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On a Bivariate Bounded Distribution: Properties and Estimation

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Abstract: Recently, a new distribution with bounded support called unit Gompertz has been derived by taking an exponential transformation from the parent Gompertz distribution. This distribution has right-skewed (unimodal) and reversed-J shaped density. Moreover, the hazard rate has constant, increasing, bathtub and upside-down bathtub. In this paper, the bivariate extension for this new distribution is introduced and its properties are discussed in detail. The new bivariate model is of the Marshall–Olkin type. The estimation problem for the model's unknown parameters has been considered using MLE and Bayesian estimation; fortunately, the Bayes estimators are theoretically obtained in explicit forms. Furthermore, the Bayesian estimators are computed using MCMC method. Two real data sets have been applied to the bivariate unit Gompertz distribution. Some simulations are carried out to see the performances of the estimators. Absolutely continuous bivariate versions of this new distribution are obtained and some of its properties are also discussed.

Keywords: Gompertz distribution; Unit Gompertz distribution; Product moments; Maximum likelihood estimation; Bayesian Estimation; MCMC.

Mathematics Subject Classification: 46T30, 62F15.

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1. Introduction

If Y is a non -negative random variable following Gompertz distribution, then its pdf is given as

$$f_G(y;\ \alpha,\beta) = \alpha\beta e^{\alpha} \exp\{\beta y - \alpha e^{\beta y}\}$$

where $y > 0$, $\alpha > 0$ and $\beta > 0$.

Mazucheli [10] derived the unit Gompartz (UG) distribution from the Gompartz (G) distribution using the transformation $X = e^{-Y}$ .Where $Y \sim G(\alpha,\beta)$ then, X has an $UG(\alpha,\beta)$ with cdf and pdf

$$F_{UG}(x;\alpha,\beta) = e^{-\alpha\,(x^{-\beta}-1)},$$

$$f_{UG}(x;\alpha,\beta) = \alpha\beta x^{-\beta-1}e^{-\alpha\,(x^{-\beta}-1)}.$$

respectively, where $0 < x < 1$, $\alpha,\beta > 0$.

The hazard function for $UG(\alpha,\beta)$is given as

$$h_{UG}(x;\alpha,\beta) = \frac{\alpha\beta x^{-\beta-1}e^{-\alpha\,(x^{-\beta}-1)}}{1 - e^{-\alpha\,(x^{-\beta}-1)}}.$$

It is clear that, $\lim_{x\to 1} h_{UG}(x;\alpha,\beta) = \alpha\beta$. So, Mazucheli [10] deduced that the hazard function is monotonically increasing for $\alpha > 0$ and $\beta \geq 1$, and it has bathtub shapes when $\alpha \leq 0.5$. So, he concluded that the UG distribution has one of the advantages over the Gompertz distribution that the latter cannot model phenomena showing an upside-down bathtub hazard function. Mazucheli [10] introduced this distribution, having two shape parameters, as alternative to beta and Kumaraswamy distributions for more details see his paper.

The study of dependent variables is influential in many practical problems. In economic studies, for example, the relationship between years of education and personal income, personal income and expenditure, inflation, and unemployment. In biological studies, the age at death of the parent and child in a genetic study; blindness in the left and right eye; the relationship between a patient's blood pressure and body weight and the failure time of the left and right kidney. In engineering studies, the lifetime of a twin-engine plane is examined, as are warranty policies based on failure time and warranty servicing time. In addition, various applications such as the shock model, competing risks model, stress model, maintenance model, and longevity model are available.

Because it considers all different cases of the random variables, the bivariate Marshall-Olkin family of distributions is very important for understanding and analyzing the failure time of two variables interacting together. There are several papers dealing with bivariate Marshall-Olkin models. Sarhan and Balakrishnan [18] introduced a bivariate distribution based on exponential and generalized exponential distributions, now known as Sarhan-Balakrishnan bivariate (SBBV) distribution. They derived several interesting properties of their distribution but the marginal distributions of SBBV distribution are not in known forms. Kundu and Gupta [5] expanded on Marshall-Olkin idea by introducing the bivariate generalized exponential (BVGE) distribution, resulting in marginal distributions that are generalized exponential distributions. They presented several properties of this distribution and discussed maximum likelihood estimation of unknown parameters. Sarhan [19] proposed a new bivariate distribution known as the bivariate generalized Rayleigh (BVGR). The new distribution has generalized Rayleigh marginal distributions. The hazard rate functions of the BVGR's marginals can be increasing, decreasing, or bathtub shaped, giving the BVGR distribution greater applicability than other distributions. Sarhan [19] investigated several interesting properties of this distribution and used the maximum likelihood and Bayes methods to estimate the unknown parameters. Many authors, including El-Gohary et al. [3], Kundu and Gupta [7] and others, discussed the Marshall-Olkin idea for various distributions.

Barreto-Souza and Lemonte [1] discussed the bounded support bivariate distribution. They proposed the bivariate Kumaraswamy (BVK) distribution, whose marginals are Kumaraswamy distributions, based on Marshall and Olkin's [9] idea. The goal of this paper is to present another bounded support bivariate distribution. The new distribution is introduced as a bivariate extension of the UG distribution such that its marginals follow univariate UG distributions. The proposed bivariate models are shown to have a singular part in their structure.

The rest of the paper is structured as follows: Section 2 introduces the BUG distribution and provides representations for the cumulative distribution function (cdf) and probability density function (pdf). Section 3 discusses some of the basic properties of this model. In Section 4, point and interval estimation for BUG distribution are provided. Section 5 presents two empirical applications for illustration purposes. Section 6 discusses a simulation study. Section 7 introduces an absolutely continuous BUG distribution. Section 8 provides a conclusion to the paper.

2. Model Description

The following is the definition of the Marshall-Olkin bivariate unit Gompertz distribution: Let U_1, U_2 and U_3 be three independent random variables such that $U_i \sim UG(\alpha_i,\beta)$ $i = 1,2,3$. Define $X_i = Max(U_i, U_3)$ $i = 1,2$. Then, the bivariate vector (X_1, X_2) has BUG distribution with parameters $(\alpha_1, \alpha_2, \alpha_3, \beta)$, denoted by $BUG(\alpha_1, \alpha_2, \alpha_3, \beta)$. Then, the joint cdf of (X_1, X_2)takes the form:

$$F_{BUG}(x_1, x_2) = F_{UG}(x_1; \alpha_1) F_{UG}(x_2; \alpha_2) F_{UG}(x_3; \alpha_3)$$

$$F_{BUG}(x_1, x_2) = Exp\{-\alpha_1(x_1^{-\beta} - 1) - \alpha_2(x_2^{-\beta} - 1) - \alpha_3(x_3^{-\beta} - 1)\}$$

where $x_3 = \min(x_1, x_2)$.

Proposition 1: If $(X_1, X_2) \sim BUG(\alpha_1, \alpha_2, \alpha_3, \beta)$. Then, the joint cdf of (X_1, X_2)can be written as

$$F_{BUG}(x_1, x_2) = \begin{cases} F_{UG}(x_1; \alpha_{13}) F_{UG}(x_2; \alpha_2), & x_1 < x_2 \\ F_{UG}(x_1; \alpha_1) F_{UG}(x_2; \alpha_{23}), & x_1 > x_2 \\ F_{UG}(x; \alpha_{123}), & x_1 = x_2 \end{cases}$$

where $\alpha_{13} = \alpha_1 + \alpha_3$, $\alpha_{23} = \alpha_2 + \alpha_3$ and $\alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3$.

