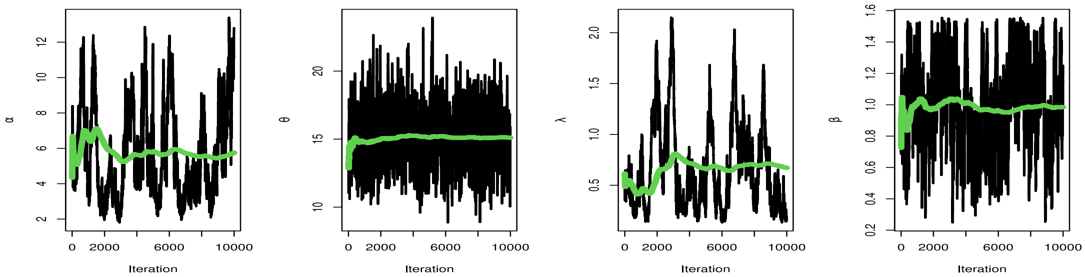

Figure 2: Trace plot and convergence line of parameters of *IEL* distribution based on SS-PALT model: Data I and $\tau = 0.5$

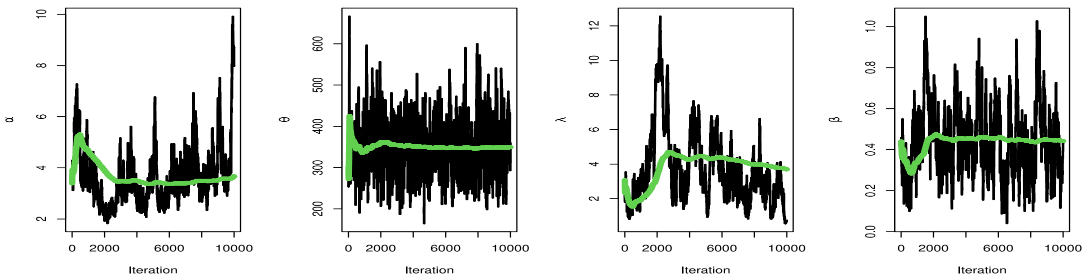

Figure 3: Trace plot and convergence line of *IEL* parameters based on SS-PALT model: Data I and $\tau = 0.7$

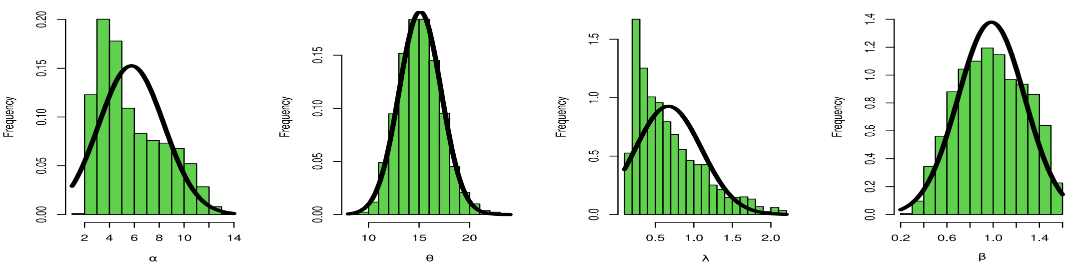

Figure 4: Posterior plot of *IEL* parameters based on SS-PALT model: Data I and $\tau = 0.5$

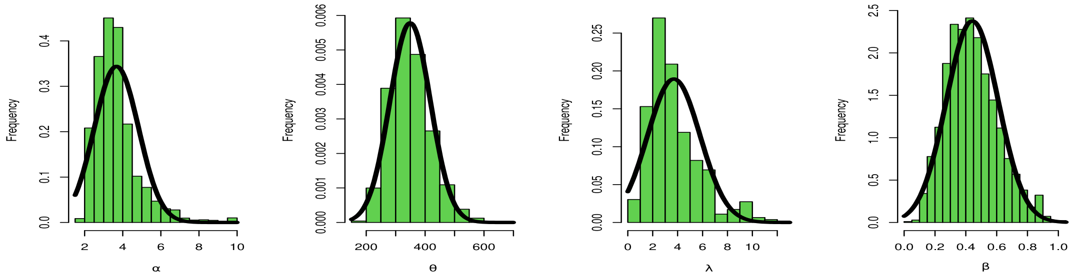

Figure 5: Posterior plot of *IEL* parameters based on SS-PALT model: Data I and $\tau = 0.7$

