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# DETECTION OF SINUSOIDAL SIGNALS IN FREQUENCY DOMAIN

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## ABSTRACT

Periodogram is a simple and efficient way to detect and estimate parameters of sinusoidal signals. In this paper, we evaluate the detection performance of the periodogram and its variants such as Bartlett, Welch methods, and a proposed Bartlett-based method. Performance evaluation through the Receiver Operating Characteristics (ROCs) of the mentioned methods are presented and compared. Previous studies showed that, the standard periodogram used to give the best detection performance. In the present work, analytical derivation and simulation results showed that, the proposed method gives a better detection performance outperforming the standard periodogram at the expense of tolerable reduction of the frequency resolution.

## 1. INTRODUCTION

The problem of detection and frequency estimation of sinusoidal signals from a finite number of noisy discrete time observations has applications in many fields. It has been widely used in sonar and radar where we need to detect moving targets. The periodogram is a common method for using the Fast Fourier Transform (FFT) to calculate the power spectral density (PSD) and consequently detect the signal by comparing the peak of the PSD with a predefined threshold level. Its variants include the Bartlett and Welch methods. The periodogram for the N-point sequence *x(n)* is given by [1]:

$$S_x(f_k) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\exp(-j2\pi f_k)\right|^2 \qquad , k = 0,1, ..., N-1. \qquad (1)$$

Where frequency $f_k = k / N$ . The values at other frequencies can be evaluated by either zero padding or interpolation. In the Bartlett method, the sequence *x(n)* is divided into *K* nonoverlapping segments, where each segment has length *M.* For each segment, the periodogram is computed and the Bartlett power spectrum estimate is obtained by averaging the periodograms for the *K* segments. By so doing, the variance in the periodogram estimate is reduced by a factor *K* but at the expense of reducing the frequency resolution by *K*.

Welch had modified the Bartlett's procedure by allowing the data segments to be overlapped and at the same time to be multiplied by a window function prior to computing the periodogram. The overlapping is used for further reducing the periodogram variance while the windowing is applied to reduce the spectral leakage associated with finite observation intervals [2].

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In this paper, standard periodogram with and without windowing, Bartlett and Welch methods are compared in order to determine the best detector for a single sinusoid in the presence of additive white Gaussian noise. A method derived from Bartlett method is proposed and compared with the other mentioned methods. Theoretical development for single sinusoid detection, false alarm and detection probabilities, using periodogarm, Bartlett, and the proposed method are given in section 2. Simulation results are presented in section 3 to corroborate the analytical derivations and to evaluate the detection performance of the different periodograms with the proposed method.

## 2. DETECTION PERFORMANCE

The problem of detecting a single real sinusoid from a finite number of noisy discrete time observations $x(n)$ is formulated as a test between two statistical hypotheses:

$$\begin{aligned} H_0 &: x(n) = q(n) \quad , n = 0, 1, \ldots, N-1 \\ H_1 &: x(n) = \alpha \sin(2\pi f_o n + \phi) + q(n) \end{aligned} \tag{2}$$

Where $q(n)$ is an additive white Gaussian noise with zero mean and variance $\sigma_q^2$ while $\alpha$, $f_o$, and $\phi$ are unknown constants which represent the sinusoid amplitude, frequency and phase respectively. For simplicity, we assume that $\phi = 0$. The aim is to find the ROC which is determined by the false alarm probability $P_{FA} = P(D_1/H_o)$ and the detection probability $P_D = P(D_1/H_1)$, where $D_1$ represents the decision of hypothesis $H_1$. Consequently, the Probability Density Functions (pdfs) of $S_x(f_k)$ are required. In the following sections, we shall derive these requirements for the periodogram, Bartlett, and the proposed method.

