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# Position Control of Flexible Manipulator Using Nonlinear $H_\infty$ with State-Dependent Riccati Equation

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## Abstract
The paper is concerned with the control of the tip position of a single-link flexible manipulator. The non-linear model of the manipulator is derived and tested, assuming the number of model shape functions is two. It is known that the Assumed Modes Method introduces uncertainty to the model by neglecting higher order dynamics. There are other sources of uncertainty, such as friction. In addition, the model is non-linear. Therefore, for the next task, which is the controller design, the $H_\infty$ approach is proposed to deal efficiently with uncertainties, and the non-linear nature of the problem is addressed by the use of State Dependent Riccati Equation (SDRE) technique. Following the SDRE approach, the state-feedback non-linear control law is derived which minimizes a quadratic cost function. This solution is then mapped into the $H_\infty$ optimization problem. The resulting control law has been tested with the simulation model of the flexible manipulator and the results are discussed in the paper.

## 1. Introduction
Flexible manipulator systems offer several advantages comparing to the traditional rigid manipulators. The advantages include: relatively smaller actuators, lower overall mass, faster response, lower energy consumption, and, in general, lower overall cost [7]. However, the control and positioning of the flexible manipulator system is more difficult than of the rigid one. It has to take into account both the rigid body degree of freedom, and the elastic degrees of freedom. The efficiency of a single-link flexible manipulator moving at high speed and having a payload is highly dependent on its dynamic behavior and therefore of its elastic degrees of freedom. It is important to recognize the flexible nature of the manipulator and to construct a mathematical model for the system that accounts for the interactions with the actuators and the payload.
In this paper, a single link flexible manipulator is considered. Lagrangian Mechanics and the Assumed Mode Method have been used to derive a proposed dynamic model of the manipulator having a revolute joint. The link has been considered as an Euler-Bernoulli beam subjected to large angular displacement. The kinematics of a single link flexible manipulator is described based on the equivalent rigid link system and a transformation matrix method. The overall motion of the flexible link manipulator consists of the rigid body motion, which is defined by the joint angle, and the elastic motion, which is defined by the first two modal coordinates. The application of Lagrangian equation yields two sets of equations. The first set is associated with the Rigid Body degrees of freedom, and is represented by a second order non-linear differential equation. The second set is associated with the Elastic degrees of freedom and is represented by two second order non-linear differential equations. These two sets of equations of motion are coupled.

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The control problem for the flexible-link manipulator is complicated because the dynamics of the system are highly nonlinear and complex. Therefore, several control design methods have emerged, including Feedback Linearization [14], Variable Structure Control [14], Control Lyapunov Functions [15], Recursive Backstepping, Quantitative Feedback Theory [5] and Nonlinear $H_\infty$ Control [4, 12]. Although $H_\infty$ techniques were originally proposed for linear systems, the approach has been further extended to nonlinear systems. These techniques are adopted and expanded in this paper because they can provide a robustness of the controller. In particular, we will consider the State Dependent Riccati Equation (SDRE) technique [6, 8], and, following the approach outlined in [1, 2, 3], we will design the non-linear $H_\infty$ controller incorporating SDRE. Some preliminary results of combining the $H_\infty$ controller with SDRE have been reported in [16]. This paper extends these results by analyzing the differences between the proposed method and more traditional SDRE technique and by assessing the system performance The main contribution of this paper is in adopting the full state feedback nonlinear $H_\infty$ SDRE approach to the needs of the flexible manipulator system and then proving the value of this approach through tests on a fairly complex nonlinear simulation model. The rest of the paper is organized as follows: Section 2 presents the design of the nonlinear SDRE controller and the nonlinear regulator problem. In section 3, based on the theory introduced in section 2, the design of the nonlinear $H_\infty$ SDRE controller for a class of nonlinear control systems is explained. Section 4 provides a brief description of the dynamic model for a single-link flexible manipulator, and discusses application of the nonlinear controllers, introduced earlier, to this problem. In section 5 simulation tests for the single-link flexible manipulator are presented. Concluding remarks are given in section 6.

