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Estimation of the Topp-Leone Alpha Power Weibull Distribution Based on Lower Record Values

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Abstract: This article focuses on obtaining the maximum likelihood and Bayes estimators of the Topp-Leone alpha power Weibull distribution based on lower record values. The unknown parameters are estimated by applying the maximum likelihood approach based on lower record values. Also, the Bayesian method is utilized using gamma distribution as an informative prior to obtain the Bayes estimators for the parameters of the Topp-Leone alpha power Weibull distribution based on lower record values. The Bayes estimators are derived under the squared error loss function as a symmetric loss function and the linear-exponential loss function as an asymmetric loss function based on lower record values. Finally, a numerical illustration is presented to demonstrate and evaluate the theoretical results where the numerical demonstration includes a simulation study and an application on real data set. The maximum likelihood estimates are evaluated using Monte Carlo Simulation via Mathematica 11 and the Bayes estimates are calculated by applying Markov Chain Monte Carlo through R programming language.

Keywords: Topp-Leone alpha power Weibull distribution, Lower record values, Maximum likelihood estimation, Bayesian estimation.
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1. Introduction

Record values and the associated statistics have been received increasing attention over the past decade for instance data relating to sports, economics, oil, mining surveys, industry, life testing studies, weather, extreme weather and air pollution including extreme temperatures, the exceedances of flood peaks, and pollutant concentrations deviating considerably from expected average levels.

Chandler [7] presented the first study of record values and its basic properties. Balakrishnan et al. [6] proposed some recurrence relationships for single and double moments of lower record values from the Gumbel distribution. Coles and Tawn [9] studied a daily rainfall series for modeling the extremes of a rainfall process in the perspective of record values. Wang et al. [31] provided approaches to constructing exact confidence intervals for unknown parameters in the family of proportional reversed hazard distributions based on lower record values. Seo and Kim [24] applied non-Bayesian and Bayesian approaches to inference for the Gumbel distribution based on lower record values. Seo and Kim [25] established an objective Bayesian analysis method for the two-parameter Rayleigh distribution based on record values. Seo and Kim [26] presented an entropy inference method based on an objective Bayesian approach when observed record values have a two-parameter logistic distribution.

There are two types of record values. If the observation is greater than all the previous observations, it is called the upper record. Conversely, if the observation is smaller than all the previous observations, then it is called the lower record. For more demonstration if $X_1, X_2, \ldots, X_n$ is a sequence of independent identically distributed random variables with *cumulative distribution function* (cdf); $F(x)$ and *probability density function*; (pdf); $f(x)$, then an observation X_j is called an upper record value if its value exceeds that of all previous observations. Thus, X_j is an upper record if $X_j > X_i$ for every $j > i$ where $j, i \geq 1$. While the observation X_j will be the lower record if its value is smaller than all its previous observations, that is, if $X_j < X_i$ for every $j > i$.

Lower records are of great interest and importance in many areas of real life applications involving data relating to weather, sports (athletic events), industry, economics, biomedical sciences, engineering, the environmental sciences, actuarial sciences, management sciences, social sciences and life testing; for example the lowest stock markets figure, also they are useful in reliability theory, meteorology, hydrology, seismology, and mining [See Khan and Faizan [**?**]].

Considering the first m lower record values $X_{L(1)}, X_{L(2)}, \ldots, X_{L(m)}$, the joint distribution of the first m lower record values which was given by Arnold et al. [5] is as follows:

$$f\left(x_{L(1)}, x_{L(2)}, \ldots, x_{L(m)}\right) = f\left(x_{L(m)}\right) \prod_{i=1}^{m-1} \frac{f\left(x_{L(i)}\right)}{F\left(x_{L(i)}\right)}, \quad -\infty < x_{L(1)}, \ldots, x_{L(m)} < \infty. \tag{1.1}$$

Many authors studied the lower record values, for example, Sultan [30] studied the estimation of the parameters of the inverse Weibull distribution based on lower record values. Soliman et al. [29] presented the *maximum likelihood* (ML) and Bayesian estimation under *squared error* (SE) and zero one loss functions of the unknown parameters for the inverse Rayleigh distribution based on lower record values. Bayesian and non-Bayesian estimation of the Topp-Leone distribution based on lower record values were considered by El-Sayed et al. [12]. Dey et al. [10] proposed the ML and Bayesian estimation of the unknown parameters of the generalized exponential distribution based on lower record values. Sindi et al. [27] introduced exponentiated general class of distributions and obtained the ML, Bayesian and empirical Bayesian estimators based on lower record values. Singh et al. [28] derived the exact explicit expressions for the single and product moments of generalized Lindley distribution based on lower record values.