### 2.1. The Standard periodogram

The standard periodogram detector performs detection by computing the PSD as given by equation l, selecting the largest coefficient, and compares it with a threshold $V_T$. If the peak coefficient is larger than $V_T$, then $H_I$ is accepted and $H_O$ is chosen otherwise. Here, because of the symmetric power spectrum of real signals, we only need to compute the spectral coefficients for $k = 1, 2, \ldots, N/2 - 1$. For simplicity, we assume that $f_o \in (0, 0.5)$ and corresponds exactly to one of the FFT bins, $k_0$, such that $f_o = k_0 / N$.

Denoting $S_{x_0}(k/N)$ and $S_{x_1}(k/N)$ as the spectral coefficients for the noise only case and the signal present case respectively. From equation 1, we can write [1]:

$$S_{x_1}(f_o) = A^2 + B^2 \tag{3}$$

Where,

$$A = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left[\alpha \sin(2\pi f_o n) + q(n)\right] \cos(2\pi f_o n)$$

$$, B = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left[\alpha \sin(2\pi f_o n) + q(n)\right] \sin(2\pi f_o n)$$

Such that A & B are independent Gaussian random variables of identical variances equal to $\sigma^2 = \sigma_q^2/2$, where $\sigma_q^2$ is the mean power of the noise, q(n) [3]. It can be shown that, the pdf of $S_{x_1}(k_0)$ is a noncentral chi-square distribution of the form [4]:

$$P_s(u) = \frac{1}{2\sigma^2}\ \exp\left(-\frac{s^2+u}{2\sigma^2}\right) I_o\left(\sqrt{u}\ \frac{s}{\sigma^2}\right) \tag{4}$$

Where, $I_o$ is the modified Bessel function of the first kind, and S is the noncentrality parameter and equals to $\sqrt{m_1^2 + m_2^2}$ , such that, $m_1$, $m_2$ are the statistical averages of A and B respectively.

The pdfs of the remaining $S_{x_1}(k)$, $k \neq k_0$, and all $S_{x_0}(k)$ are of central chi-square distribution of the form [4]:

$$P_q(u) = \frac{1}{2\sigma^2}\ \exp\left(-\frac{u}{2\sigma^2}\right) \tag{5}$$

The derived expressions for the false alarm and detection probabilities are as follows [1]:

$$P_{FA} = 1 - \left[\int_0^{v_T} P_q(u)\,du\right]^{(N/2)-1}$$

$$P_{FA} = 1 - \left(1-\exp\left(-v_T/2\sigma^2\right)\right)^{N/2-1} \tag{6}$$

$$P_D = 1 - \left(\left[\int_0^{v_T} P_q(u)\,du\right]^{(N/2)-2} \int_0^{v_T} P_s(u)\,du\right)$$

$$P_D = 1 - \left(1-\exp\left(-v_T/2\sigma^2\right)\right)^{N/2-2}\left(1-Q_1\left(\frac{s}{\sigma}/\frac{\sqrt{v_T}}{\sigma}\right)\right) \tag{7}$$

Where, $V_T$ is the predefined threshold to achieve certain level of false alarm, and $Q_1$ is the Marcum Q function [5].

### 2.2. The Bartlett Method

When the Bartlett method is used instead of the standard periodogram, the number of the power coefficients is reduced to *M,* because of averaging, and the spectrum of Bartlett method is defined as [1]:

$$S_{xa}(m) = \frac{1}{N}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{M-1} x_i(n)\exp\left(-j\frac{2\pi nm}{M}\right)\right|^2 \tag{8}$$

Where, $x_i(n) = x(iM + n)$, $m = 0, 1, \ldots, M-1$, $i = 0, 1, \ldots, K-1$.