## 2. Nonlinear Control Using SDRE

Assume that the non-linear system is described by the following state-space equation:

$$\begin{aligned} \dot{x} &= f(x) + b(x)u \\ y &= g(x) \end{aligned} \tag{1}$$

where the state $x \in R^n$ , the control $u \in R^m$, and $f, b, g \in C^k, k \geq 1$. It is assumed that $f(0) = 0$, so that the origin is an equilibrium point of the open loop system. The SDRE method relies on being able to write the system dynamics (1) in a point-wise linear fashion, i.e. having a state-dependent coefficient (SDC) form.

$$\begin{aligned} \dot{x} &= A(x)x + B(x)u \\ y &= H(x)x \end{aligned} \tag{2}$$

So that $f(x) = A(x)x$ , $b(x) = B(x)$ and $g(x) = H(x)x$.

It is known that there is infinite number of ways to represent the nonlinear system in the SDC form. Associated with the SDC form the following definitions apply:

- $\{H(x), A(x)\}$ is observable (detectable) parameterization of the nonlinear system (in a given region $\Omega$) if the pair $\{H(x), A(x)\}$ is point-wise observable (detectable) in the linear sense for all $[x \in \Omega]$.
- $\{A(x), B(x)\}$ is controllable (stabilizable) parameterization of the nonlinear system (in a given region $\Omega$) if the pair $\{A(x), B(x)\}$ is point-wise controllable (stabilizable) in the linear sense for all $[x \in \Omega]$.

In the nonlinear quadratic regulator problem the aim is to minimize the infinite horizon cost function [9] of the form:

$$J = \frac{1}{2} \int_{t_0}^{\infty} \left[ x^T Q(x) x + u^T R(x) u \right] dt \tag{3}$$

where $Q(x)=H^T(x)H(x)\geq 0$, and $R(x)>0$ for all $x$, subject to the nonlinear constraint (2). It is assumed that $R,Q\in C^k, k\geq 1$. We seek a stabilizing solution in the form $u=L(x)x$ where the nonlinear feedback gain $L$ is a matrix function of the state vector ($x$). The above formulation is analogous to linear quadratic regulator (LQR) theory [10] except that the matrices $Q$, $R$ and $L$ all have elements that are allowed to be functions of the state $x$.

The SDRE approach for obtaining a suboptimal solution of the problem (3), (2) is:

- Use direct parametrization to bring the nonlinear dynamics to the form of SDC (2).
- Solve the State Dependent Riccati Equation, to obtain $P(x)$:

$$A^T(x)P(x)+P(x)A(x)-P(x)B(x)R^{-1}(x)B^T(x)P(x)+Q(x)=0 \tag{4}$$

Accepting only symmetrical and positive semi-definite solutions, i.e. $P(x)=P^T(x)\geq 0\ \forall\, x$.

- Construct the nonlinear feedback controller:

$$u=-R^{-1}(x)B^T(x)P(x)x \tag{5}$$

For some special cases, these equations can be solved analytically to produce an equation for $u$ as a function of $x$. Otherwise, they can be solved numerically at a sufficiently high sampling rate. It is clear from (5), that full state feedback has to be available in order to construct the control $u$ using this method. The approach of the local stability of the closed loop system resulting from using the SDRE nonlinear regulator technique is outlined in [2].

## 3. Combining Nonlinear H∞ Control with SDRE Method

In this section, the nonlinear $H_\infty$ suboptimal control [11], is combined with the SDRE method. Two formulations of this approach are available: the state-feedback and the output feedback [2]. Here we concentrate only on the state-feedback, as this method is used for the application to the flexible manipulator.