Some references in the field of the lower record values include Ahsanullah [1], Hassan *et al.* [13], Nanda et al. [20], Mohamed [19], Chaturvedi and Malhotra [8], Alawady et al. [3], Alam et al. [2], and AL-Essa et al. [4].

Mahdavi and Kundu [17] presented a method to add an extra parameter to a family of distributions, such an addition of parameters makes the resulting distribution richer and more flexible for modeling data. The suggested method is called alpha power transformation and it is useful to incorporate skewness into a family of distributions. The alpha power transformation method was applied to many distributions by many researchers, such as Nassar et al.[22], Dey et al.[11], Nadarajah and Okorie [21], Mead et al.[18], Nassar et al.[23], Hozaien et al. [14].

The cdf and pdf of the *Topp-Leone alpha power Weibull* (TLAW) distribution given by Hozaien et al. [14] is

$$G(x;\underline{\Omega}) = \left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right]^{\theta}, \tag{1.2}$$

and

$$g(x;\underline{\Omega}) = \frac{2\theta\lambda ln\alpha}{\beta(\alpha-1)}\left(\frac{x}{\beta}\right)^{\lambda-1} e^{-\left(\frac{x}{\beta}\right)^{\lambda}}\alpha^{1-e^{-\left(\frac{x}{\beta}\right)^{\lambda}}}\left(1-\frac{\alpha^{1-e^{-(x/\beta)^{\lambda}}}-1}{\alpha-1}\right)\left[1-\left(1-\frac{\alpha^{1-e^{-(x/\beta)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right]^{\theta-1}, x>0;\ \theta,\alpha,\lambda,\beta>0,\ \alpha\neq 1, \tag{1.3}$$

where $\underline{\Omega} = (\theta, \alpha, \lambda, \beta)$.

This paper is organized as follows: In Section 2, ML estimators for the unknown parameters of the TLAW distribution are obtained based on lower record values. The Bayes estimators of the unknown parameters are derived under SE and *linear exponential* (LINEX) loss functions based on lower record values in Section 3. A numerical illustration through simulation study is introduced in Section 4 and an application with a real data set is given in Section 5. In Section 6, a general conclusion is presented.

2. Maximum Likelihood Estimation

In this section, estimation of the unknown parameters of the TLAW distribution using maximum likelihood method based on lower record values, is obtained.

Let $X_{L(1)}, X_{L(2)}, \ldots, X_{L(m)}$ is the lower record values from the TLAW distribution, substituting (1.2) and (1.3) in (1.1), then the likelihood function based on lower record values is

$$\begin{aligned} L\left(\underline{\Omega};\underline{x}_L\right) =& \left(\frac{2\theta\lambda ln\alpha}{\beta(\alpha-1)}\right)^{m}\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right)^{\theta}\prod_{i=1}^{m}\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right)^{-1} \\ &\times\prod_{i=1}^{m}\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1} e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right). \end{aligned} \tag{2.1}$$

The log likelihood denoted by ℓ is given by

$$\begin{aligned} \ell \equiv lnL\left(\underline{\Omega};\underline{x}_L\right) =& mln\left(\frac{2\theta\lambda ln\alpha}{\beta(\alpha-1)}\right)+\theta ln\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right)+(\lambda-1)\sum_{i=1}^{m}ln\left(\frac{x_{L(i)}}{\beta}\right)-\sum_{i=1}^{m}\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda} \\ &+ln\alpha\sum_{i=1}^{m}\left(1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}\right)+\sum_{i=1}^{m}ln\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)-\sum_{i=1}^{m}ln\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right). \end{aligned} \tag{2.2}$$

The ML estimators of $\underline{\Omega} = (\alpha, \lambda, \beta, \theta)$ can be obtained by differentiating (2.2) with respect to α, λ, β and θ and equating to zero, then

$$\frac{\partial \ell}{\partial \lambda} = \frac{m}{\hat{\lambda}} + \frac{\frac{2\hat{\theta} ln\hat{\alpha}}{\hat{\alpha}-1} ln\left(\frac{x_{L(m)}}{\hat{\beta}}\right)\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}} e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}} \hat{\alpha}^{1-e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}}} \left(1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)}{1-\left(1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)^2} + \sum_{i=1}^{m} l\left(\frac{x_{L(i)}}{\hat{\beta}}\right) - \sum_{i=1}^{m}\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}} ln\left(\frac{x_{L(i)}}{\hat{\beta}}\right)$$

$$+ ln\hat{\alpha}\sum_{i=1}^{m}\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}} e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}} ln\left(\frac{x_{L(i)}}{\hat{\beta}}\right) - ln\hat{\alpha}\sum_{i=1}^{m}\frac{\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}} e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}} \hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}} ln\left(\frac{x_{L(i)}}{\hat{\beta}}\right)}{\hat{\alpha}-\hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}} \tag{2.3}$$