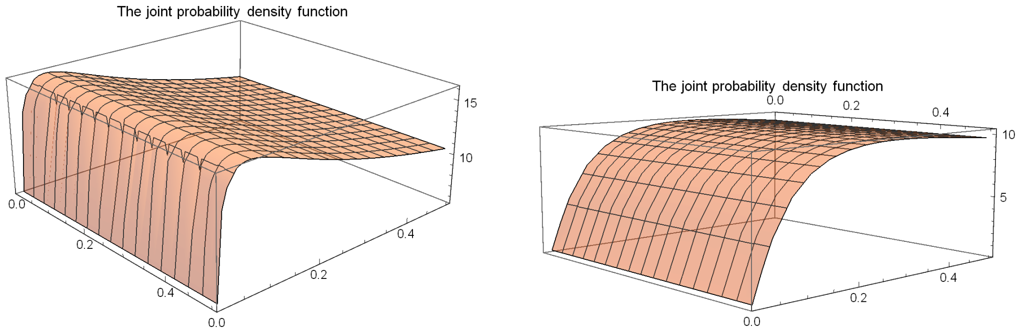
Proposition 2: If $(X_1, X_2) \sim BUG(\alpha_1, \alpha_2, \alpha_3, \beta)$ Then, the joint pdf of (X_1, X_2) is given as

$$f_{BUG}(x_1, x_2) = \begin{cases} f_{UG}(x_1; \alpha_{13}) f_{UG}(x_2; \alpha_2), & x_1 < x_2 \\ f_{UG}(x_1; \alpha_1) f_{UG}(x_2; \alpha_{23}), & x_1 > x_2 \\ \frac{\alpha_3}{\alpha_{123}} f_{UG}(x; \alpha_{123}), & x_1 = x_2 = x \end{cases}$$

The joint pdf can take various shapes depending on the parameter values. Different shapes of the joint pdf for different sets of parameters values are provided in Figure 1.

3. Basic Properties

This section discusses some BUG distribution properties such as reliability functions, statistical measures, marginal and conditional densities, and product moments.



(a) Set (1): $(\alpha_1, \alpha_2, \alpha_3, \beta) = (0.5,\ 3.5,\ 0.5,\ 0.\ 2)$ **(b) Set (2):** $(\alpha_1, \alpha_2, \alpha_3, \beta) = (0.5,\ 3,\ 0.3,\ 0.3)$

Figure 1. The joint probability density function of the BUG distribution

3.1. Reliability and Reversed Hazard Functions

The joint survival function of BUG distribution is

$$S_{BUG}(x_1, x_2) = \begin{cases} F_{UG}(x_1; \alpha_{13})\left[F_{UG}(x_2; \alpha_2) - 1\right] + \overline{F}_{UG}(x_2; \alpha_{23}), & x_1 < x_2 \\ F_{UG}(x_2; \alpha_{23})\left[F_{UG}(x_1; \alpha_1) - 1\right] + \overline{F}_{UG}(x_1; \alpha_{13}), & x_1 > x_2 \\ \overline{F}_{UG}(x_2; \alpha_{123}), & x_1 = x_2 \end{cases}$$

The reversed hazard function of BUG distribution is

$$r(x_1, x_2) = \begin{cases} r_{UG}(x_1; \alpha_{13})\, r_{UG}(x_2; \alpha_2), & x_1 < x_2 \\ r_{UG}(x_1; \alpha_1)\, r_{UG}(x_2; \alpha_{23}), & x_1 > x_2 \\ r_{UG}(x; \alpha_3), & x_1 = x_2 = x \end{cases}$$

where $r_{UG}(x; \alpha) = \beta\alpha x^{-(\beta+1)}$.

3.2. Factorization Property

The BUG distribution has an absolute continuous and a singular parts. The BUG's joint cdf can be factored into absolutely continuous and singular parts as shown below.

$$F_{UG}(x_1, x_2) = \frac{\alpha_{12}}{\alpha_{123}} F_a(x_1, x_2) + \frac{\alpha_3}{\alpha_{123}} F_s(x_3),$$

where $x_3 = \min(x_1, x_2)$,
$F_s(x_3) = F_{UG}(x; \alpha_{123})$ and

$$F_a(x_1, x_2) = \frac{\alpha_{123}}{\alpha_{12}} F_{UG}(x_1; \alpha_1)\, F_{UG}(x_2; \alpha_2)\, F_{UG}(x_3; \alpha_3) - \frac{\alpha_3}{\alpha_{12}} F_{UG}(x; \alpha_{123}).$$

One can note that:$F_s(.,.)$and $F_a(.,.)$ are the singular and the absolutely continuous part respectively.

As a result, the BUG's pdf can be factored into absolutely continuous and singular parts, as shown below.

$$f_{BUG}(x_1, x_2) = \frac{\alpha_{12}}{\alpha_{123}} f_a(x_1, x_2) + \frac{\alpha_3}{\alpha_{123}} f_s(x_3)$$

where

$$f_a(x_1, x_2) = \frac{\alpha_{123}}{\alpha_{12}} \begin{cases} f_{UG}(x_1; \alpha_{13}) f_{UG}(x_2; \alpha_2), & x_1 < x_2 \\ f_{UG}(x_1; \alpha_1) f_{UG}(x_2; \alpha_{23}), & x_1 < x_2 \end{cases}$$

and $f_s(x_3) = f_{UG}(x; \alpha_{123})$.

Clearly, here $f_a(x_1, x_2)$and $f_s(x_3)$ are the absolutely continuous and singular parts respectively.

3.3. The Mode and Median

The absolute continuous BUG distribution's median is given below.

$$\left(\frac{\alpha_{123}}{\ln 2 + \alpha_{123}}\right)^{1/\beta}$$

The absolutely continuous BUG distribution mode is as follows:
$\{\left(\frac{\beta\alpha_{13}}{\beta+1}\right)^{1/\beta}, \left(\frac{\beta\alpha_{2}}{\beta+1}\right)^{1/\beta}\}$ and $\{\left(\frac{\beta\alpha_{1}}{\beta+1}\right)^{1/\beta}, \left(\frac{\beta\alpha_{23}}{\beta+1}\right)^{1/\beta}\}$.

3.4. Marginal and Conditional Densities

Proposition 3: If $(X_1, X_2) \sim BUG(\alpha_1, \alpha_2, \alpha_3, \beta)$. Then,

1. $X_i \sim UG(\alpha_{i3})$, such that $\alpha_{i3} = \alpha_i + \alpha_3$ and $i = 1, 2$.
2. max $(X_1, X_2) \sim UG(\alpha_{123})$.
3. The conditional density of X_i given $X_j = x_j$, $i \neq j$ is as follows:

$$f_{i/j}\left(x_i/x_j\right) = \begin{cases} f^{(1)}_{i/j}\left(x_i/x_j\right), & x_i < x_j \\ f^{(2)}_{i/j}\left(x_i/x_j\right), & x_i > x_j \\ f^{(3)}_{i}\left(x_i\right), & x_i = x_j \end{cases}$$

where

$$f^{(1)}_{i/j}\left(x_i/x_j\right) = \beta\frac{\alpha_{13}}{\alpha_{123}} x_i^{-(\beta+1)} \exp\left\{\alpha_3\left(x_j^{-\beta} - 1\right) - \alpha_{13}\left(x_i^{-\beta} - 1\right)\right\}$$

$$f^{(2)}_{i/j}\left(x_i/x_j\right) = \beta\alpha_1 x_i^{-(\beta+1)} \exp\left\{-\alpha_1\left(x_i^{-\beta} - 1\right)\right\}$$

$$f^{(3)}_{i}\left(x_i\right) = \frac{\alpha_3}{\alpha_{23}} x_i^{-(\beta+1)} x_j^{(\beta+1)} \exp\left\{\alpha_{23}\left(x_j^{-\beta} - 1\right) - \alpha_{123}\left(x_i^{-\beta} - 1\right)\right\}.$$

Shapes of the pdf of X_i for different values of α_{i3} and β are provided in Figure 2. Figure 3 shows some plots of the conditional pdf's of X_1 given $X_2 = x_2$ for selected values of x_2 ($x_2 = 0.2, 0.5, 0.8$) and different values of parameters.