In this case, the spectral coefficients are expressed as:

$$S_{xa}(m) = \sum_{i=0}^{k-1} A_i^2(m) + B_i^2(m) \tag{9}$$

Where, $A_i(m) = \frac{1}{\sqrt{N}} \sum_{n=0}^{M-1} x_i(n) \cos\left(\frac{2\pi nm}{M}\right)$

$, B_i(m) = \frac{1}{\sqrt{N}} \sum_{n=0}^{M-1} x_i(n) \sin\left(\frac{2\pi nm}{M}\right)$

Where, $A_i$ and $B_i$ are independent Gaussian random variables. By further assuming that $f_o = m_0 / M$, $m_0 \in (1, M/2-1)$, and following the previous derivation, it can be shown that the pdf of $S_{xa_1}(m_0)$ is a noncentral chi-square distribution of the form:

$$P_{S_a}(u) = \frac{1}{2\sigma_a^2}\left(\frac{u}{S_a^2}\right)^{(K-1)/2} \exp\left(-\frac{S_a^2 + u}{2\sigma_a^2}\right) I_{k-1}\left(\sqrt{u}\,\frac{S_a}{2\sigma_a^2}\right) \quad (10)$$

Where, $\sigma_a^2 = \sigma_q^2 / (2K)$ & $S_a = \alpha\sqrt{M}/2$

The pdfs of the remaining $S_{xa_1}(m)$ and all $S_{xa_0}(m)$ are of central chi-square distribution of the form:

$$P_{q_a}(u) = \frac{1}{2\sigma_a^{2K}\, 2^K\, \Gamma(K)}\, u^{K-1} \exp\left(-\frac{u}{2\sigma_a^2}\right) \quad (11)$$

Where, $\Gamma(K)$ is the gamma function. As a result, the false alarm and detection probabilities of single sinusoid detection using the Bartlett method, $P_{FAa}$, and $P_{Da}$, are given by [1]:

$$P_{FAa} = 1 - \left(\int_o^{v_T} P_{q_a}(u)\,du\right)^{(M/2)-2} \quad (12)$$

$$P_{Da} = 1 - \left(\int_o^{v_T} P_{q_a}(u)\,du\right)^{(M/2)-2} \int_o^{v_T} P_{S_a}(u)\,du \quad (13)$$

### 2.3. The Proposed Method

The proposed method is derived from the Bartlett method. The main difference is that instead of computing the periodogram of each segment and then performs averaging over the K segments; we add the spectrum components of all segments first. Then we perform magnitude, squaring, and division over N. The average power spectrum of this proposed method is defined as:

$$S_{xc}(m) = \frac{1}{N}\left| \sum_{i=0}^{K-1} \sum_{n=0}^{M-1} x_i(n) \exp\left(-j\frac{2\pi nm}{M}\right) \right|^2 \quad (14)$$

Where, $x_i(n) = x(iM + n)$, $m = 0, 1, \ldots, M-1$, $i = 0, 1, \ldots, K-1$.

By so doing, we expect an increase of the obtained spectrum level due to useful signal compared to Bartlett method.

To derive an expression for the false alarm and detection probabilities of this method, we follow the same analysis used in the standard periodogram and Bartlett methods. In our case, the spectrum coefficients are defined as:

$$S_{xc}(m) = A_c^2 + B_c^2 \quad (15)$$

Where, $$A_c = \frac{1}{\sqrt{N}} \sum_{i=0}^{K-1} \sum_{n=0}^{M-1} x_i(n) \cos\left(\frac{2\pi \, n \, m}{M}\right)$$

$$, B_c = \frac{1}{\sqrt{N}} \sum_{i=0}^{K-1} \sum_{n=0}^{M-1} x_i(n) \sin\left(\frac{2\pi \, n \, m}{M}\right)$$

It can be proved [3] that A and B are independent Gaussian Random variables of identical variances of $\sigma_c^2 = \frac{\sigma_q^2}{2}$. Then we follow [4] to derive the pdf of $S_{xc}(m_0)$,denoted by $P_{sc}(u)$, as:

$$P_{sc}(u) = \frac{1}{2\sigma_c^2} \exp\left(-\frac{S_c + u}{2\sigma_c^2}\right) I_o\left(\sqrt{u}\,\frac{S_c}{\sigma_c^2}\right) \quad (16)$$