$$- \frac{2ln\hat{\alpha}}{\hat{\alpha}-1}\sum_{i=1}^{m}\frac{\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}} e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}} \hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}} ln\left(\frac{x_{L(i)}}{\hat{\beta}}\right)\left(1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)}{1-\left(1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)^2} = 0,$$

$$\frac{\partial \ell}{\partial \beta} = \frac{-m}{\hat{\beta}} - \frac{\frac{2\hat{\theta}\hat{\lambda} ln\hat{\alpha}}{\hat{\beta}^{\hat{\lambda}+1}(\hat{\alpha}-1)} x_{L(m)}^{\hat{\lambda}} e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}} \hat{\alpha}^{1-e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}}} \left(1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)}{1-\left(1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)^2} - \frac{\hat{\lambda}-1}{\hat{\beta}^m} + \frac{\hat{\lambda}}{\hat{\beta}^{\hat{\lambda}+1}}\sum_{i=1}^{m} x_{L(i)}^{\hat{\lambda}} - \frac{\hat{\lambda} ln\hat{\alpha}}{\hat{\beta}^{\hat{\lambda}+1}}\sum_{i=1}^{m} x_{L(i)}^{\hat{\lambda}} e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}$$

$$+ \frac{\hat{\lambda} ln\hat{\alpha}}{\hat{\beta}^{\hat{\lambda}+1}}\sum_{i=1}^{m}\frac{x_{L(i)}^{\hat{\lambda}} e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}} \hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}}{\hat{\alpha}-\hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}} + \frac{2\hat{\lambda} ln\hat{\alpha}}{\hat{\beta}^{\hat{\lambda}+1}(\hat{\alpha}-1)}\sum_{i=1}^{m}\frac{x_{L(i)}^{\hat{\lambda}} e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}} \hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}} \left(1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)}{1-\left(1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)^2} = 0, \tag{2.4}$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{m}{\hat{\alpha} ln\hat{\alpha}} - \frac{m}{\hat{\alpha}-1} + \frac{2\hat{\theta}\left(1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)\left(\frac{\hat{\alpha}^{-e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}\left(1-e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}}\right)}{(\hat{\alpha}-1)} - \frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{(\hat{\alpha}-1)^2}\right)}{1-\left(1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)^2} + \frac{1}{\hat{\alpha}}\sum_{i=1}^{m}\left(1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}\right)$$

$$-\sum_{i=1}^{m}\frac{\frac{\hat{\alpha}^{-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}\left(1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}\right)}{(\hat{\alpha}-1)} - \frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{(\hat{\alpha}-1)^2}}{1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}} - 2\sum_{i=1}^{m}\frac{\left(1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)\left(\frac{\hat{\alpha}^{-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}\left(1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}\right)}{(\hat{\alpha}-1)} - \frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{(\hat{\alpha}-1)^2}\right)}{1-\left(1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(i)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)^2} = 0, \tag{2.5}$$

and

$$\frac{\partial \ell}{\partial \theta} = \frac{m}{\hat{\theta}} + ln\left(1-\left(1-\frac{\hat{\alpha}^{1-e^{-\left(\frac{x_{L(m)}}{\hat{\beta}}\right)^{\hat{\lambda}}}}-1}{\hat{\alpha}-1}\right)^{2}\right) = 0. \tag{2.6}$$

To obtain the ML estimates, (2.3)-(2.6) can be solved numerically using Monte Carlo Simulation via Mathematica 11.

The ML estimators for the reliability and hazard rate functions can be obtained using the invariance property of the ML estimators by replacing the parameters with their ML estimators in the reliability and hazard rate functions.

3. Bayesian Estimation

In this section, Bayesian approach is applied to obtain the estimators of the unknown parameters of the TLAW distribution based on lower record values. Bayes estimators are obtained using two different loss functions, the SE loss function as a symmetric loss function and LINEX loss function as an asymmetric loss function.

Considering informative prior for the unknown parameters $\underline{\Omega} = (\alpha, \lambda, \beta, \theta)$ which are assumed to be independent random variables and each has gamma distribution. Then the joint prior distribution of the parameters