3.5. Product Moments

According to proposition (3) the marginal distributions of the vector (X_1, X_2) are UG distributions, then the moments of X_1 and X_2 can be obtained directly from the following marginals:
$E\left(X_1^r\right) = \alpha_{13}^{\frac{r}{\beta}} e^{\alpha_{13}} \Gamma\left(1 - \frac{r}{\beta}, \alpha_{13}\right)$ and $E\left(X_2^r\right) = \alpha_{23}^{\frac{r}{\beta}} e^{\alpha_{23}} \Gamma\left(1 - \frac{r}{\beta}, \alpha_{23}\right)$.

Where $\Gamma(.,.)$ is the upper incomplete gamma function and $\frac{r}{\beta} < 1$. Now the product moments of UG distribution will be presented.

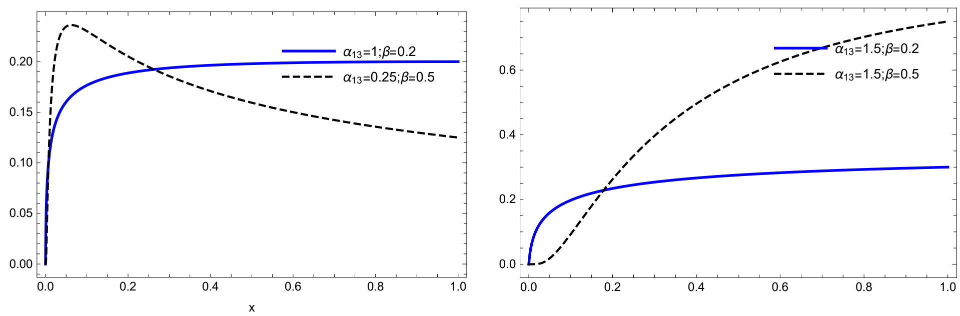


Figure 2. The probability density function of the marginal distribution of X1

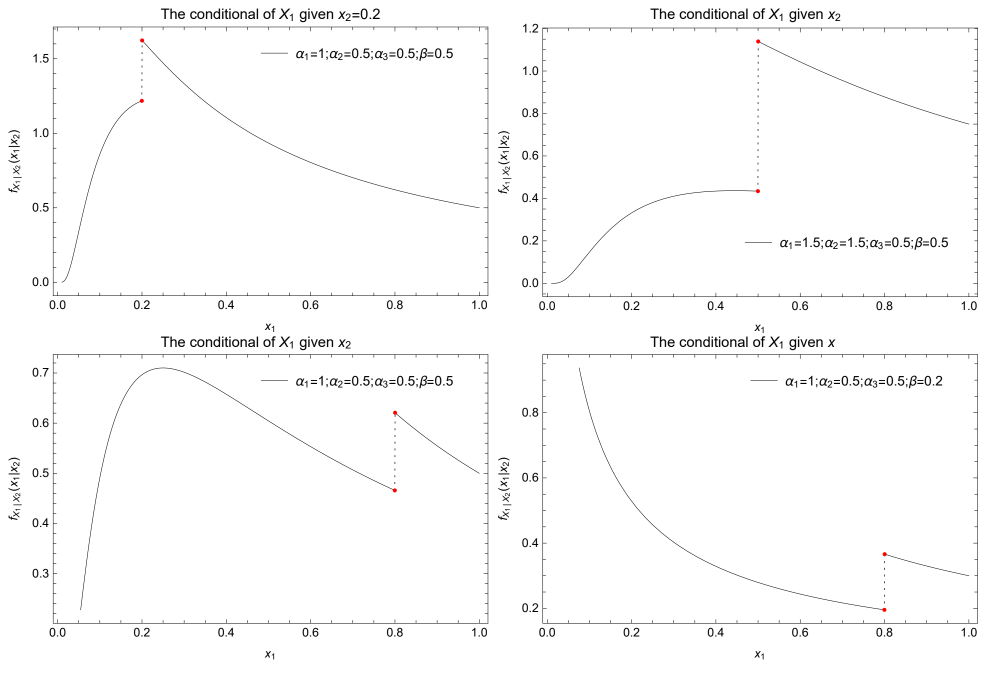


Figure 3. The conditional probability density function of X_1 given $X_2 = x_2$ at different sets of the parameters

Proposition 4. The r^{th} and s^{th} joint moments of the X_1 and X_2, denoted by $\mu'_{r,s}$ is given by

$$E\left(X_1{}^r X_2{}^s\right) = e^{\alpha_{123}} \alpha_{13}^{\frac{r}{\beta}} \alpha_2^{\frac{s}{\beta}} \Gamma\left(1-\frac{r}{\beta}\right) \Gamma\left(1-\frac{s}{\beta},\alpha_2\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_{13}^{k+1} \Gamma\left(2-\frac{r+s}{\beta}+k,\alpha_2\right)}{k!\left(1-\frac{r}{\beta}+k\right) \alpha_2^{k-\frac{r+s}{\beta}}}$$

$$+e^{\alpha_{123}} \alpha_{23}^{\frac{s}{\beta}} \alpha_1^{\frac{r}{\beta}} \Gamma\left(1-\frac{s}{\beta}\right) \Gamma\left(1-\frac{r}{\beta},\alpha_1\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_{23}^{k+1} \Gamma\left(2-\frac{r+s}{\beta}+k,\alpha_1\right)}{k!\left(1-\frac{s}{\beta}+k\right) \alpha_1^{k-\frac{r+s}{\beta}}}$$

$$+e^{\alpha_{123}} \alpha_3 \, \alpha_{123}^{\frac{r+s}{\beta}-1} \, \Gamma\left(1-\frac{r+s}{\beta},\alpha_{123}\right),$$

where $\Gamma(.)$ is the complete gamma function and the product moments are exist for $\frac{r+s}{\beta} < 1$.

4. Estimation of BUG Distribution

The estimation of the unknown parameters for the BUG distribution using maximum likelihood and Bayesian estimation is considered in the following two subsections.

4.1. Maximum Likelihood Estimation

Suppose $\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$ be a random sample from $BUG(\alpha_1, \alpha_2, \alpha_3, \beta)$. distribution. Consider the following notations:

$$I_1 = \{i; x_{1i} < x_{2i}\}, \quad I_2 = \{i; x_{1i} > x_{2i}\}, \quad I_3 = \{x_{1i} = x_{2i} = x_i\}, \quad I = I_1 \bigcup I_2 \bigcup I_3,$$

$|I_1| = n_1, \quad |I_2| = n_2, \quad |I_3| = n_3, \quad$ and $n_1 + n_2 + n_3 = n$.

The log-likelihood function of the sample of size n takes the form

$$L(\theta) = (2n_1 + 2n_2 + n_3)\log\beta + n_1\log\alpha_{13} + n_1\log\alpha_2 + n_2\log\alpha_{23} + n_3\log\alpha_3$$

$$-\alpha_{13}\sum_{i=1}^{n_1}(x_{1i}^{-\beta}-1) - \alpha_2\sum_{i=1}^{n_1}(x_{2i}^{-\beta}-1) - \alpha_1\sum_{i=1}^{n_2}(x_{1i}^{-\beta}-1)$$

$$-\alpha_{23}\sum_{i=1}^{n_2}(x_{2i}^{-\beta}-1) - \alpha_{123}\sum_{i=1}^{n_3}(x_i^{-\beta}-1) - (\beta+1)\,\gamma(x_{1i}, x_{2i}, x_i).$$

where $\gamma(x_{1i}, x_{2i}, x_i) = \sum_{i=1}^{n_1}\log x_{1i} + \log x_{2i} + \sum_{i=1}^{n_2}\log x_{1i} + \log x_{2i} + \sum_{i=1}^{n_3}\log x_i$.

