ALEXANDRIA JOURNAL OF SCIENCE AND TECHNOLOGY

Alexandria Journal of Science and Technology

Article

On a delay implicit functional integro-differential equation

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KEYWORDS: measure of noncompactness; Darbo fixed point Theorem; existence of solutions; compact in measure; uniqueness of the solution; continuous dependence.

Received: November 22, 2023
Accepted: January 04, 2024
Published: January 16, 2024

ABSTRACT: In this work, we use the De Blasi measure of noncompactness and Darbo fixed point Theorem to study the existence of solutions for an initial value problem of a delay implicit functional integro-differential equation. The sufficient condition for the uniqueness of the solution will be given. The continuous dependence of the unique solution on some data will be studied.

1. INTRODCTION

The study of implicit differential and integral equations has received much attention over the last 30 years or so. For papers studying such kind of problems (see \cite{15,16,17,18})

and the references therein.

For the theoretical results concerning the existence of solutions, in the classes of continuous or integrable functions, you can see Bana´s [18–21]. Each of these monographs contains some existence results, and the main objective is to present a technique to obtain some results concerning various integral equations.

Here we are concerning with the initial value problem of the delay implicit functional integro-differential equation

$$\frac{dx}{dt} = f\left(t, \frac{dx}{dt}, \int_0^{\phi(t)} g\big(s, x(s)\big)\mathrm{ds}\right), \ \text{a.e } t \in (0,1) \qquad (1)$$

with the initial data

$$x(0) = x_0, \qquad (2)$$

Let $\frac{dx}{dt} = y(t)$, then the solution of the problem (1)-(2) can be given by

$$x(t) = x_0 + \int_0^t y(s)ds, \qquad (3)$$

where y is the solution of the functional integral equation

$$y(t) = f\left(t, y(t), \int_0^{\phi(t)} g\left(s, x_0 + \int_0^t y(\theta)d\theta\right)\right). \qquad (4)$$

We study the existence of nondecreasing solutions $y \in L_1[0,1]$ of the integral equation (4) will be studied by the De Blasi measure of noncompactness [1], Hausdorff measure of noncompactness χ[2] and Darbo fixed point Theorem [4]. The sufficient condition for the uniqueness of the solution will be given. The continuous dependence of the unique solution on the initial data x_0 and on the functions g and ϕ will be studied.

Consequently, the existence of absolutely continuous solution $x \in AC[0,1]$, the unique solution and the continuous dependence of the unique solution on the initial data x_0 and on the functions g and ϕ of the problem (1)-(2) will be studied.

We arrange our article just like that: Section 2 the Some properties and theories used. In Section 3, contains the solvability for the existence of the solutions x $\in AC[0,1]$. Moreover, we study the unique solution $x \in AC[0,1]$ of the problem (1)-(2) and its continuous dependence on the initial data x_0 and on the functions g and ϕ of the problem (1)-(2). Some examples in Section 4.

2. Preliminaries

Let $L_1 = L_1(I)$ be the class of Lebesgue integrable functions on the interval , $I = [0,1]$, with the standard norm

$$\| x \|_1 = \int_0^1 |x(t)|dt.$$

Theorem 1. Let x be a bounded subset of L_1. Assume that there is a family of measurable subsets $(\Omega_c) 0 \leq c \leq b-a$ of the interval (a,b) such that meas $\Omega_c = c$. If for every $c \in [0, b-a]$ and for every

$$x \in X, \qquad x(t_1) \leq x(t_2), \qquad (t_1 \in \Omega_c, t_2 \notin \Omega_c)$$

then, the set $\boldsymbol{x}$ is compact in measure.

Online ISSN: 2974-3273

DOI: 10.21608/AJST.2024.250598.1021

3. Continuous dependence

Consider now the initial value problem (1)-(2) under the following assumptions:

(i) $\phi: I \to I$, $\phi(t) \le t$ is continuous and increasing.