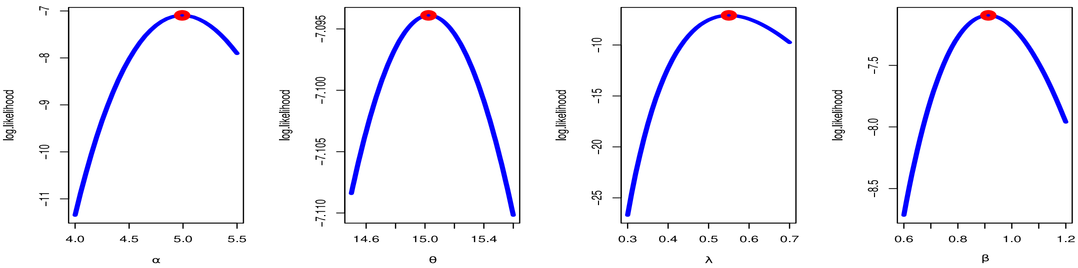

Figure 6: Profile likelihood of *IEL* parameters based on SS-PALT model: Data I and $\tau = 0.5$

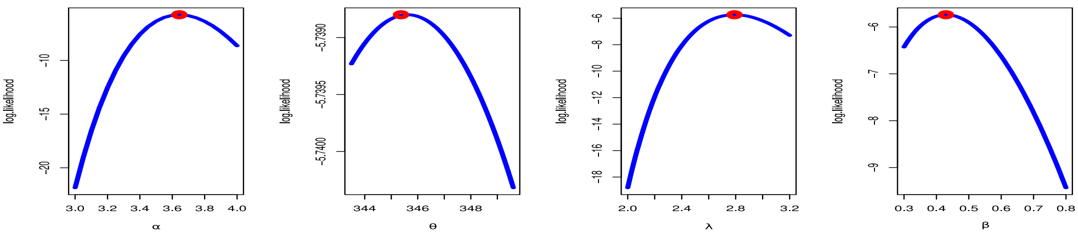

Figure 7: Profile likelihood of *IEL* parameters based on SS-PALT model: Data I and $\tau = 0.7$

Data set II: The information may be found at (https://covid19.who.int) which shows the COVID-19 drought mortality rate for Canada for 36 days, from 10 April to 15 May 2020. The information is provided and was taken from Almetwally (2021). The data can be displayed as follows: 3.1091, 3.3825, 3.1444, 3.2135, 2.4946, 3.5146, 4.9274, 3.3769, 6.8686, 3.0914, 4.9378, 3.1091, 3.2823, 3.8594, 4.0480, 4.1685, 3.6426, 3.2110, 2.8636, 3.2218, 2.9078, 3.6346, 2.7957, 4.2781, 4.2202, 1.5157, 2.6029, 3.3592, 2.8349, 3.1348, 2.5261, 1.5806, 2.7704, 2.1901, 2.4141, 1.9048.

The (K-S) statistic (with associated P-value) is obtained to demonstrate the reliability of the *IEL* model. First, *MLE*s for parameters, respectively, are 43.7007(9.8057), 39.1437(24.2393) and 0.3079(0.2692), with their SEs, and the K-S(P-value) is 0.1098(0.7782). The *IEL* distribution fits the COVID-19 data, according to this finding.

Table 5: *MLE* and Bayesian by estimates values and SE with different accretion time: Data II

		MLE		Bayesian	
τ		estimates	SE	estimates	SE
2	alpha	42.6126	342.0720	26.8167	13.1466
	theta	24.8002	23.0712	24.8810	1.4565
	lambda	0.3048	2.6435	0.6416	0.2316
	beta	1.2656	0.8443	1.4805	0.4234
	S	0.9429		0.9858	
	H	0.2101		0.0690	
2.5	alpha	41.4235	459.6048	77.3586	43.4865
	theta	10.4623	10.1949	10.4569	0.6422
	lambda	0.2623	3.1058	0.1995	0.1331
	beta	2.0576	1.1395	1.9539	0.3684
	S	0.8444		0.9728	
	H	0.5519		0.1233	