$\underline{\Omega} = (\alpha, \lambda, \beta, \theta)$ is given by

$$\pi(\underline{\Omega}) = \frac{r_2^{r_1} r_4^{r_3} r_6^{r_5} r_8^{r_7}}{\Gamma r_1 \Gamma r_3 \Gamma r_5 \Gamma r_7}\theta^{r_1-1}\alpha^{r_3-1}\lambda^{r_5-1}\beta^{r_7-1}e^{-(r_2\theta+r_4\alpha+r_6\lambda+r_8\beta)}, \underline{\Omega} > \underline{0},\ r_i > 0,\ \text{i} = 1, ..., 8, \tag{3.1}$$

where $r_i = r_1, r_2, \ldots, r_8$ are the hyper parameters of the prior distributions. Combining (2.1) with (3.1), then the joint posterior distribution of $\underline{\Omega} = (\alpha, \lambda, \beta, \theta)$ is given by

$$\begin{aligned}\pi\left(\underline{\Omega} \mid \underline{x}\right) &= k\, L\left(\underline{\Omega}|\underline{x}_L\right)\pi(\underline{\Omega}) \\ &= k\,\varphi(\underline{\Omega}, r_i)\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right)^{\theta}\prod_{i=1}^{m}\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1}e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right) \\ &\times\prod_{i=1}^{m}\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right)^{-1},\end{aligned} \tag{3.2}$$

where

$$\varphi(\underline{\Omega}, r_i) = \left(\frac{2\lambda\theta ln\alpha}{\beta(\alpha-1)}\right)^{m}\frac{r_2^{r_1} r_4^{r_3} r_6^{r_5} r_8^{r_7}}{\Gamma r_1 \Gamma r_3 \Gamma r_5 \Gamma r_7}\theta^{r_1-1}\alpha^{r_3-1}\lambda^{r_5-1}\beta^{r_7-1}e^{-(r_2\theta+r_4\alpha+r_6\lambda+r_8\beta)},$$

and

$$k^{-1} = \int_{\underline{\Omega}} \frac{\varphi(\Omega, r_i)\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right)^{\theta}\prod_{i=1}^{m}\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1}e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)}{\prod_{i=1}^{m}\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^{2}\right)}\, d\underline{\Omega},$$

where $\int_{\underline{\Omega}} = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty$, and $d\underline{\Omega} = d\theta d\alpha\, d\lambda\, d\beta$.

The marginal posterior distribution of the parameter Ω_j can be obtained as follows:

$$\pi^*\left(\Omega_j \mid \underline{x}\right) = \int_{\underline{\Omega}_i} \pi\left(\underline{\Omega} \mid \underline{x}\right) d\underline{\Omega}_i, \qquad i \neq j, \qquad i, j = 1,\ 2,\ 3,\ 4.$$

Then the posterior density of θ is

$$\pi\left(\theta \mid \underline{x}\right) = k \int_0^\infty \int_0^\infty \int_0^\infty \frac{\varphi(\Omega, r_i)\left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]^\theta \prod_{i=1}^m \left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1} e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda} \alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}} \left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)}{\prod_{i=1}^m \left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]} d\alpha\, d\lambda\, d\beta. \tag{3.3}$$

The posterior density of α is

$$\pi\left(\alpha \mid \underline{x}\right) = k \int_0^\infty \int_0^\infty \int_0^\infty \frac{\varphi(\Omega, r_i)\left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]^\theta \prod_{i=1}^m \left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1} e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda} \alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}} \left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)}{\prod_{i=1}^m \left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]} d\theta\, d\lambda\, d\beta. \tag{3.4}$$

The posterior density of λ is

$$\pi\left(\lambda \mid \underline{x}\right) = k \int_0^\infty \int_0^\infty \int_0^\infty \frac{\varphi(\Omega, r_i)\left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]^\theta \prod_{i=1}^m \left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1} e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda} \alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}} \left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)}{\prod_{i=1}^m \left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]} d\theta\, d\alpha\, d\beta, \tag{3.5}$$

and the posterior density of β is

$$\pi\left(\beta \mid \underline{x}\right) = k \int_0^\infty \int_0^\infty \int_0^\infty \frac{\varphi(\Omega, r_i)\left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]^\theta \prod_{i=1}^m \left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1} e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda} \alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}} \left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)}{\prod_{i=1}^m \left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]} d\theta\, d\alpha\, d\lambda. \tag{3.6}$$

3.1. Bayesian estimation under squared error loss function

One of the most popular used loss functions is the SE loss function. It is commonly used with Bayesian estimation because it does not require complicated numerical calculation. It is a symmetric loss function that assigns equal weight to overestimation as well as underestimation of the parameters under consideration. The Bayes estimators of the parameters $\underline{\Omega} = (\alpha, \lambda, \beta, \theta)$ under the SE loss function are the means of their marginal posterior distributions. Hence using the marginal posterior distributions in (3.3)-(3.6), the Bayes estimators can be obtained as given below

$$\theta^*_{SE} = E\left(\theta \mid \underline{x}\right) = \int_\theta \theta\pi\left(\theta|\underline{x}\right) d\theta$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{k\theta\varphi\left(\Omega, r_i\right)\left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]^\theta \prod_{i=1}^m \left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1} e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)}{\prod_{i=1}^m\left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]} d\alpha\, d\lambda\, d\beta d\theta, \tag{3.7}$$