(ii) $f: I \times R \times R \to R^{+}$ is a Carath´eodory function which is measurable t $\in I$ for

any $x, y \in R \times R$ and continuous in $x, y \in R \times R$ for all t $\in I$ and there exists a measurable and bounded function $m_1 : I \to R$ and a positive constant b_1 such that

$$|f(t,x,y)| \le m_1 + b_1(|x| + |y|).$$

Moreover, f is nondecreasing for every nondecreasing x, y i.e. for almost all $t_1, t_2 \in I^2$

such that $t_1 \le t_2$ and for all $x(t_1) \le x(t_2)$ and $y(t_1) \le y(t_2)$ implies

$$\mathrm{f}(t_1, x(t_1), y(t_1)) \le \mathrm{f}(t_2, x(t_2), y(t_2)).$$

(iii) $g: I \times I \times R \to R^{+}$ is a Carath´eodory function which is measurable in $t \in I$ for any $x \in R$ and continuous in $x \in R$ for all $t \in I$. Moreover, there exist an integrable function $m_2 : I \to R$ and a positive constant b_2 such that

$$|g(t,s,x)| \le m_2 + b_2|x|.$$

(iv) $b_1 + b_1 b_2 < 1$.

Now, the following lemma can be proved.

Lemma 1. The problem (1)-(2) is equivalent to the integral equation

$$x(t) = x_0 + \int_0^t y(s)ds,$$

Where

$$y(t) = f\left(t, y(t), \int_0^{\phi(t)} g\left(s, x_0 + \int_0^t y(\theta)d\theta\right)\right).$$

Now, we have the following existences theorem.

Theorem 2. Let the assumptions (i)-(iv) be satisfied, then the equation (4) has at least one nondecreasing solution $x \in \mathrm{AC}(I)$.

Proof. Let Q_r be the closed ball of all nondecreasing function on I

$$Q_r = \{ y \in L_1(I) : \| y \| \le r\},\ r = \frac{\|m_2\|_1 b_1 + b_1 b_2 |x_0| + \|m_1\|_1}{1-(b_1+b_1 b_2)}$$

and the supper position operator F

$$Fy(t) = f\left(t, y(t), \int_0^{\phi(t)} g\left(s, x_0 + \int_0^t y(\theta)d\theta\right)\right).$$

Then, we deduce that F transforms the nondecreasing functions into functions of the same type.

From our assumptions and by Theorem 1. Q_r is compact in measure.

Now, let $y \in Q_r$, then

$$|Fy(t)| = | f(t, y(t), \int_0^{\phi(t)} g(s, x_0 + \int_0^t y(\theta)d\theta))|$$

$$\le |m_1(\mathrm{t})| + b_1\left(|y(t)| + \left|\int_0^{\phi(t)} g\left(s, x_0 + \int_0^t y(\theta)d\theta\right)\right|\right)$$

$$\le |m_1(\mathrm{t})| + b_1 |y(t)| + b_1 \int_0^{\phi(t)} |m_2(\mathrm{s})|\mathrm{ds} + b_1 b_2 |x_0| \int_0^{\phi(t)} \mathrm{ds}$$

$$+ b_1 b_2 \int_0^{\phi(t)} \int_0^s y(\theta)d\theta \mathrm{ds}$$

$$\le |m_1(\mathrm{t})| + b_1 |y(t)| + b_1 \| m_2 \|_1 + b_1 \mathrm{b}2|x_0| + b_1 b_2 \| y \|$$

$$\le |m_1(\mathrm{t})| + b_1 |y(t)| + b_1 \| m_2 \|_1 + b_1 \mathrm{b}2|x_0| + b_1 b_2 r$$

and

$$\int_0^1 |Fy(t)|dt = \int_0^1 |m_1(\mathrm{s})|\mathrm{ds} + b_1 \int_0^1 |y(s)|ds + (b_1 \| m_2 \| + b_1 b_2 |x_0| + b_1 b_2 \mathrm{r}) \int_0^1 dt,$$

Then

$$\| Fy \|_1 \le \| m_1 \|_1 + b_1 \| y \|_1 + \| m_2 \|_1 b_1 + b_1 b_2 |x_0| + b_1 b_2 r$$

$$\le \| m_1 \|_1 + b_1 r + \| m_2 \|_1 b_1 + b_1 b_2 |\mathrm{x}_0| + b_1 b_2 r = r.$$