The likelihood equations are

$$\frac{n_1}{\widehat{\alpha_{13}}} + \frac{n_2}{\widehat{\alpha_1}} - \sum_{I_1 \cup I_2} A\left(x_{1i};\widehat{\beta}\right) - \sum_{I_3} A\left(x_i;\widehat{\beta}\right) = 0,$$

$$\frac{n_1}{\widehat{\alpha_2}} + \frac{n_2}{\widehat{\alpha_{23}}} - \sum_{I_1 \cup I_2} A\left(x_{2i};\widehat{\beta}\right) - \sum_{I_3} A\left(x_i;\widehat{\beta}\right) = 0,$$

$$\frac{n_1}{\widehat{\alpha_{13}}} + \frac{n_2}{\widehat{\alpha_{23}}} + \frac{n_3}{\widehat{\alpha_3}} - \sum_{I_1} A\left(x_{1i};\widehat{\beta}\right) - \sum_{I_2} A\left(x_{2i};\widehat{\beta}\right) - \sum_{I_3} A\left(x_i;\widehat{\beta}\right) = 0,$$

and

$$\frac{2n_1+2n_2+n_3}{\widehat{\beta}} - \gamma(x_{1i}, x_{2i}, x_i) + \widehat{\alpha}_{13}\sum_{I_1} B\left(x_{1i};\widehat{\beta}\right) + \widehat{\alpha}_2 \sum_{I_1} B\left(x_{2i};\widehat{\beta}\right) + \widehat{\alpha}_1 \sum_{I_2} B\left(x_{1i};\widehat{\beta}\right)$$

$$+\widehat{\alpha}_{23}\sum_{I_2} B\left(x_{2i};\widehat{\beta}\right) + \widehat{\alpha}_{123}\sum_{I_3} B\left(x_i;\widehat{\beta}\right) = 0,$$

where $A(x;\beta) = (x^{-\beta} - 1)$ and $B(x;\beta) = x^{-\beta}\log x$.

The numerical solutions for these equations are considered to obtain $\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\alpha}_3$ and β as be shown in section 6.

The asymptotic variance –covariance (var-cov) matrix can be written as follows

$$I(\underline{\theta})^{-1} = \left.\begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{bmatrix}^{-1}\right|_{\Theta=\hat{\Theta}}$$

where

$$I_{11} = -\frac{\partial^2 \ln L}{\partial \alpha_1^2}|_{\Theta=\hat{\Theta}} = \frac{n_1}{\hat{\alpha}_{13}^2} + \frac{n_2}{\hat{\alpha}_1^2}, I_{22} = -\frac{\partial^2 \ln L}{\partial \alpha_1^2}|_{\Theta=\hat{\Theta}} = \frac{n_1}{\hat{\alpha}_2^2} + \frac{n_2}{\hat{\alpha}_{23}^2}, I_{13} = -\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \alpha_3}|_{\Theta=\hat{\Theta}} = \frac{n_1}{\hat{\alpha}_{13}^2},$$

$$I_{23} = -\frac{\partial^2 \ln L}{\partial \alpha_2 \partial \alpha_3}|_{\Theta=\hat{\Theta}} = \frac{n_2}{\hat{\alpha}_{23}^2}, I_{33} = -\frac{\partial^2 \ln L}{\partial \alpha_3^2}|_{\Theta=\hat{\Theta}} = \frac{n_1}{\hat{\alpha}_{13}^2} + \frac{n_2}{\hat{\alpha}_{23}^2} + \frac{n_3}{\hat{\alpha}_3^2},$$

$$I_{14} = -\left.\frac{\partial^2 \ln L}{\partial \alpha_1 \partial \beta}\right|_{\Theta=\hat{\Theta}} = -\sum_{I_1} B(x_{1i}, \hat{\beta}) - \sum_{I_2} B(x_{1i}, \hat{\beta}) - \sum_{I_3} B(x_i, \hat{\beta}),$$

$$I_{24} = -\left.\frac{\partial^2 \ln L}{\partial \alpha_2 \partial \beta}\right|_{\Theta=\hat{\Theta}} = -\sum_{I_1} B(x_{2i}, \hat{\beta}) - \sum_{I_2} B(x_{2i}, \hat{\beta}) - \sum_{I_3} B(x_i, \hat{\beta}),$$

$$I_{34} = -\left.\frac{\partial^2 \ln L}{\partial \alpha_3 \partial \beta}\right|_{\Theta=\hat{\Theta}} = -\sum_{I_1} B(x_{1i}, \hat{\beta}) - \sum_{I_2} B(x_{2i}, \hat{\beta}) - \sum_{I_3} B(x_i, \hat{\beta}),$$

and

$$I_{44} = -\left.\frac{\partial^2 \ln L}{\partial \beta^2}\right|_{\Theta=\hat{\Theta}} = \frac{(2n_1+2n_2+n_3)}{\hat{\beta}^2} + \hat{\alpha}_{13}\sum_{I_1} C(x_{1i}, \hat{\beta}) + \hat{\alpha}_2 \sum_{I_1} C(x_{2i}, \hat{\beta})$$
$$+ \hat{\alpha}_1 \sum_{I_2} C(x_{1i}, \hat{\beta}) + \hat{\alpha}_{23}\sum_{I_2} C(x_{2i}, \hat{\beta}) + \hat{\alpha}_{123}\sum_{I_3} C(x_i, \hat{\beta}).$$

where $C(x;\beta) = x^{-\beta}[\log x\,]^2$.

Now, The asymptotic normality results will be considered to obtain the asymptotic confidence intervals of α_1,α_2,$\alpha_3$and β It can be stated as follows

$\sqrt{n}\ [(\hat{\alpha}_1 - \alpha_1), (\hat{\alpha}_2 - \alpha_2), (\hat{\alpha}_3 - \alpha_3), (\hat{\beta} - \beta)] \to N_4(0\ ,\ I(\Theta)^{-1})$ as $n \to \infty$

Where $I^{-1}(\underline{\theta})$ is the variance-covariance matrix, $\hat{\Theta} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \beta)$. and $\Theta = (\alpha_1, \alpha_2, \alpha_3, \beta)$. $I^{-1}(\Theta)$ is estimated by $I^{-1}(\Theta)$;

The asymptotic variance-covariance matrix that defined above and this can be used to obtain the asymptotic confidence intervals of $\alpha_1, \alpha_2, \alpha_3$ andβ.

4.2. Bayesian Estimation

The explicit Bayes estimators under the squared error loss function are obtained. When the shape parameter β is known, we assume the same conjugate prior on α_1, α_2 and α_3 as considered by Kundu and Gupta [6] as follows:

Assume that α_1, α_2 and α_3 are independent and distributed as gamma, we have

$$\pi_i(\alpha_i) = \frac{b^{a_i}}{\Gamma(a_i)} \alpha_i^{a_i-1} e^{-b_i\alpha_i}\ ,\ i = 1, 2, 3\ , \alpha_i > 0$$

The joint prior density of α_1, α_2 and α_3 is given as follows

$$\pi_0(\alpha_1, \alpha_2, \alpha_3) = \prod_{i=1}^{3} \frac{b^{a_i}}{\Gamma(a_i)} \alpha_i^{a_i-1} e^{-b_i\ \alpha_i}$$

Now, suppose $\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$is a random sample from $BUG(\alpha_1, \alpha_2, \alpha_3, \beta)$ distribution. Consider the following notations:

$D = \{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$, $\Theta = (\alpha_1, \alpha_2, \alpha_3)$ and $n = n_1 + n_2 + n_3$.