The non-linear system considered in this section can have a more general form than that of equation (1). In particular, in addition to the control input $u$, we consider the exogenous input signal $w\in\Re^{m_1}$ which may include tracking commands and/or disturbances. Also, two vectors of output signals are defined: $z$ is the controlled output and $y$ is the measured output. Therefore, the system is described as:

$$\dot{x}=f(x)+b_1(x)w+b_2(x)u \tag{6}$$

$$z=g_1(x)+d_1(x)u \tag{7}$$

$$y=g_2(x)+d_2(x)w \tag{8}$$

where all of the functions are smooth (*i.e.*, $C^1$), $d_1(x)$, and $d_2(x)$ have full rank, and $f(0)=0$, $g_2(0)=0$ and $b_2(x)\neq 0$ for all $x$.

Notice that the measured output $y$ will not be used for the case of state-feedback design. The above model can be represented in the State Dependent Coefficient form as follows:

$$\dot{x}=A(x)x+B_1(x)w+B_2(x)u \tag{9}$$

$$z=H_1(x)x+D_1(x)u \tag{10}$$

$$y=H_2(x)x+D_2(x)w \tag{11}$$

It is assumed that $(A,B_1),(A,B_2)$ and $(H_1A),(H_2A)$ are pointwise stabilizable and detectable in the linear sense, respectively, for $[x\in\Omega]$, where $\Omega$ is the region of interest (which may be the entire space).

The control task can be formulated as follows: For a given $\gamma \geq 0$, the system (9)-(10) is required to have the $L_2$ - gain between the signals $z$ and $u$ , less than or equal to γ, i.e.:

$$\int_0^T \left\| z(t) \right\|^2 dt \leq \gamma^2 \int_0^T \left\| w(t) \right\|^2 dt \tag{12}$$

for all $T \geq 0$ and all $w \in L_2(0,T)$. If a controller can be found such that the closed loop system is internally stable and such that the inequality (12) is satisfied, the exogenous signals will be locally attenuated by γ. The inequality (12) can be satisfied by solving the nonlinear max-min differential game problem.

$$\max_{w \in L_{2+}} \min_{u \in L_{2+}} \frac{1}{2} \int_0^\infty \left\| z(t) \right\|^2 - \gamma^2 \left\| w(t) \right\|^2 dt \tag{13}$$

subject to the constraints (9) and (10).

Invoking the Kwakernaak's Lemma [17], the max-min problem of equation (13) can be substituted by an equivalent LQ problem. In a linear case, this would lead to a design of a LQ controller by solving a suitably modified Riccati equation. In the non-linear case considered here, a suitably modified State Dependent Riccati Equation has to be solved for *P(x)*:

$$A^T(x)P(x) + P(x)A(x) - P(x)\left[B_2(x)B_2^T(x) - \frac{1}{\gamma^2}B_1(x)B_1^T(x)\right]P(x) + H_1^T(x)H_1(x) = 0$$

The parameter γ must be assumed sufficiently large in order to obtain $P(x) > 0 \;\forall\; x$. The nonlinear $H_\infty$ state feedback control is then constructed as:

$$u(x) = -B_2^T(x)P(x)\,x \tag{14}$$

The local stability of the closed loop system is determined by the following theorem from [2].

**Theorem:** Consider (9), (10) and assume $z \in \Re^s, H_1(0) = 0$. Also assume that all mappings in (9), (10) are $C^\infty$ and that $\{H_1(0), A(0)\}$ is detectable and $\{H_1(0), B_1(0)\}$ is stabilizable. Then the state feedback SDRE design procedure given by (14) yields a local solution to the nonlinear $H_\infty$ control problem (13).