To get the value of $S_c = \sqrt{E^2(A_c) + E^2(B_c)}$ , where, $E()$ means expectation, we proceed as follow:

$$E(A_c) = E\left\{\frac{\alpha}{\sqrt{N}} \sum_{i=0}^{K-1} \sum_{n=0}^{M-1} \sin(2\pi f_o n) \cos(2\pi f_o n) + \frac{1}{\sqrt{N}} \sum_{i=0}^{K-1} \sum_{n=0}^{M-1} q(n) \cos(2\pi f_o n)\right\}$$

Since q(n) is white Gaussian process with zero mean, so, the expectation of the second term in the above equation equals zero.

$$E(A_c) = \frac{\alpha}{\sqrt{N}} \sum_{i=0}^{K-1} \sum_{n=0}^{M-1} \sin(2\pi f_o n) \cos(2\pi f_o n)$$

$$= \frac{\alpha}{\sqrt{N}} \sum_{i=0}^{K-1} \sum_{n=0}^{M-1} \frac{1}{2}[\sin(4\pi f_o n) + \sin(0)]$$

$$E(A_c) = \frac{\alpha}{2\sqrt{N}} \sum_{i=0}^{K-1} \sum_{n=0}^{M-1} \sin(4\pi f_o n) = 0 \quad (17)$$

Using the same manner:

$$E(B_c) = \frac{\alpha}{\sqrt{N}} \sum_{i=0}^{K-1} \sum_{n=0}^{M-1} \sin(2\pi f_o n) \sin(2\pi f_o n)$$

$$E(B_c) = \frac{\alpha}{2\sqrt{N}} \sum_{i=0}^{K-1} \sum_{n=0}^{M-1} \left[\cos(0) - \cos(4\pi f_o n)\right]$$

$$= \frac{\alpha}{2\sqrt{N}} \sum_{i=0}^{K-1} \sum_{n=0}^{M-1} 1 - \frac{\alpha}{2\sqrt{N}} \sum_{i=0}^{K-1} \sum_{n=0}^{M-1} \cos(4\pi f_o n)$$

$$E(B_c) = \frac{\alpha KM}{2\sqrt{N}} = \frac{\alpha\sqrt{N}}{2} \tag{18}$$

Consequently, the value of $S_c$ is given by $S_c = \frac{\alpha}{2}\sqrt{N}$

The pdfs of the remaining $S_{xc_1}(m)$ and all $S_{xc_0}(m)$ are of central chi-square distribution of the form:

$$P_{qc} = \frac{1}{2\sigma_c} \exp\left(-\frac{u}{2\sigma_c^2}\right) \tag{19}$$

The derived expressions for the false alarm and detection probabilities are as follows:

$$P_{FAc} = 1 - \left(\int_o^{v_T} P_{qc}(u)\,du\right)^{(M/2)-1}$$

$$P_{FAc} = 1 - \left(1 - \exp\left(-v_T/2\sigma_c^2\right)\right)^{(M/2)-1} \tag{20}$$

$$P_{Dc} = 1 - \left(\int_o^{v_T} P_{qc}(u)\,du\right)^{(M/2)-2} \left(\int_o^{v_T} P_{sc}(u)\,du\right)$$

$$P_{Dc} = 1 - \left(1 - \exp\left(-v_T/2\sigma_c^2\right)\right)^{(M/2)-2} \left(1 - Q_1\left(\frac{S_c}{\sigma_c}, \frac{\sqrt{v_T}}{\sigma_c}\right)\right) \tag{21}$$

By performing simple calculations in equations 6, 7, 20, and 21, for a threshold value, $V_T$, of 10.3 and K= 4 segments, the false alarm and detection probabilities of the proposed method are $10^{-3}$ and 89% respectively, while those of the standard periodogram are $4 \times 10^{-3}$ and 89% respectively. It is of interest to test the performance of the proposed method for different segment values. By choosing K=2 segments and for the same value of $V_T$, $P_{FAc}$ and $P_{Dc}$ were found to be $2 \times 10^{-3}$ and 89% respectively. Consequently, the detection performance of the proposed method outperforms that of the standard periodogram for the assumed conditions by reducing the false alarm probability by a factor of K, while maintaining the probability of detection but at the expense of reducing the frequency resolution by a factor of K. We applied this proposed method in a radar Moving Target Detection (MTD) algorithm, the results obtained were very promising [6].