$$\alpha^*_{SE} = E\left(\alpha \mid \underline{x}\right) = \int_\alpha \alpha\pi\left(\alpha|\underline{x}\right) d\alpha$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{k\alpha\varphi\left(\Omega, r_i\right)\left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]^\theta \prod_{i=1}^m \left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1} e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)}{\prod_{i=1}^m\left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]} d\theta\, d\lambda\, d\beta d\alpha, \tag{3.8}$$

$$\lambda^*_{SE} = E\left(\lambda \mid \underline{x}\right) = \int_\lambda \lambda\pi\left(\lambda|\underline{x}\right) d\lambda$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{k\lambda\varphi\left(\Omega, r_i\right)\left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]^\theta \prod_{i=1}^m \left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1} e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)}{\prod_{i=1}^m\left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]} d\theta\, d\alpha\, d\beta d\lambda, \tag{3.9}$$

and

$$\beta^*_{SE} = E\left(\beta \mid \underline{x}\right) = \int_\beta \beta\pi\left(\beta|\underline{x}\right) d\beta$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{k\beta\varphi\left(\Omega, r_i\right)\left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]^\theta \prod_{i=1}^m \left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1} e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)}{\prod_{i=1}^m\left[1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^\lambda}}-1}{\alpha-1}\right)^2\right]} d\theta\, d\alpha\, d\lambda d\beta. \tag{3.10}$$

The Bayes estimates for the parameters θ, α, λ and β can be obtained by solving (3.7)-(3.10) numerically.

3.2. *Bayesian estimation under linear exponential loss function*

The use of a symmetric loss function might be unsuitable for different estimation problems, since in life testing, over estimation may be more serious than under estimation or vice versa. Under the LINEX loss function, the Bayes estimators of the parameters $\underline{\Omega} = (\alpha, \lambda, \beta, \theta)$ can be derived using (3.3)-(3.6) as follows:

$$\Omega^*_{LINEX}=\frac{-1}{\delta}ln\left[E\left(e^{-\delta\Omega_j}\mid \underline{x}\right)\right],$$

where

$$E\left(e^{-\nu\Omega_j}\mid \underline{x}\right)=\int_{\psi_j} e^{-\delta\Omega_j}\pi\left(\Omega_j\mid \underline{x}\right)d\Omega_j.$$

Hence

$$\theta^*_{LINEX} = \frac{-1}{\delta}lnE\left(e^{-\delta\theta}|\underline{x}\right) = \frac{-1}{\delta}ln\int_{\theta} e^{-\delta\theta}\pi\left(\theta|\underline{x}\right)d\theta$$

$$= \frac{-1}{\delta}ln\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{ke^{-\delta\theta}\varphi(\Omega,r_i)\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^2\right)^{\theta}\prod_{i=1}^{m}\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1}e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)}{\prod_{i=1}^{m}\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^2\right)}d\alpha\, d\lambda\, d\beta d\theta, \tag{3.11}$$

$$\alpha^*_{LINEX} = \frac{-1}{\delta}lnE\left(e^{-\delta\alpha}|\underline{x}\right) = \frac{-1}{\delta}ln\int_{\alpha} e^{-\delta\alpha}\pi\left(\alpha|\underline{x}\right)d\alpha$$

$$= \frac{-1}{\delta}ln\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{ke^{-\delta\alpha}\varphi(\Omega,r_i)\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^2\right)^{\theta}\prod_{i=1}^{m}\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1}e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)}{\prod_{i=1}^{m}\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^2\right)}d\theta\, d\lambda\, d\beta d\alpha, \tag{3.12}$$

$$\lambda^*_{LINEX} = \frac{-1}{\delta}lnE\left(e^{-\delta\lambda}|\underline{x}\right) = \frac{-1}{\delta}ln\int_{\lambda} e^{-\delta\lambda}\pi\left(\lambda|\underline{x}\right)d\lambda$$

$$= \frac{-1}{\delta}ln\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{ke^{-\delta\lambda}\varphi(\Omega,r_i)\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^2\right)^{\theta}\prod_{i=1}^{m}\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1}e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)}{\prod_{i=1}^{m}\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^2\right)}d\theta\, d\alpha\, d\beta d\lambda, \tag{3.13}$$

and

$$\beta^*_{LINEX} = \frac{-1}{\delta}lnE\left(e^{-\delta\beta}|\underline{x}\right) = \frac{-1}{\delta}ln\int_{\beta} e^{-\delta\beta}\pi\left(\beta|\underline{x}\right)d\beta$$

$$= \frac{-1}{\delta}ln\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{ke^{-\delta\beta}\varphi(\Omega,r_i)\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(m)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^2\right)^{\theta}\prod_{i=1}^{m}\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda-1}e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)}{\prod_{i=1}^{m}\left(1-\left(1-\frac{\alpha^{1-e^{-\left(\frac{x_{L(i)}}{\beta}\right)^{\lambda}}}-1}{\alpha-1}\right)^2\right)}d\theta\, d\alpha\, d\lambda d\beta. \tag{3.14}$$

Equations (3.11)-(3.14) can be solved numerically to obtain the Bayes estimates for the parameters θ, α, λ and β under the LINEX loss function.