# 4. Application to the Flexible Manipulator

## 4.1 Dynamic Model of the Flexible Manipulator

The flexible manipulator consist of a flexible beam positioned horizontally (axes x, y) with an electric motor attached to one end of the beam whereas the other end of the beam moves freely. The model of the flexible manipulator is obtained on basis of Lagrange equations of motion [7], which may be written as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \quad i = 1,2 \tag{15}$$

where $T$ is the kinetic energy, $V$ potential energy, $q_i$ generalized coordinate, and $Q_i$ generalized force. The application of Lagrange equation yields two sets of equations. The first set is associated with the rigid body degree of freedom defined by $\theta$, and the other set is associated with the elastic degrees of freedom defined by $\delta_i$. These two sets of equations of motion are nonlinear, coupled, second order ordinary differential equations. The generalized coordinates are shown in Figure 1.

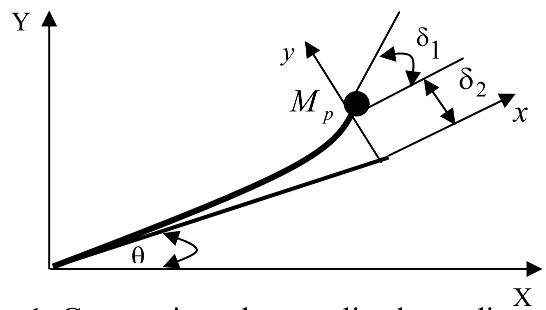



Fig. 1. Geometric and generalized coordinates of a flexible link

Under the Assumed Modes Method, retaining a finite number, $m = 2$ of modes, the dynamic equations are derived as:

$$M(\theta,\delta_1,\delta_2)\begin{bmatrix}\ddot{\theta}\\ \ddot{\delta}_1\\ \ddot{\delta}_2\end{bmatrix}+\begin{bmatrix}h_1(\dot{\theta},\delta_1,\dot{\delta}_1,\delta_2,\dot{\delta}_2)+F(\dot{\theta})\\ h_2(\dot{\theta},\delta_1,\delta_2,\dot{\delta}_2)\\ h_3(\dot{\theta},\delta_1,\delta_2,\dot{\delta}_1)\end{bmatrix}+\begin{bmatrix}0&0&0\\0&k_1&0\\0&0&k_2\end{bmatrix}\begin{bmatrix}\theta\\ \delta_1\\ \delta_2\end{bmatrix}$$
$$+\begin{bmatrix}D_1&0&0\\0&D_2&0\\0&0&D_3\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \dot{\delta}_1\\ \dot{\delta}_2\end{bmatrix}=\begin{bmatrix}u\\0\\0\end{bmatrix} \quad (16)$$

where $\delta = [\delta_1, \delta_2]^T \in \Re^2$ is the deflection vector, $\theta \in \Re$ is the joint variable, $M$ represents the inertia matrix, $h = [h_1 \;\; h_2 \;\; h_3]^T$ represents the vector of the Coriolis and centrifugal forces, $F$ is the Coulomb friction, $u$ is the control input torque, integer $m$ is the number of flexible modes (or equivalently the number of mode shape functions), in our model $m = 2$. $D = [D_1\ 0\ 0; 0\ D_2\ 0; 0\ 0\ D_3] \in \Re^{(m+1)*(m+1)}$ represents the viscous structural damping matrix, and $K = [0\ 0\ 0; 0\ k_1\ 0; 0\ 0\ k_2] \in \Re^{(m+1)*(m+1)}$ represents the stiffness matrix. Assuming that the beam deflection $d$ is small compared to the link length $L$, the normalized output may be written as $y = \theta + (d/l)$ with, $d = \sum_{i=1}^{m} \alpha_i \phi_i(l)\ \delta_i$, where $\phi_i(l)$ represents the $i^{th}$ mode shape and $\alpha_i$ represents a constant which when defining the normalized tip position (denoted by $y_t(t)$) is set to $\alpha_i = 1$. Therefore, $y_t(t)$ is given by:

$$y_t(t) = \theta + \frac{1}{l}\sum_{i=1}^{m}\phi_i(l)\delta_i \quad (17)$$

For the purpose of design, simulation, and control the dynamic equations of flexible-link manipulator can be represented in the state-space form. A state vector is defined as: $x(t) = [x_1(t) \cdots x_6(t)]^T$ where, $[x_1(t) \cdots x_6(t)]^T = [\theta\ \dot{\theta}\ \delta_1\ \dot{\delta}_1\ \delta_2\ \dot{\delta}_2]^T$ therefore, model (16) can be written as:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\end{bmatrix}=\begin{bmatrix}x_2\\ L_1(x,u)\\ x_3\\ L_2(x,u)\\ x_5\\ L_3(x,u)\end{bmatrix} \text{ and } \begin{bmatrix}L_1(x,u)\\ L_2(x,u)\\ L_3(x,u)\end{bmatrix}=[M]^{-1}\begin{bmatrix}-h_1(\dot{\theta},\delta,\dot{\delta})-D_1x_2-F(\dot{\theta})+u\\ -h_2(\dot{\theta},\delta,\dot{\delta})-D_2x_4-K_1x_3\\ -h_3(\dot{\theta},\delta,\dot{\delta})-D_3x_6-K_2x_5\end{bmatrix} \quad (18)$$

## 4.2 Application of $H_\infty$ Approach with SDRE Technique to the Tip Position Control

Considering the form of the model used for $H_\infty$ control design (equations (9) - (11)), the flexible manipulator system can be represented as below:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1(x_1,x_2,x_3,x_4,x_5,x_6,w) + g_1(x_3,x_5)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2(x_1,x_2,x_3,x_4,x_5,x_6,w) + g_2(x_3,x_5)u \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= f_3(x_1,x_2,x_3,x_4,x_5,x_6,w) + g_3(x_3,x_5)u \\
e &= x_1 - x_{ref} \\
z &= [e,u]^T
\end{aligned} \tag{19}$$

In equation (19) $w$ is the vector of uncertainties that represent the deviations of parameters from their nominal values.

For instance, the inertia matrix is a function of the load mass $m_p$. Therefore, in deriving the state-space equations, we need to take into account that the uncertainty on the load mass does not propagate throughout the system dynamics. To consider this uncertainty, one may assume that [3]:

$$m_p = m_{p0}(1 + w_1) \tag{20}$$

where $m_{p0}$ is the nominal value of the load mass and $w_1$ is an $L_2$ bounded disturbance acting on it. Note that there are several parameters that may have uncertain values, for instance: the amplitude of the sigmoidal function that models the Coulomb friction, the value of the hub damping for each joint and the value of structural damping due to link flexibility. In the nonlinear $H_\infty$ via SDRE technique the objective is to attenuate the disturbances on the controlled output when exogenous inputs have bounded energy. Therefore, any bounded signal with a compact support can enter the system as a disturbance. Consequently, in the non-affine model (19) with respect to the exogenous input $w$, all deviations must be $L_2$ bounded.

Based on the results developed in the preceding sections, the procedure for designing a nonlinear robust regulator for a flexible-link manipulator is as follows:

Step 1. Construct the state space model as in (6) - (8)
Step 2. Parameterize the model in SDC form (9) - (11)
Step 3. Solve the state feedback nonlinear $H_\infty$ SDRE algebraic Riccati equation for P.
Step 4. Construct the nonlinear $H_\infty$ feedback control via (14).