## 3 COMPUTER SIMULATIONS

Computer experiments were performed to verify the theoretical derivations. The performance of different periodograms, including the windowed periodogram, the Bartlett and Welch methods, and the proposed method in single sinusoid detection are evaluated. The variance of the additive white Gaussian noise was fixed to unity while different Signal-to-Noise-Ratios (SNRs) were produced by properly scaling the sinusoidal amplitude, $\alpha$.. *T*he phase $\phi$ was set to 0. The total sample size *N* was 256 and the segment length *M* had a value of 64. Results for detection were averaged over 10,000 independent trials.

Figs. 1 and 2 show the experimental and theoretical ROCs in detecting a real sinusoid in the presence of additive white Gaussian noise using the standard periodogram, the Bartlett (K=4), and the proposed (K=4) methods. The simulation results were obtained by using the method suggested in [7] and used effectively in [8-9]. Two values of SNR, namely, -9 dB and -12 dB were tried. The frequency $f_o$ was chosen as 0.25**.** It can be seen that the simulation results agreed in a good sense with the theoretical derivation. The reason that all experimental curves are approximately above that of theoretical ones is the finite number of trials. It can be seen that the proposed method provides a better detection performance than the standard periodogram and the Bartlett method.

Fig. 3 plots the theoretical ROCs of the periodogram and the Bartlett method for different segment length M at SNR=-9dB. It is observed that the detection performance decreased as the number of segment K increased. On the other hand, and to show the superiority of the proposed method, Fig. 4 plots the theoretical ROCs of the periodogram and the proposed method for different segment length M at SNR=-9dB. It is observed that the detection performance of the proposed method increased as the number of segment K increased outperforming that of the periodogram.

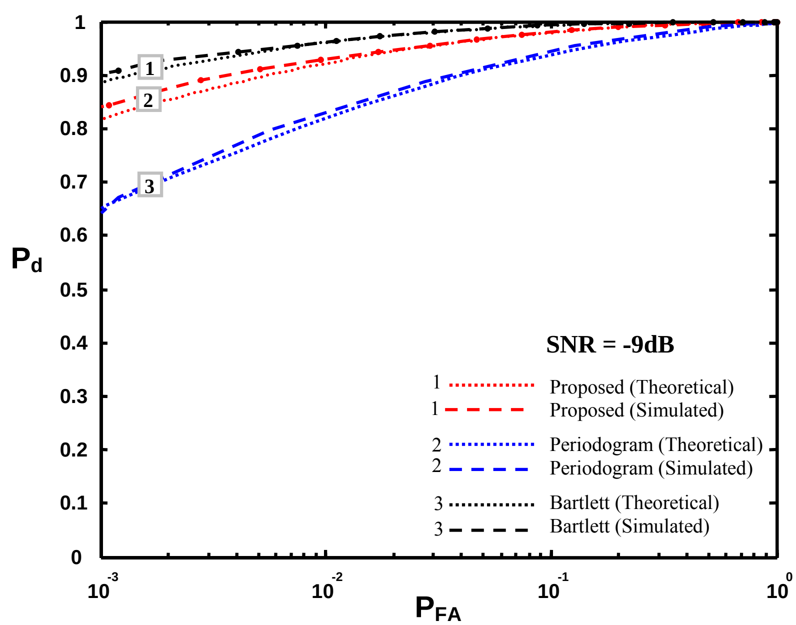


**Fig.1** Theoretical and simulated ROCs for detecting single sinusoid with frequency $f_o$ = 0.25 and SNR= -9dB using periodogram, Bartlett, and The proposed method.