4. Simulation Study

In this section, a simulation study is carried out to illustrate the performance of the ML and Bayes estimates under SE and LINEX loss functions based on lower record values from the TLAW distribution. The performance of the ML and Bayes estimates of the parameters and relative absolute bias (RABs) and Estimated Risks (ERs) are computed based on lower record values through Monte Carlo simulation study according to the following steps:

1. The population parameter values of $\underline{\Omega} = (\theta, \alpha, \lambda, \beta)$ are used to generate random samples of size n from the TLAW distribution observing that if U is uniform distribution (0,1), then

$$t_{ij(TLAW)} = \beta\left\{-ln\left(1 - \frac{ln\left[(\alpha-1)\left(1-\left(1-(U_{ij})^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right)+1\right]}{ln\alpha}\right)\right\}^{\frac{1}{\lambda}}$$

, is TLAW ($\underline{\Omega}$) distribution.

1. For each sample size n, consider the first observation is the first lower record value t_1 denoting it by R_1 and the second observation t_2 denoting it by R_2which is smaller than the maximum record ($t_1 > t_2$) and if $t_1 \leq t_2$ ignore it and repeat until you get a sample of record values (m).
2. For the number of the surviving units t and the population parameter values of the shape parameters, the ML estimates of the parameters θ, α, λ and β are obtained using Monte Carlo simulation. The calculations are performed using Mathematica 11.
3. The Bayes estimates of the parameters under SE and LINEX loss functions are evaluated for the number of the surviving units t based on the population parameter values of the parameters $\theta, \alpha, \lambda, \beta$ and the hyper parameters of the prior distribution applying *Markov chain Monte Carlo* (MCMC) method of simulation. The computations are performed using R programming language.
4. Tables 1 and 2 display the RABs and ERs of the averages of the ML estimates for the parameters θ, α, λ and β from the TLAW distribution based on lower records where the population parameter values are ($\alpha = 1.2, \ \lambda = 0.4, \ \beta = 0.5, \ \theta = 1.6$) and ($\alpha = 1.5, \ \lambda = 1.5, \ \beta = 1.5, \ \theta = 1.5$) based on samples of m= 3, 6, 9 and *number of replications* (NR)= 1000,
   where
   $$RAB = \frac{Absolute\,(estimate - true\ value)}{true\ value},$$
   and
   $$ER = \frac{1}{1000}\sum_{i=1}^{1000}(estimate - true\ value)^2.$$
5. Tables 4 and 5 present the averages of the Bayes estimates, the RABs and ERs using gamma distribution as informative prior for the parameters. The estimates are obtained under the SE and LINEX loss functions based on samples of lower record values m=4, 6 and m=4, 8, and NR = 10000,

Concluding Remarks

1. From Tables 1 and 2, one can observe that the RABs and ERs of the ML averages for the parameters α, β, θ and λ decreases when the sample size of lower records m increases.
2. From Tables 3 and 4, it is noted that the RABs and ERs of the Bayes averages for the parameters α, β, θ and λ decrease when the sample of m increases.

Table 1. ML averages, relative absolute biases and estimated risks of the parameters from TLAW distribution based on lower records: (α = 1.2, λ = 0.4, β = 0.5, θ = 1.6)

m	**Parameter**	**Average**	**RAB**	**ER**
3	α	1.9576	0.6313	0.1913
	λ	0.3226	0.1934	0.002
	β	0.7069	0.4138	0.0143
	θ	2.077	0.2981	0.0759
6	α	1.9019	0.5849	0.1642
	λ	0.5778	0.4446	0.0105
	β	0.7062	0.4125	0.0141
	θ	1.4109	0.1181	0.0119
9	α	1.8061	0.5051	0.0918
	λ	0.513	0.2825	0.0032
	β	0.5368	0.0737	0.0003
	θ	1.6531	0.0332	0.0007

Table 2. ML averages, relative absolute biases and estimated risks of the parameters from TLAW distribution based on lower records: (α = 1.5, λ = 1.5, β = 1.5.θ = 1.5)

m	**Parameter**	**Average**	**RAB**	**ER**
3	α	2.1577	0.4384	0.1442
	λ	1.8059	0.2039	0.0312
	β	1.6046	0.0697	0.0036
	θ	1.391	0.0727	0.004
6	α	2.1603	0.4402	0.10901
	λ	1.8457	0.2305	0.0299
	β	1.5727	0.0485	0.0013
	θ	1.8795	0.253	0.036
9	α	2.2997	0.5331	0.0799
	λ	1.9446	0.2964	0.0247
	β	1.4732	0.0179	0.0001
	θ	1.5955	0.0636	0.0011