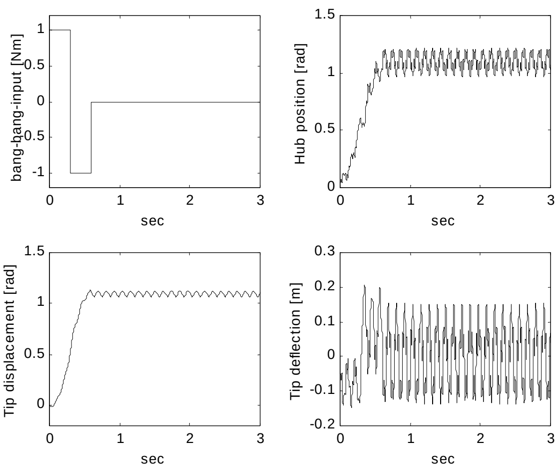
## 5. Simulations Tests

As mentioned earlier, the main objective is to control the tip position of a single-link flexible manipulator robustly. The purpose of the simulation is to demonstrate the performance of the developed model and controller algorithm in analyzing the effects of manipulator flexibility, and payload on the dynamic behavior of the system.

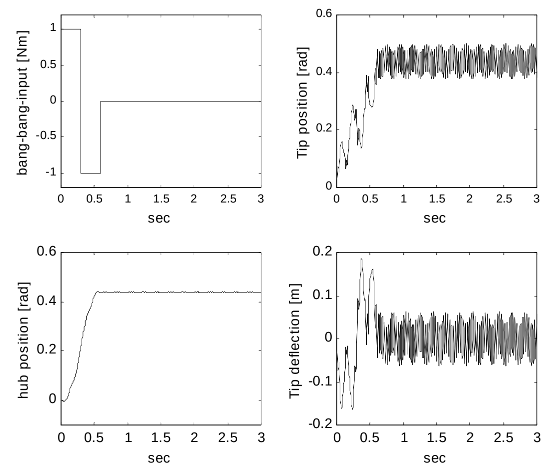
### 5.1 Simulation set-up and open loop response

Simulations were performed in Matlab/Simulink using Runge-Kutta, fourth-order numerical integration. It has been assumed that the flexible link rotates on the horizontal plane *i.e.*, the axis of rotation is vertical. The geometric and mass properties of the flexible manipulator are: length $L = 1m$, mass density $\rho = 7842\, kg/m^3$, Young Modulus $E = 2\cdot 10^{11}\, Nm$, area moment of inertia $I = 20\cdot 10^{-11}\, m^4$, cross-sectional area $A = 9\cdot 10^{-9}\, m^2$, and Link's mass $m = 0.24\, kg$.

Before developing the control design, we study the open loop response of the flexible manipulator system. The flexible manipulator is excited with a bang-bang input torque profile of amplitude 1 [Nm], shown in Fig, 2. This torque was applied at the hub of the manipulator. The system variables considered here are: the joint angle $\theta$, the tip deflection ν, and the tip position $y$ with no payload as shown in Fig, 2. Next, the effect of the payload (fixed at the free end) has been investigated, by calculating the dynamic response of the manipulator with payload to manipulator mass ratio, $m_p/m = 0.5$. **Fig, 3**, shows the dynamic response for $m_p/m = 0.5$. It can be seen that the increase of the payload will be accompanied by an increase of the elastic displacement and the residual vibration after performing a maneuver.