Now, let $\{y_n\} \subset Q_r$, and $y_n \to y$, then

$$Fy_n(t) = f\left(t, y_n(t), \int_0^{\phi(t)} g\left(s, x_0 + \int_0^t y_n(\theta)d\theta\right)\right)$$

and

$$\lim_{n\to\infty} Fy(t) = \lim_{n\to\infty} f\left(t, y_n(t), \int_0^{\phi(t)} g\left(s, x_0 + \int_0^t y_n(\theta)d\theta\right)\right).$$

Applying Lebesgue dominated convergence Theorem [15], then from our assumptions we get

$$\lim_{n\to\infty} Fy_n(t) = f\left(t, \lim_{n\to\infty} y_n(t), \int_0^{\phi(t)} g\left(s, x_0 + \int_0^t \lim_{n\to\infty} y_n(\theta)d\theta\right)\right)$$

$$= f\left(t, y(t), \int_0^{\phi(t)} g\left(s, x_0 + \int_0^t y(\theta)d\theta\right)\right) = Fy(t).$$

This means that $Fy_n(t) \to Fy(t)$. Hence the operator F is continuous.

Taking Ω be a non empty subset of Q_r. Let $\epsilon > 0$ be fixed number and take a measurable set $D \subset I$ such that measure $D \le \epsilon$. Then, for any $y \in \Omega$,

$$\| Fy \|_{L_1(D)} = \int_D |Fy(t)|\,dt \le \int_D f\left(t, y(t), \int_0^{\phi(t)} g\left(s, x_0 + \int_0^t y(\theta)\,d\theta\right)\right)ds$$

$$\le \int_D |m_1(\mathrm{t})|\mathrm{dt} + b_1 \int_D |y(t)|dt + b_1 \int_D \int_0^{\phi(t)} g(s, x_0 + \int_0^s |y(\theta)|\,d\theta\,ds$$

$$\le \int_D |m_1(t)|dt + b_1 \int_D |y(t)|dt + b_1 \int_D \int_0^{\phi(t)} (|m_2(s)| + b_2 |x_0 \int_0^s y(\theta)|d\theta)ds$$

$$\le \int_D |m_1(t)|dt + b_1 \int_D |y(t)dt| + b_1 \int_D \int_0^{\phi(t)} |m_2(s)|\,dsdt$$

$$+ b_1 b_2 |x_0| \int_D \int_0^{\phi(t)} dsdt + b_1 b_2 \int_D \int_0^{\phi(t)} \int_0^s |y(\theta)|\,d\theta dsdt.$$

But for $D \subset I$ with meas $D < \epsilon$, we have

$$\lim_{e\to 0} \sup \int_D |m_1(\mathrm{t})|\mathrm{dt} = 0, \lim_{e\to 0} \sup \int_D |m_2(\mathrm{t})|\mathrm{d} = 0 \text{ and } \lim_{e\to 0} \sup b_1 b_2 |x_0| \int_D \phi(\mathrm{t}) = 0.$$

Then applying the De Blasi measure of noncompactness [1,2,4,8].

$$\beta(Fy(t)) \le \beta(y(t))(b_1 + b_1 b_2)$$

and

$$\beta(F\Omega) \le \beta(\Omega(t))(b_1 + b_1 b_2)$$

Then implies

$$\chi(F\Omega) \le (b_1 + b_1 b_2)\chi(\Omega),$$

Where χ is the Hausdorff measure of noncompactness [1,2,4,8]. Since b1 $b_1 + b_1 b_2 < 1$, from Darbo fixed point Theorem [4] F is a contraction with regard to the measure of noncompactness χ [4] and has at least one fixed point in $y \in Q_r$. Then there exist at least one solution $y \in L_1(I)$ of equation (4). Consequently there exists at least one solution

$x \in AC(I)$ of the problem (1)-(2).