Then the Likelihood function can be written as

$$L(D|\Theta) = Exp(\log L(D|\Theta))$$

$$L(D|\Theta) = \beta^{2n_1+2n_2+n_3}\alpha_{13}^{n_1}\ \alpha_{23}^{n_2}\ \ \alpha_2^{n_1}\alpha_1^{n_2}\alpha_3^{n_3}.Exp\{-\alpha_{13}Z_1(\beta) - \alpha_2 Z_2(\beta) - \alpha_1 Z_3(\beta)$$
$$-\alpha_{23}Z_4(\beta) - \alpha_{123}Z_5(\beta) - (\beta+1)Z\}.$$

$$L(D|\Theta) \propto \sum_{j=1}^{n_1}\sum_{k=1}^{n_2} \binom{n_1}{j}\binom{n_2}{k} \alpha_1^{j+n_2}\ \ \alpha_2^{k+n_1}\alpha_3^{n-j-k} Exp(-\alpha_1 T_1 - \alpha_2 T_2 - \alpha_3 T_3)\ .$$

where

$$Z_1(\beta) = \sum_{i=1}^{n_1}(x_{1i}^{-\beta} - 1), Z_2(\beta) = \sum_{i=1}^{n_1}(x_{2i}^{-\beta} - 1), Z_3(\beta) = \sum_{i=1}^{n_2}(x_{1i}^{-\beta} - 1),$$

$$Z_4(\beta) = \sum_{i=1}^{n_4}(x_{2i}^{-\beta} - 1), Z_5(\beta) = \sum_{i=1}^{n_3}(x_i^{-\beta} - 1),\ T_1 = Z_1(\beta) + Z_3(\beta) + Z_5(\beta)$$
$$, T_2 = Z_2(\beta) + Z_4(\beta) + Z_5(\beta)\ , T_3 = Z_1(\beta) + Z_4(\beta) + Z_5(\beta)\ ,$$

and $Z = \sum_{i=1}^{n_1} \log x_{1i} + \log x_{2i} + \sum_{i=1}^{n_2} \log x_{1i} + \log x_{2i} + \sum_{i=1}^{n_3} \log x_i$,

Since $f(D, \Theta) = \pi_0(\Theta) L(D|\Theta)$ and $f(D) = \int f(D|\Theta)\, d\Theta = \int \pi_0(\Theta) L(D|\Theta) d\Theta$

Hence the joint posterior density function of $\Theta = (\alpha_1, \alpha_2, \alpha_3)$ given the data D, denoted by $\pi_1(\Theta|D)$ can be written as

$$\pi_1(\Theta|D) = \frac{f(D, \Theta)}{f(D)}$$

$$\pi_1(\Theta|D) \propto \sum_{j=1}^{n_1}\sum_{k=1}^{n_2} A_{jk}\, Gamma\left[\alpha_1;\ a_{1j},\ b_1+T_1\right] Gamma\ [\alpha_2;\ a_{2k},\ b_2+T_2]$$

$$\times Gamma\ [\alpha_3;\ a_{3jk},\ b_3+T_3],$$

where $A_{ij} = \frac{C_{ij}}{\sum_{j=1}^{n_1}\sum_{k=1}^{n_2} C_{jk}}$, and $C_{jk} = \binom{n_1}{j}\binom{n_2}{k} \cdot \frac{\Gamma(a_{1j})}{[b_1+T_1]^{a_{1j}}} \cdot \frac{\Gamma(a_{2jk})}{[b_2+T_2]^{a_{2k}}} \cdot \frac{\Gamma(a_{3jk})}{[b_3+T_3]^{a_{3jk}}}$.
$a_{1j} = a_1 + j + n_2$, $a_{2k} = a_2 + k + n_1$ and $a_{3jk} = a_3 + n + k + j$.

Therefore, under the assumption of independence of $\alpha_1, \alpha_2$and α_3 and β is assumed to be known. It is possible to get the Bayes estimators of $\alpha_1, \alpha_2$and α_3 explicitly under the square error loss function as follows:

$$\widetilde{\alpha}_1 = \frac{1}{b_1+T_1}\sum_{j=1}^{n_1}\sum_{k=1}^{n_2} A_{jk}\, a_{1j},$$

$$\widetilde{\alpha}_2 = \frac{1}{b_2+T_2}\sum_{j=1}^{n_1}\sum_{k=1}^{n_2} A_{jk}\, a_{2k},$$

and $\widetilde{\alpha}_3 = \frac{1}{b_3+T_3}\sum_{j=1}^{n_1}\sum_{k=1}^{n_2} A_{jk}\, a_{3jk}$.

But in the case of β is unknown, we can't have explicit form of the Bayes estimators so we can use one of approximation methods to obtain the Bayes estimates of the four unknown parameters for the BUG distribution. One of these methods is called Markov Chain Monte Carlo (MCMC) that generates random draws from the joint posterior distribution, see Gelman et al. [4]. The popular MCMC method is Metropolis–Hasting algorithm which can be written as follows:

1. Set the number of random draws to be generated, say m.
2. Select an initial value of $\boldsymbol{\theta}$, say $\boldsymbol{\theta}^{(0)}$.
3. For i = 1, 2, ..., m, repeat the steps below:

(i) Generate $\boldsymbol{\theta}^*$ from multivariate normal with mean $\boldsymbol{\theta}^{(i-1)}$.and variance-covariance Σ.
(ii) Calculate the ratio $\kappa = min\{1, \frac{\pi(\boldsymbol{\theta}*|data)}{\pi(\boldsymbol{\theta}^{(i-1)}|data)}\}$.
(iii) Create a random value u on (0, 1) using a uniform distribution.
(iv) If κ ≥ u put $\boldsymbol{\theta}^{(i)} = \boldsymbol{\theta}^*$, otherwise put $\boldsymbol{\theta}^{(i)} = \boldsymbol{\theta}^{(i-1)}$.

After removing the first m_0 burn-in draws and using the remaining $m - m_0$ as the chosen draws from the joint posterior distribution, the Bayes estimate of θ_j is

$$\widehat{\theta}_j = \sum_{i=m_0+1}^{m} \frac{\theta_j^{(i)}}{m-m_0}, \quad j = 1,\ 2,\ 3,\ 4.$$

Moreover, for $0 < v < 1$, the lower and upper bounds of the $100(1-v)\%$ Bayesian probability interval of θ_j can be obtained by taking the (v /2)100th and $(1 - v/2)100th$ percentiles of the sequence of the $m - m_0$ draws.

5. Data Analysis

In this section, The BUG distribution is applied to two real data sets to see how the BUG distribution works in practice.

5.1. *Data Set 1: UEFA Champion's League data*

This data set was obtained from Meintanis [11] and is shown in Table 1. It is explained as follows: the data represents the football (soccer) data where at least one goal scored by the home team and at least one goal scored directly from a penalty kick, foul kick or any other direct kick (all of them together will be called as kick goal) by any team have been considered. Here X_1 represents the time in minutes of the first kick goal scored by any team and X_2 represents the first goal of any type scored by the home team. In this case all possibilities are contained, for example $X_1 < X_2$ or $X_1 > X_2$ or $X_1 = X_2 = X$. Many authors have reanalyzed this data such as Kundu and Gupta [5], Barreto-Souza and Lemonte [1], Sarhan [19], Muhammed [12, 13, 15, 14], Mandouh [8] and others Here, these data will be applied to the BUG distribution. All the data points were divided by 90 (once a professional soccer match totals 90 min) to ensure that the data belong to the interval (0, 1), that is, we model the proportion of time that any team and the home team scored the first kick goal.