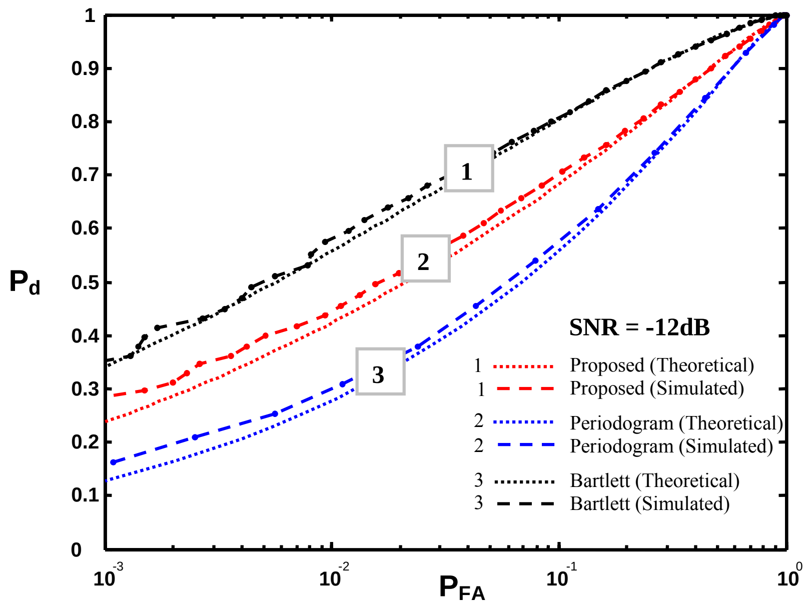


**Fig.2** Theoretical and simulated ROCs for detecting single sinusoid with frequency fo = 0.25 and SNR= -12 dB using periodogram, Bartlett, and The proposed method.

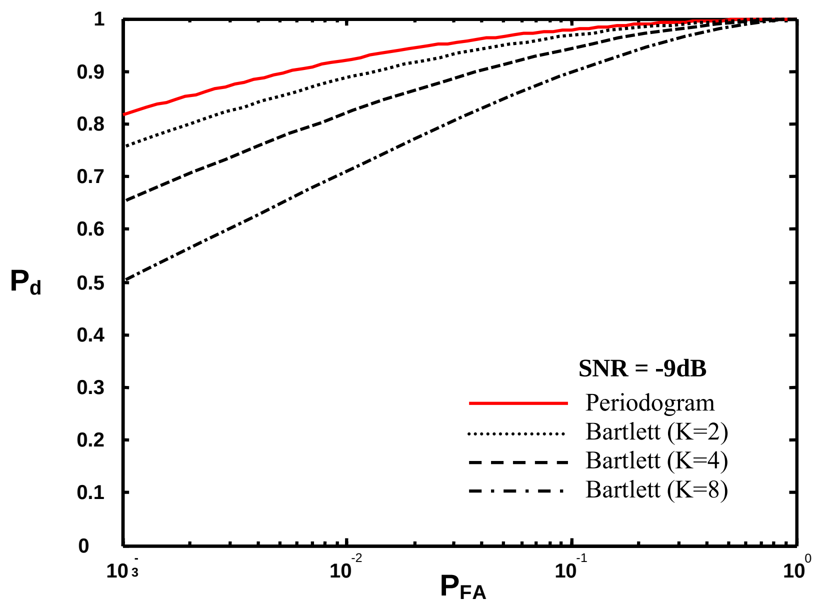


**Fig.3** Comparison of theoretical ROCs of the periodogram and Bartlett method with different segment lengths for detecting single sinusoid with frequency $f_o$ = 0.25 and SNR = -9dB.

Fig. 5 compares the detection performance of different simulated forms of periodograms with the proposed method for $f_o$ = 0.25 and SNR = -9dB. In the windowed periodogram, the Hann window function was used. Again, the proposed method gave the best detection performance followed by the periodogram, Welch, Bartlett, and the windowed periodogram.

To conclude, the proposed method provides a better detection performance for single sinusoid detection outperforming the conventional periodogram and all its variants for the pre-assumed conditions.