Fig, 2 Bang-Bang input torque of the manipulator without payload




Fig, 3. Bang-Bang input torque of the manipulator with payload 0.5 m

## 5.2 Tests with SDRE and H∞ SDRE controllers

Based on the results obtained in the preceding sections, the nonlinear $H_\infty$ SDRE controller technique was designed and implemented in Matlab/Simulink to control the output of the single-link flexible manipulator. For comparison purposes, the attenuation factor, was fixed as $\gamma = 10$. This is one of the parameters affecting the performance of the closed-loop system. This section reports some simulation results obtained for the single-link flexible arm described above via SDRE and $H_\infty$ SDRE controller technique without the addition of a payload at the free end. Fig. 4, shows the closed loop output response of the tip position, and tip deflection for a step input with amplitude of 1.0 [*rad*] for both nonlinear SDRE and nonlinear $H_\infty$ SDRE controllers with fixed state and control input weighting matrices $Q$ and $R$. As can be observed, a considerably good tracking, and small settling time of the tip position for the step input is achieved. The tip deflection is completely damped after 0.65 sec for nonlinear $H_\infty$ SDRE controller and after 1.0 sec for nonlinear SDRE technique. By adjusting the weighting matrices $Q$ and $R$ SDRE controller can achieve faster response, near, but not better than the nonlinear $H_\infty$ SDRE controller. For purpose of comparison Fig. 5, shows the hub displacement, tip displacement, control input torque, and tip deflection of the flexible manipulator system for three different cases of nonlinear $H_\infty$ SDRE design with different weighting state. Indeed, we see that the nonlinear $H_\infty$ SDRE regulator is an effective way of directly handling unstable non-minimum phase systems. It is simple to adjust the control and state weighting matrices, and, also, it offers significant design flexibility while yielding closed loop stability. As in other optimal control algorithms the controlled output may be weighted with respect to the disturbance for obtaining a faster response. Since the cost function is of quadratic type, increasing the weighting on the output state result in a more damped response, and more emphasis on rise time, decreasing the weighting on the output state result in a more overshoot response, and lower rise time as shown in Fig. 5.

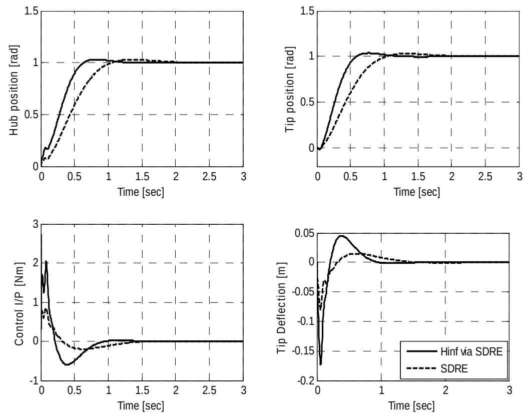



Fig. 4. Step response of flexible manipulator for SDRE and $H_\infty$ via SDRE for fixed weighting

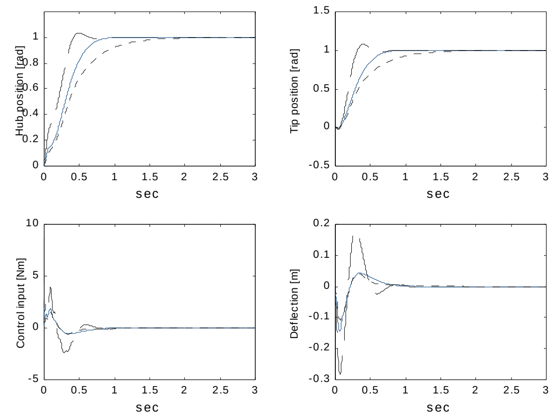



Fig. 5. Step response of flexible manipulator using $H_\infty$ via SDRE with varying weighting state

## 5.3 Control of Flexible Manipulator in the Presence of Varying Payloads

In previous sections control of the flexible manipulator was set without the effect of payload at the free end. However, the payload is a very important parameter for the design and control of a flexible manipulator. Changes in payload mass result in changes in the dynamic performance of the arm, an important objective of the manipulator mechanical and control design is to increase its payload [13].

The effect of the payload has been investigated for open loop torque control profile in Section 0.1, by calculating the dynamic response of the manipulator assuming different payload to manipulator mass ratio, *mp/m* (Fig, 3). It is anticipated, however, that the increase of payload will be accompanied by an increase of the elastic displacement and the residual vibration after performing a manoeuvre. In this section the nonlinear $H_\infty$ SDRE controller is applied to control the flexible manipulator system for three-different ratios, of the payload mass to the mass of the arm, *mp/m* = 0, 0.25, 0.5 , where: *mp* mass of the payload, and *m* mass of the flexible link. For the purpose of comparison, we use the same attenuation factor $\gamma$, and the state weighting $Q$ respectively: as for the case without payload, in previous section. But we now change the fixed payload mass at the free end of the flexible manipulator. To facilitate comparison between cases, we start the simulations from the same initial conditions, again using the same sample interval 5 *msec*, and we use $A(x)$ parameterization for all simulation cases. As a final basis of comparison, in Fig, 6, we show the plots for the three cases. In this figure, the dotted lines represent the case without payload, solid lines represent the case, *mp/m* = 0.25, and dashed lines represent case, *mp/m* = 0.5. As expected, all of the outputs of the hub displacement, and tip displacement of the three cases are asymptotically approaching the value of one, as desired.