Table 3. Bayes averages, relative absolute biases and estimated risks of TLAW distribution based on lower records

m	Loss functions	Parameter	Average	RAB	ER
4	SE	α	1.20203	0.00168	0.00164
		λ	0.70011	0.00015	4.70E-06
		β	0.50138	0.00277	0.00077
		θ	0.89899	0.00113	0.00041
	LINEX	α	1.20098	0.00082	0.00039
		λ	0.6979	0.003	0.00176
		β	0.50152	0.00304	0.00092
		θ	0.89886	0.00126	0.00051
6	SE	α	1.19934	0.00055	0.00018
		λ	0.70007	0.00012	2.28E-06
		β	0.49929	0.00143	0.0002
		θ	0.89934	0.00073	0.00017
	LINEX	α	1.19977	0.00019	0.00002
		λ	0.69971	0.00041	0.00003
		β	0.50143	0.00286	0.00081
		θ	0.90026	0.00029	0.00003

Table 4. Bayes averages, relative absolute biases and estimated risks of TLAW distribution based on lower records

m	Loss functions	Parameter	Average	RAB	ER
4	SE	α	1.60087	0.00055	0.00031
		λ	1.29924	0.00059	0.00023
		β	1.20055	0.00046	0.00012
		θ	1.40091	0.00065	0.00033
	LINEX	α	1.59898	0.00064	0.00042
		λ	1.30073	0.00056	0.00021
		β	1.19896	0.00087	0.00044
		θ	1.4004	0.00028	0.00006
8	SE	α	1.60043	0.00038	0.00015
		λ	1.29993	0.00005	0.0000019
		β	1.20012	0.00018	0.0000057
		θ	1.40074	0.00053	0.00022
	LINEX	α	1.59912	0.00055	0.00031
		λ	1.3007	0.00054	0.00019
		β	1.19905	0.0008	0.00036
		θ	1.40036	0.00026	0.00005

5. Application

In this section, a real dataset is used to illustrate the flexibility and applicability of the TLAW distribution based on lower record values. The data given by Linhart and Zucchini [16] represents the failure times of the air-conditioning system of an airplane: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95. Kolmogorov–Smirnov test showed that the TLAW distribution fits this data well, and the p-value is 0.9541.

Figures 1, 2, 3, 4, and 5 show the PP-plot, QQ-plot, fitted pdf of the TLAW distribution, boxplot and the empirical scaled TTT-transform plot, respectively.

Table 5 displays the ML estimates of the parameters and *standard errors* (St.E) for the real dataset based on lower records. The Bayes estimates of the parameters and St.E for the real dataset based on lower records are given in Table 6.