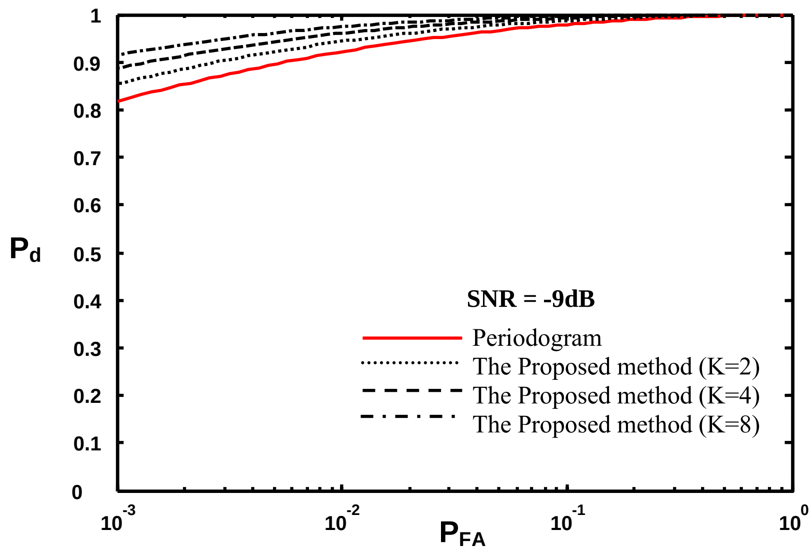


**Fig.4** Comparison of the theoretical ROCs of the periodogram and the proposed method with different segment lengths for detecting single sinusoid with frequency $f_o$ = 0.25 and SNR = -9dB.

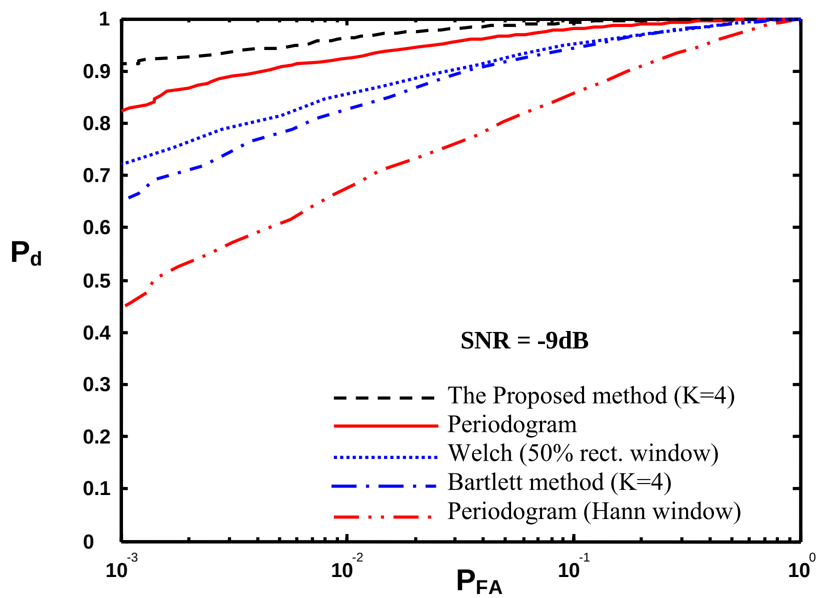


**Fig.5** Comparison of the simulated ROCs for detecting single sinusoid with frequency $f_o$ = 0.25 and SNR = -9dB using the proposed method, periodogram, windowed periodogram, Bartlett, and Welch methods.

## 4 CONCLUSIONS

The theoretical performance of detecting a real sinusoid immersed in additive white Gaussian noise using the standard periodogram, Bartlett method, and a proposed Bartlett-based method are derived and confirmed by computer simulations. Also, computer simulations were provided to compare the detection performances of the proposed method with the standard periodogram and its variants including windowed periodogram and Welch method. For the assumed conditions and when the sinusoid frequency corresponds to one of the spectral bins, the proposed method gives the best detection performance over all the other mentioned methods by reducing the false alarm probability by a factor of K less than that of the standard periodogram, while maintaining the probability of detection but at the expense of reducing the frequency resolution by a factor of K.

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