Two things are immediately apparent from the figure. Note first of all that increasing the mass ratio, *mp/m*, increases the settling time. The second thing is the increase in the amplitude of overshoot as the ratio, *mp/m* increase. However, overall, the increase in the payload was handled sufficiently well by the nonlinear $H_\infty$ SDRE controller.

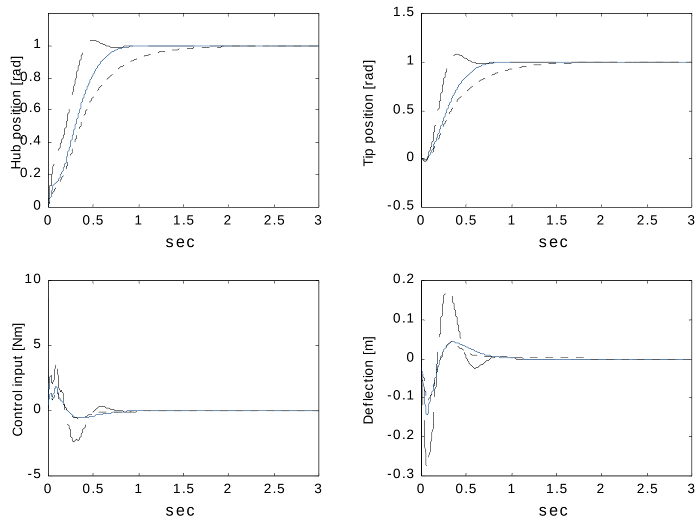



Fig. 6. Step response of flexible manipulator
using $H_\infty$ via SDRE with varying weighting state

## 6. Conclusions

The Lagrange mechanics and the assumed mode method have been used to derive a proposed dynamic model of a single-link flexible manipulator having a revolute joint. The model is valid for an arbitrary number of deflection modes. The model may be used to investigate the motion of the manipulator in the horizontal and vertical planes. The proposed model has been used to investigate the effect of two main design parameters, the payload, and the open loop control torque profile. The results of the investigation show that as long as the rest-to-rest rotational maneuver is considered, the payload has a dominant effect on the elastic deflection of the manipulator. In general, in a flexible-link manipulator, the system parameters may not be known exactly a priori. Consequently, this will introduce significant uncertainties in the robot's dynamic model. The uncertainties considered in this paper are the deviations of parameters from their nominal values. The focus was on providing a theoretical basis for the control of nonlinear systems via the state feedback nonlinear $H_\infty$ via State-Dependent Riccati Equation techniques, which, have proven quite successful in a number of simulated applications, including the control of single-link flexible manipulator. The proposed control methodology is based on minimizing the effect of the disturbance on the tip position.

Extra design degrees of freedom arising from the non-uniqueness of the State Dependent Coefficient parameterization can be utilized to enhance controller performance and the nonlinear $H_\infty$ via State-Dependent Riccati Equation method does not cancel beneficial nonlinearities. It was shown that the proposed model and controller, under certain relatively mild conditions, renders the origin a globally asymptotically stable equilibrium point. Additional results in the paper show that the regulator is near optimal.

Throughout this paper, it was assumed that all the states of the plant were available for measurement. Obviously some of these states are available via standard sensors (such as hub angle, hub velocity and tip position). Other states may require more sophisticated sensors or the introduction of observers.

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