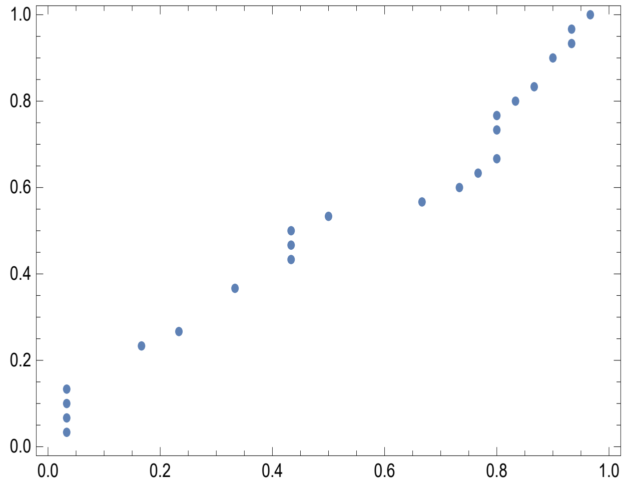

Figure 1. PP plot of the TLAW distribution for the real dataset

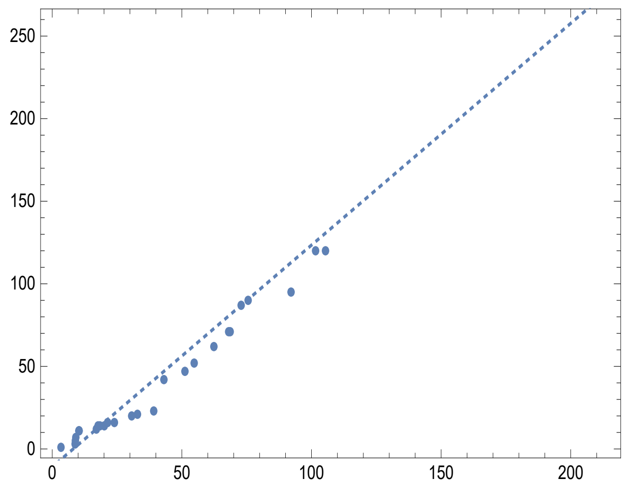

Figure 2. QQ plot of the TLAW distribution for the real dataset

From Figures 1, 2, 3, 4, and 5 one can conclude that the plot of the empirical scaled TTT-transform of real data set, implies that this data has a modified bathtub hazard function. Also, the boxplot shows that this data is right skewed. P-P plot, Q-Q plot and the fitted TLAW distribution plots indicate that the TLAW distribution provides a better fit to this data.

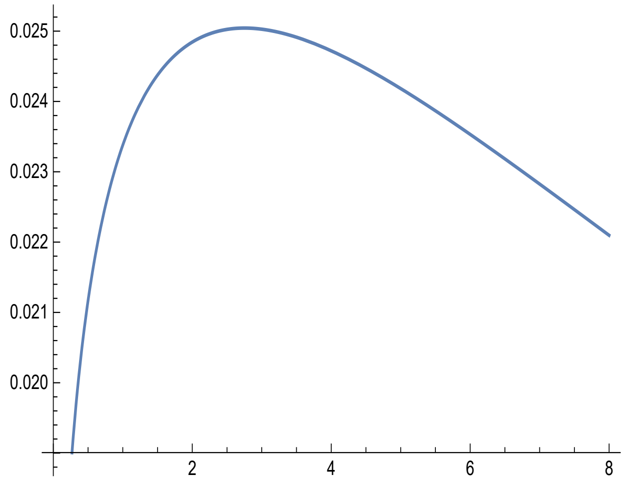

Figure 3. Fitted pdf of the TLAW distribution for the real dataset

Table 5. ML estimates of the parameters and St.E of TLAW distribution for the real data based on lower records

m	Parameter	ML estimate	St.E
5	α	1.23308	0.1192
	λ	0.25249	0.01062
	β	0.72824	0.05104
	θ	0.79554	0.08844

Table 6. Bayes estimates of the parameters and St.E from TLAW distribution for the real data based on lower records

m	Loss functions	Parameter	Bayes estimate	St.E
5	SE	α	1.09934	0.00016
		λ	0.69968	0.0001
		β	0.49835	0.00016
		θ	0.89939	0.00017
	LINEX	α	1.09761	0.00016
		λ	0.70058	0.00016
		β	0.4994	0.00013
		θ	0.90039	0.00009

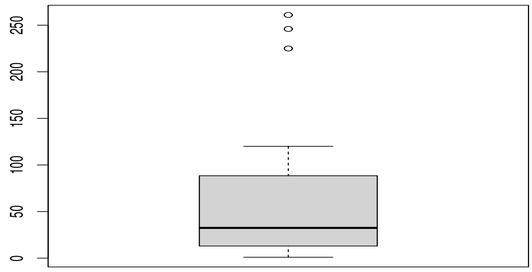

Figure 4. Boxplot for the real data set

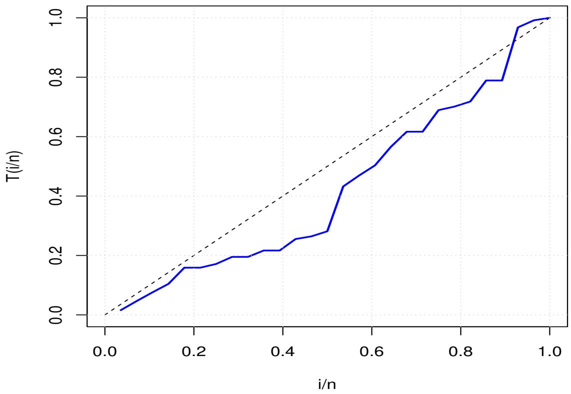

Figure 5. The empirical scaled TTT-transform plot for the real data set

6. Conclusion

In this paper, estimation for the unknown parameters of the TLAW distribution based on lower record values is discussed. The unknown parameters are estimated by applying the ML method based on lower record values. Also, the Bayes estimators are derived under the SE loss function and the LINEX loss function based on lower record values. Monte Carlo simulation is conducted to demonstrate the theoretical results of the ML estimates and the Bayes estimates are obtained by applying MCMC simulation. Finally, an application on real data set is applied.

As a future potential work, we may consider the ML and Bayesian prediction from TLAW distribution based on record values. Also, we may study the ML and Bayesian estimation for TLAW based on Type II censored sample Finally, we may characterize the ML and Bayesian prediction for TLAW based on Type II censored sample.

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