The Kolmogorov-Smirnov distances between the fitted marginals and the empirical distribution functions for X_1 , X_2 and $\max(X_1, X_2)$ with UG (1.133, 0.192), UG (1.897, 0.192) and UG (2.392, 0.192) are (0.508), (0.18), and (0.282), respectively. That gives an indication that the BUG model may be used to analyze this data set. Moreover, the Akaike information criterion (AIC), Bayesian information criterion (BIC), the consistent Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) are calculated for BUG model to be respectively as follows: (34.084), (39.688), (35.684), (35.877).

To test whether BUG distribution fits the data or not, one can use the two-dimensional Kolomogorov-Sminrov test of goodness of fit (Peacock [17]. Using the computational environmental R peacock package, we obtain the value of test statistic as 0.4054 with p value 0.1. Based on this p value, we cannot reject the null hypothesis that the data came from the BUG distribution at 0.05 level of significance

The MLE, the length of 95% confidence intervals (CIL) and the variance covariance matrix for $\alpha_1, \alpha_2, \alpha_3$ and β is calculated for this data set using BUG model as shown in Table 2.

5.2. *Data Set 2: Cholesterol levels*

This data set contains cholesterol levels at 5 and 25 weeks after treatment in 30 patients. Before analyzing this data, All the data points are divided by 400 to guarantee that the data belong to the interval (0, 1). This data set was used by Muhammed [14, 16] and it is represented in Table 3. Again, in this case all possibilities are exist.

The Kolmogorov-Smirnov distances between the fitted marginals and the empirical distribution functions for X_1 , X_2 and $\max(X_1, X_2)$ with UG (1.26, 1.051), UG (1.885,1.051) and UG (2.535, 1.051) are (0.791), (1.644), and (2.727), respectively. That gives an indication that the BUG model may be used to analyze this data set. In addition, the two-dimensional Kolomogorov-Sminrov test of goodness of fit (Peacock [17] with test statistics 0.4 and p value 0.2213, we cannot reject the null hypothesis that the data came from the BUG distribution at 0.05 level of significance. Moreover, AIC, BIC, CAIC and HQIC are calculated for BUG model to be respectively as follows: (-75.938), (-70.333), (-74.338), (-74.145). Moreover, The MLE, the length of 95% confidence intervals (CIL) and the variance covariance matrix for $\alpha_1, \alpha_2, \alpha_3$ and β is calculated for this data set using BUG model as shown in Table 4.

The Bayes estimation of the four unknown parameters for the two data sets based on gamma priors are considered. The posterior descriptive summaries of interest, such as the posterior mean, median, standard deviation, and 95% Bayesian credible ranges, are provided in Table 5.

6. Simulation Study

In this section, the results of a Monte Carlo simulation study were introduced for showing the performance of MLE of the model parameters. The evaluation of the MLEs was performed based on the following quantities for each sample size: the Average Estimates (AE), Relative Absolute Bias (RAB), the Mean Squared Error (MSE) and Confidence Interval Length (CIL) are estimated from $R = 1000$ replications for for $\widehat{\alpha}_1, \widehat{\alpha}_2$and $\widehat{\alpha}_3$ and $\widehat{\beta}$ the sample size has been considered at $n = 30, 50,\ 100, 150$ and 200, and some values for the parameters $\alpha_1, \alpha_2$and α_3 and β have been considered.

Algorithm to generate from BUG distribution

Step 1. Generate U_1, U_2 and U_3 from Uniform$(0, 1)$.

Step 2. Calculate $Z_1 = \left(\frac{\alpha}{\alpha - \log U_1}\right)^{1/\beta}$,$Z_2 = \left(\frac{\alpha}{\alpha - \log U_2}\right)^{1/\beta}$ and $Z_3 = \left(\frac{\alpha}{\alpha - \log U_3}\right)^{1/\beta}$.

Step3. Obtain $X_1 = \max(Z_1, Z_3)$ and $X_2 = \max(Z_2, Z_3)$.

Step4. Define the indicator functions as

$$\delta_{1i} = \begin{cases} 1\,; & x_{1i} < x_{1i} \\ 0; & otherwise \end{cases},\ \delta_{2i} = \begin{cases} 1\,; & x_{1i} > x_{1i} \\ 0; & otherwise \end{cases}\ \ and\ \ \delta_{3i} = \begin{cases} 1\,; & x_{1i} = x_{1i} \\ 0; & otherwise \end{cases}.$$

Step5. The corresponding sample size n must satisfy $n = n_1 + n_2 + n_3$

Such that $n_1 = \sum_{i=1}^{n} \delta_{1i}, \qquad n_2 = \sum_{i=1}^{n} \delta_{2i}\ \ and \qquad n_3 = \sum_{i=1}^{n} \delta_{1i}.$

Using the MATHCAD program, a 1000 data set is generated for different choices of sample sizes that employed to solve the nonlinear likelihood equations. It can be noted from Table 6 and Table 7 that the estimates work well and MSE and RAB decreases as the sample size increases.

In addition, using gamma priors, we compute the Bayes estimates of the unknown parameters described in the previous section. R package is used to compute these estimates. We conduct 10000 simulations and replicate the process 1000 times for computing the average estimates (AE), RAB and MSEs Assuming gamma priors with hyperparameters are equal 0.5 and multivariate normal as proposal distribution. The average estimates (AE), RAB and MSEs are provided Tables 8-9 and one can note that as sample size increases, RAB and MSE decrease in most cases.

7. Absolutely Continuous Bivariate Unit Gompartz model

Based on Block and Basu [2] idea, an absolutely continuous bivariate Gompartz (BUG_{ac}) distribution will be introduced by removing the singular part from the Marshall-Olkin bivariate bivariate Gompartz and remaining only the absolutely continuous part.

A random vector (Y_1, Y_2) follows a BUG_{ac} distribution if its pdf is given by

$$f_{Y_1,Y_2}(y_1, y_2) \qquad = c.\begin{cases} f_{UG}(y_1; \alpha_{13}) \cdot f_{UG}(y_2; \alpha_2) & if \quad y_1 < y_2 \\ f_{UG}(y_1; \alpha_1) \cdot f_{UG}(y_2; \alpha_{23}) & if \quad y_1 > y_2 \end{cases},$$

where $c = \frac{\alpha_{12}}{\alpha_{123}}$ is normalizing constant.

We denote $(Y_1, Y_2) \sim BUG_{ac}(\alpha_1, \alpha_2, \alpha_3, \beta)$ if (X_1, X_2) has a BUG distribution, then (X_1, X_2) given $X_1 \neq X_2$ has a BUG_{ac} distribution.

Proposition 5. Let $(Y_1, Y_2) \sim BUG_{ac}(\alpha_1, \alpha_2, \alpha_3, \beta)$, then

1. The associated failure function is

$$F_{Y_1,Y_2}(y_1, y_2) = \frac{\alpha_{123}}{\alpha_{12}} F_{UG}(y_1; \alpha_1)\; F_{UG}(y_2; \beta_2)\, F_{UG}(y; \alpha_3) - \frac{\alpha_3}{\alpha_{12}} F_{UG}(y; \alpha_{123});$$

where $y = \min(y_1, y_2)$. Furthermore,

1. The marginal failure functions are given by

$$F_{Y_1}(y_1) = \frac{\alpha_{123}}{\alpha_{12}} F_{UG}(y_1; \alpha_{13}) - \frac{\alpha_3}{\alpha_{12}} F_{UG}(y_1; \alpha_{123})$$

$$F_{Y_2}(y_2) = \frac{\alpha_{123}}{\alpha_{12}} F_{UG}(y_2; \alpha_{23}) - \frac{\alpha_3}{\alpha_{12}} F_{UG}(y_2; \alpha_{123})$$

2. The marginal pdfs associated with the cdf function given above are as follows

$$f_{Y_1}(y_1) = c f_{UG}(y_1; \alpha_{13}) - c\frac{\alpha_3}{\alpha_{123}} f_{UG}(y_1; \alpha_{123}), \quad y_1 > 0$$

and

$$f_{Y_2}(y_2) = c f_{UG}(y_2; \alpha_{23}) - c\frac{\alpha_3}{\alpha_{123}} f_{UG}(y_2; \alpha_{123}), \quad y_2 > 0.$$

Note that: Unlike those of the BUG distribution, the marginals of the BUG_{ac} distribution are not BUG distributions. If $\beta_3 \to 0^+$, then Y_1 and Y_2 follow BUG distributions and in this case, Y_1 and Y_2 become independent.