3.1 Uniqueness of the solution

Now, consider the following assumptions:

(ii)* $f : I \times R \times R \to R$ is measurable in $t \in I\ \forall\, x, y \in R$ and satisfies Lipschitz condition,

$$|f(t,x,y) - f(t,x_1,y_1)| \leq b_1(|x - x_1| + |y - y_1|),\ t \in I\ \forall x, y, x_1, y_1 \in R$$

and $\int_0^1 |f(t,0,0)|dt$ exists. Moreover, f is nondecreasing for every nondecreasing x, y i.e. for almost all $t_1, t_2 \in I^2$such that $t_1 \leq t_2$ and for all $x(t_1) \leq x(t_2)$ and $y(t_1) \leq y(t_2)$ impales $f\big(t_1, x(t_1), y(t_1)\big) \leq f\big(t_2, x(t_2), y(t_2)\big)$.

(iii)* $g : I \times R \to R$ is measurable $in\ t \in I$ and satisfies Lipschitz condition,

$$|g(t,x) - g(t,y)| \leq b_2|x - y|, \qquad t \in I, \qquad x, y \in R.$$

From the assumption (ii)* we have

$$|f(t,x,y)| - |f(t,0,0)| \leq |f(t,x,y) - f(t,0,0)| \leq b_1(|x| + |y|) \leq |f(t,0,0)| + b_1(|x| + |y|)$$

and

$$|f(t,x,y)| \leq \| m_1 \|_1 + b_1(|x| + |y|),$$

Also, from the assumption (iii)* we get

$$|g(t,x)| - |g(t,0)| \leq |g(t,x) - g(t,0)| \leq b_2|x|,$$

and

$$|g(t,x)| \leq |g(t,0)| + b_2|x|$$

and

$$|g(t,x)| \leq \| m_2 \|_1 + b_2|x|.$$

So, we have proved the following Lemma.

Lemma 2. The assumptions (ii) * and (iii) * implies the assumptions (ii) and (iii) respectively.

Theorem 3. Let the assumptions $(i), (ii)*, (iii)*\ and\ (iv)$ be satisfied. If

$$b_1 + b_1 b_2 < 1,$$

Then the solution of the problem$(1) - (2)$ is unique.

Proof. Form Lemma2 the assumptions of Theorem1 are satisfied and the solution of integral equation(4) exists. Let y_1, y_2 be two solutions in Q_r of the integral equation(4), then

$$|y_2(t) - y_1(t)| = \Big| f\Big(t, y_2(t), \int_0^{\phi(t)} (s, x_0 + \int_0^s y_2(\theta)d\theta)ds\Big) - f\Big(t, y_1(t), \int_0^{\phi(t)} (s, x_0 + \int_0^s y_1(\theta)d\theta)ds\Big)\Big|$$

$$\leq \Big| f\Big(t, y_2(t)), \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_2(\theta)d\theta\Big)ds - f\Big(t, y_2(t), \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_1(\theta)d\theta)ds\Big) + f\Big(t, y_2(t), \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_1(\theta)d\theta)ds\Big) - f\Big(t, y_1(t), \int_0^{\phi(t)} (s, x_0 + \int_0^s y_1(\theta)d\theta)ds\Big)\Big|$$

$$\leq b_1 b_2 \int_0^{\phi(t)} \int_0^s |y_2(\theta) - y_1(\theta)| d\theta ds$$

$$\leq b_1 b_2 \| y_2 - y_1 \|_1 + b_1 |y_2(t) - y_1(t)|,$$

Then

$$\int_0^1 |y_2(t) - y_1(t)| dt \leq b_1 b_2 \| y_2 - y_1 \|_1 \int_0^1 dt + b_1 \int_0^1 |y_2(t) - y_1(t)| dt$$

and

$$\| y_2 - y_1 \|_1 \leq b_1 b_2\ \| y_2 - y_1 \|_1 + b_1 \| y_2 - y_1 \|_1.$$

Hence

$$\| y_2 - y_1 \|_1 (1 - (b_1 + b_1 b_2)) \leq 0,$$

Then $y_1 = y_2$ and the solution of the integral equation(4) is unique. Consequently the solution of the problem (1)-(2) is unique.