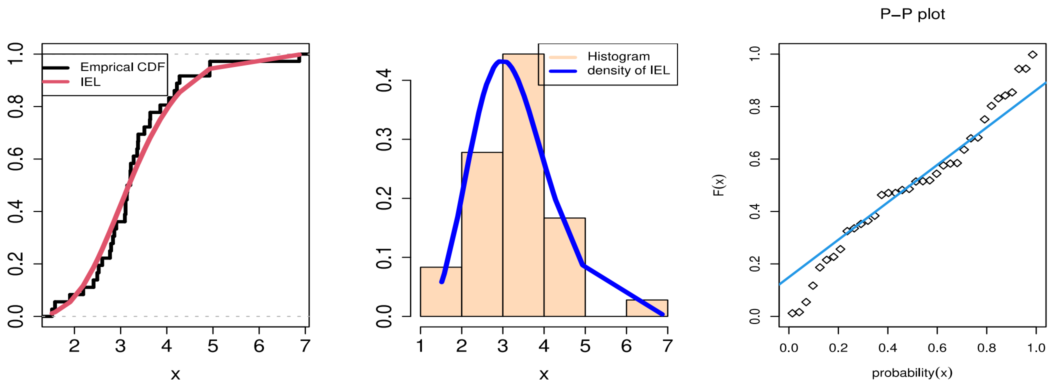

Figure 8: Estimated CDF and PDF with data also P-P plot of *IEL* distribution: Data II

Based on the SS-PALT model for data I, Figures 9 and 10 were obtained to determine whether the estimators are maximal or not for parameters of the *IEL* distribution under PTII censoring.

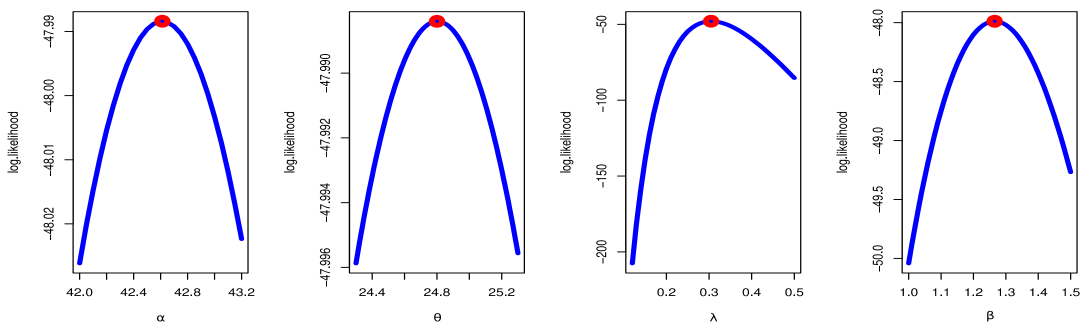

Figure 9: Profile likelihood of *IEL* parameters based on SS-PALT model: Data II and $\tau = 2$

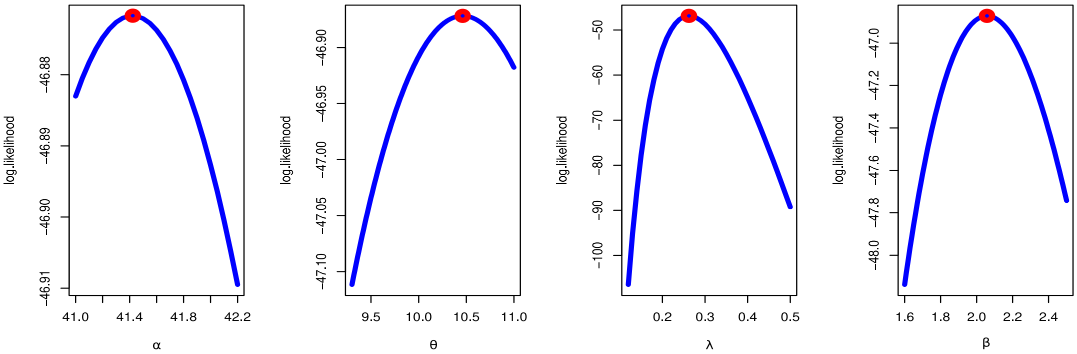

Figure 10: Profile likelihood of IEL parameters based on SS-PALT model: Data II and $\tau = 2.5$

As a hole, we drawn the following points as the conclusions:

1. As a whole, the MSE is getting smaller with the increase of n.
2. SS-PALT models under PTII censoring, the bias and MSE of the Bayesian estimates are smaller than maximum likelihood estimation under given prior information Informative.
3. As n and m are increased, the CP of the parameters is decreased.

7. Conclusions

In this paper, we provided SS-PALT models under PTII censoring with random removals when the observed data come from IEL distribution. We determined the MLEs of the obscure parameters. We inferred Bayes estimators of the parameters and the acceleration parameter using gamma informative priors based on the square error loss function. We did a recreation study to think about the execution of every one of these systems. From the simulation study, we watch that the Bayes estimates are superior to MLEs as far as MSEs. We introduced reenacted case to represent every one of the techniques for derivation examined here and additionally to bolster the conclusions drawn.

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