Proposition 6. The product moments of $(Y_1, Y_2) \sim BUG_{ac}(\alpha_1, \alpha_2, \alpha_3, \beta)$ are given by

$$E\left(X_1{}^r X_2{}^s\right) = c e^{\alpha_{123}} \alpha_{13}^{\frac{r}{\beta}} \alpha_2^{\frac{s}{\beta}} \Gamma\left(1 - \frac{r}{\beta}\right) \Gamma\left(1 - \frac{s}{\beta}, \alpha_2\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_{13}^{k+1} \Gamma\left(2 - \frac{r+s}{\beta} + k, \alpha_2\right)}{k!\left(1 - \frac{r}{\beta} + k\right) \alpha_2^{k - \frac{r+s}{\beta}}}$$
$$+ c\, e^{\alpha_{123}} \alpha_{23}^{\frac{s}{\beta}} \alpha_1^{\frac{r}{\beta}} \Gamma\left(1 - \frac{s}{\beta}\right) \Gamma\left(1 - \frac{r}{\beta}, \alpha_1\right) - \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_{23}^{k+1} \Gamma\left(2 - \frac{r+s}{\beta} + k, \alpha_1\right)}{k!\left(1 - \frac{s}{\beta} + k\right) \alpha_1^{k - \frac{r+s}{\beta}}}.$$

Proposition 7. Let $(Y_1, Y_2) \sim BUG_{ac}(\alpha_1, \alpha_2, \alpha_3, \beta)$. Then

1. The Stress- Strength parameter takes the form:

$$R = P(Y_1 < Y_2) = \frac{\alpha_1}{\alpha_{12}},$$

2. $Max(Y_1, Y_2) \sim GU(\alpha_{123})$.

8. Conclusion and future work

Recently, the unit-Gompertz (UG) distribution was introduced as a new transformed model which has right-skewed (unimodal) and reversed-J shaped density. Moreover, the hazard rate has constant, increasing, bathtub and upside-down bathtub. In this paper, a bivariate extension for this new distribution called bivariate unit Gompertz (BUG) is introduced. Some of its statistical properties are discussed. The BUG distribution is a singular distribution and it has an absolute continuous and singular parts. This model can be used in practice for non-negative and dependent random variables since the joint distribution and the joint density functions are in closed forms. The MLEs for the four unknown parameters and their approximate var-cov matrix have been obtained and some simulations are carried out. Explicit Bayesian estimators are also obtained in the case of three unknown parameters of this model. Also, the Bayesian estimators are computed using MCMC in the case of the four unknown parameters. Two real data sets have been re-analyzed and showed that the new model proposed in this study can provide a better fit to these data sets. Along the same line as Block and Basu [2], bivariate exponential model, an absolute continuous version of the BUG model also obtained and several of its properties are presented. This article focuses solely on estimating model parameters for complete data. Because censoring is a common phenomenon in reliability and survival analysis, future work could look into different types of censored data.

Table 1. UEFA Champion's League data

S.N.	X_1	X_2	S.N.	X_1	X_2	S.N.	X_1	X_2	S.N.	X_1	X_2
1	26.0	20.0	11	72.0	72.0	21	53.0	39.0	31	49.0	49.0
2	63.0	18.0	12	66.0	62.0	22	54.0	7.00	32	24.0	24.0
3	19.0	19.0	13	25.0	9.00	23	51.0	28.0	33	44.0	30.0
4	66.0	85.0	14	41.0	3.00	24	76.0	64.0	34	42.0	3.00
5	4.00	4.00	15	16.0	75.0	25	64.0	15.0	35	27.0	47.0
6	49.0	49.0	16	18.0	18.0	26	26.0	48.0	36	28.0	28.0
7	8.00	8.00	17	22.0	14.0	27	16.0	16.0	37	2.00	2.00
8	69.0	71.0	18	42.0	42.0	28	44.0	6.00			
9	39.0	39.0	19	36.0	52.0	29	25.0	14.0			
10	82.0	48.0	20	34.0	34.0	30	55.0	11.0			

Table 2. The MLE, the CIL and the Variance- Covariance matrix for BUG Model for Data Set1

Para	MLE	CIL	Var-Cov			
$\widehat{\alpha}_1$	0.495	0.091	0.0160	0.0040	-0.0050	0.0004
$\widehat{\alpha}_2$	1.258	0.249	0.0040	0.1210	-0.0120	0.0040
$\widehat{\alpha}_3$	0.638	0.148	-0.0050	-0.0120	0.0430	0.0009
$\widehat{\beta}$	0.192	0.021	0.0004	0.0040	0.0009	0.0009

References

Table 3. Cholesterol levels at 5 and 25 weeks after treatment in 30 patients.

S.N.	X_1	X_2	S.N.	X_1	X_2	S.N.	X_1	X_2
1	325.0	246.0	11	217.0	252.0	21	316.0	283.0
2	278.0	245.0	12	248.0	305.0	22	243.0	245.0
3	257.0	212.0	13	225.0	225.0	23	305.0	272.0
4	192.0	192.0	14	287.0	208.0	24	197.0	197.0
5	276.0	325.0	15	233.0	217.0	25	243.0	247.0
6	262.0	294.0	16	198.0	198.0	26	315.0	283.0
7	309.0	232.0	17	229.0	179.0	27	205.0	205.0
8	287.0	287.0	18	310.0	352.0	28	315.0	255.0
9	304.0	245.0	19	214.0	274.0	29	263.0	215.0
10	215.0	261.0	20	253.0	209.0	30	210.0	271.0

Table 4. The MLE, the CIL and the Variance- Covariance matrix for BUG Model for Data Set2

Para	MLE	CIL	Var-cov			
$\widehat{\alpha_1}$	0.65	0.154	0.046	0.073	0.009	0.063
$\widehat{\alpha_2}$	1.275	0.426	0.073	0.354	0.038	0.225
$\widehat{\alpha_3}$	0.61	0.166	0.009	0.038	0.054	0.049
$\widehat{\beta}$	1.051	0.323	0.063	0.225	0.049	0.203

Table 5. Summary results for the posterior parameters for the two data sets

for data set 2 (the acceptance rate is 87.99%)				
95% credible intervals	Standard deviation	Median	Mean	Parameter
(1.1335, 1.8130)	0.4312	1.4548	1.4591	α_1
(1.1185, 1.5341)	0.3337	1.2941	1.2918	α_2
(0.7643, 1.7933)	0.6319	1.0142	1.2109	α_3
(0.6245, 0.9103)	0.2109	0.7549	0.7792	β
for data set 1 (the acceptance rate is 70.58%)				
95% credible intervals	Standard deviation	Median	Mean	Parameter
(1.5727, 1.8571)	0.2180	1.7342	1.6932	α_1
(0.5053, 0.8320)	0.2637	0.6761	0.6933	α_2
(1.4487, 1.8840)	0.2783	1.7094	1.6446	α_3
(0.2137, 0.2538)	0.0312	0.2326	0.2343	β

Table 6. The AE, MSE, RAB and CL for BUG model with Parameters true values (0.2, 0.2, 2, 1.1) using the MLE method

Sample size	Parameters	AE	MSE	RAB	CL (Lower, Upper)
30	α_1	0.3220	0.0150	0.6110	0.401(0.122,0.523)
	α_2	0.2400	0.0016	0.2020	0.339(0.071, 0.41)
	α_3	3.4720	2.1660	0.7360	4.414(1.264, 5.679)
	β	1.4290	0.1080	0.2990	0.627(1.116, 1.724)
50	α_1	0.556	0.1270	1.7790	0.406(0.353, 0.759)
	α_2	0.517	0.1000	1.5840	0.321(0.356, 0.677)
	α_3	5.759	14.1340	1.8800	3.907(3.806,7.713)
	β	1.093	0.0001	0.0064	0.271(0.951, 1.229)
100	α_1	0.293	0.0087	0.4660	0.094(0.246, 0.34)
	α_2	0.37	0.0290	0.8480	0.136(0.301, 0.438)
	α_3	3.969	3.8780	0.9850	1.391(3.274, 4.665)
	β	1.193	0.0087	0.0850	0.158(1.115, 1.272)
150	α_1	0.199	3.0E-7	0.0029	0.033(0.183, 0.216)
	α_2	0.213	0.0002	0.0650	0.035(0.196, 0.230)
	α_3	2.233	0.0540	0.1160	0.307(2.079,2.386)
	β	1.24	0.0190	0.1270	0.108(1.185, 1.294)
200	α_1	0.251	0.0026	0.2550	0.031(0.235, 0.267)
	α_2	0.219	0.0004	0.0950	0.028(0.205, 0.233)
	α_3	2.478	0.2280	0.2390	0.263(2.346, 2.609)
	β	1.037	0.0040	0.0570	0.079(0.998, 1.076)

Table 7. The AE, MSE, RAB and CL for BUG model with parameters true values as (1.2, 1.2, 2.1, 1.2) using the MLE method

Sample size	Parameters	AE	MSE	RAB	CL (Lower, Upper)
30	α_1	1.0830	0.0140	0.0970	0.646(0.760, 1.406)
	α_2	1.1980	3.0E-5	0.0015	0.663(0.867, 1.53)
	α_3	2.5960	0.24600	0.2360	1.395(1.898, 3.293)
	β	2.4540	1.57300	1.0450	0.669(2.120,2.788)
50	α_1	1.2560	0.0031	0.0460	0.48(1.016, 1.496)
	α_2	1.3250	0.0160	0.1040	0.495(1.077, 1.572)
	α_3	3.0380	0.8790	0.4460	1.096(2.49, 3.585)
	β	1.7720	0.3270	0.4760	0.367(1.588, 1.955)
100	α_1	1.1600	0.0016	0.0330	0.203(1.059, 1.262)
	α_2	1.2020	5.0E-6	0.0019	0.215(1.094, 1.31)
	α_3	2.7740	0.4540	0.3210	0.472(2.538, 3.01)
	β	1.5840	0.1480	0.3200	0.181(1.494, 1.675)
150	α_1	1.0930	0.0110	0.0890	0.171(1.007, 1.179)
	α_2	1.0980	0.0100	0.0850	0.177(1.009, 1.186)
	α_3	2.6800	0.3360	0.2760	0.405(2.477, 2.882)
	β	1.5250	0.1060	0.2710	0.136(1.457,1.593)
200	α_1	1.1980	5.0E-6	0.0019	0.182(1.107, 1.289)
	α_2	1.1820	0.0003187	0.0150	0.182(1.091, 1.273)
	α_3	2.9700	0.75700	0.4140	0.44(2.75, 3.19)
	β	1.6490	0.20200	0.3750	0.096(1.601,1.698)

Table 8. The AE, MSE, RAB and CL for BUG Model with different parameters true values using Bayesian estimation method

Parameters true Values		(1.5, 1.5, 1.5, 0.55)			(0.5, 0.5, 0.5, 1.25)		
Sample size	Parameters	AE	RAB	MSE	AE	RAB	MSE
30	α_1	1.7011	0.1341	0.0590	1.2913	1.5827	0.6265
	α_2	2.0952	0.3968	0.3682	1.5459	2.0918	1.0943
	α_3	1.5738	0.0492	0.0245	1.2078	1.4155	0.5013
	β	0.4629	0.0743	0.0006	0.5575	0.5540	0.4796
50	α_1	1.6960	0.1306	0.0407	1.2936	1.5872	0.6301
	α_2	2.0915	0.3943	0.3516	1.5488	2.0975	1.1003
	α_3	1.5799	0.0533	0.0097	1.2126	1.4252	0.5082
	β	0.4629	0.1583	0.0078	0.5564	0.5549	0.4812
100	α_1	1.6915	0.1277	0.0393	0.7363	0.4725	0.05585
	α_2	2.0887	0.3925	0.3483	0.8441	0.6882	0.1184
	α_3	1.5691	0.0461	0.0080	0.7029	0.4058	0.0412
	β	0.4642	0.1560	0.0075	0.7897	0.3683	0.2119
150	α_1	1.6996	0.1331	0.0423	0.7361	0.4721	0.0558
	α_2	2.0973	0.3982	0.3585	0.8442	0.6884	0.1185
	α_3	1.5759	0.0506	0.0089	0.7020	0.4041	0.0409
	β	0.4625	0.1591	0.0078	0.7899	0.3681	0.2117
200	α_1	1.6938	0.1292	0.0402	0.7356	0.4711	0.0555
	α_2	2.0933	0.3956	0.3539	0.8440	0.6880	0.1184
	α_3	1.5704	0.0496	0.0081	0.7020	0.4040	0.0408
	β	0.4639	0.1566	0.0076	0.7901	0.3680	0.2116

Table 9. The AE, MSE, RAB and CL for BUG model with different parameters true values using Bayesian estimation method

Parameters true Values		(3.0, 3.0, 3.0, 0.3)			(0.3, 0.3, 0.3, 1.75)		
Sample size	Parameters	AE	RAB	MSE	AE	RAB	MSE
30	α_1	2.0478	0.3174	0.9169	0.7355	1.4518	0.1897
	α_2	2.5667	0.1444	0.1942	0.8434	1.8115	0.2954
	α_3	1.8884	0.3706	1.2486	0.7016	1.3388	0.1613
	β	0.4050	0.3501	0.0113	0.7903	0.5484	0.9211
50	α_1	2.0506	0.3165	0.9116	0.7348	1.4493	1.8907
	α_2	2.5641	0.1453	0.1965	0.8432	1.8106	0.2951
	α_3	1.9046	0.3651	1.2129	0.7012	1.3373	0.1610
	β	0.4043	0.3477	0.0111	0.7908	0.5481	0.9201
100	α_1	2.0520	0.3160	0.9087	0.7354	1.4513	0.1896
	α_2	2.5643	0.1452	0.1959	0.8435	1.8118	0.2955
	α_3	1.9063	0.3646	1.2091	0.7026	1.3420	0.1621
	β	0.4043	0.3475	0.0111	0.7902	0.5484	0.9212
150	α_1	2.0302	0.3233	0.9504	0.7350	1.4499	0.1892
	α_2	2.5531	0.1490	0.2056	0.8439	1.8130	0.2959
	α_3	1.8914	0.3695	1.2412	0.7018	1.3393	0.1615
	β	0.4063	0.3544	0.0116	0.7904	0.5483	0.9208
200	α_1	2.0508	0.3164	0.9115	0.7355	1.4518	0.1897
	α_2	2.5643	0.1452	0.1960	0.8437	1.8122	0.2956
	α_3	1.9028	0.3657	1.2168	0.7016	1.3386	0.1613
	β	0.4045	0.3482	0.0112	0.7905	0.5483	0.9206

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