3.2 Continuous dependence

Definition1. The solution of the initial value problem (1)-(2) depends continuously on the parameter x_0, if

$$\forall \epsilon > 0, \qquad \exists\, \delta(\epsilon) > 0\ s.t\ |x_0 - x_0^*| < \delta \Rightarrow \| x - x^* \| < \epsilon.$$

Where x^*

$$x^*(t) = x_0^* + \int_0^t y^*(s)ds.$$

Theorem 4. Let the assumptions of Theorem 3 be satisfied, then the unique solution of the problem (1)-(2) depends continuously on the parameter x_0.

Proof. Let $\delta > 0$ be given such that $|x_0 - x_0^*| \leq \delta$ and let x_0^* be the solution of (1)-(2) corresponding to initial value x_0^*, then

$$|x(t) - x^*(t)| = |x_0 + \int_0^t y(s)ds - x_0^* - \int_0^t y^*(s)ds|$$

$$\leq |x_0 - x_0^*| + \int_0^t |y(s) - y^*(s)| ds \leq \delta + \| y - y^* \|_1$$

But

$$|y(t) - y^*(t)| = \Big| f\Big(t, y(t), \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)\, ds\Big) - f\Big(t, y^*(t), \int_0^{\phi(t)} g(s, x_0^* + \int_0^s y^*(\theta)d\theta)\, ds\Big)\Big|$$

$$\leq \Big| f\Big(t, y(t), \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)\, ds\Big) - f\Big(t, y(t), \int_0^{\phi(t)} g(s, x_0^* + \int_0^s y^*(\theta)d\theta)\, ds\Big) + f\Big(t, y(t), \int_0^{\phi(t)} g(s, x_0^* + \int_0^s y^*(\theta)d\theta)\, ds\Big) - f\Big(t, y^*(t), \int_0^{\phi(t)} g(s, x_0^* + \int_0^s y^*(\theta)d\theta)\, ds\Big)\Big|$$

$$\leq b_1 b_2 \int_0^{\phi(t)} |x_0 - x_0^*|\, ds + b_1 b_2 \int_0^{\phi(t)} \int_0^s |y(\theta) - y^*(\theta)|\, d\theta ds + b_1 |y(t) - y^*(t)|$$

$$\leq b_1 b_2 \delta + b_1 b_2 \| y - y^* \|_1 + b_1 |y(t) - y^*(t)|,$$

Then

$$\int_0^1 |y(t) - y^*(t)|\, dt \leq b_1 b_2 \delta \int_0^1 dt + b_1 b_2 \| y - y^* \|_1 \int_0^1 dt + b_1 \int_0^1 |y(t) - y^*(t)| dt$$

and

$$\| y - y^* \|_1 \leq \delta\, b_1 b_2 + b_1 b_2\ \| y - y* \|_1 + b_1 \| y - y* \|_1.$$

Hence

$$\| y - y^* \|_1 (1 - (b_1 + b_1 b_2)) \leq \delta\, b_1 b_2,$$

Then

$$\| y - y^* \|_1 \leq \frac{\delta\, b_1 b_2}{(1 - (b1 + b_1 b_2))} = \epsilon_1$$

and

$$\| x - x^* \|_C \leq \delta + \epsilon_1 = \epsilon.$$

Definition 2. The solution x of the initial value problem (1)-(2) depends continuously on the function g, if

$\forall\epsilon > 0, \exists\, \delta(\epsilon) > 0\ s.t\ |g(t,x) - g*(t,x)| < \delta \Rightarrow \| x - x^* \| < \epsilon.$

Theorem 5. Let the assumptions of Theorem 3 be satisfied, then the unique solution of the problem (1)-(2) depends continuously on the function g.

Proof. Let δ > 0 be given such that $|g(t,x(t)) - g*(t,x(t))| \le \delta$ and let $x*$ be the solution of (1)-(2) corresponding to $g*(t,x(t))$, then

$$|x(t) - x^*(t)| = |x_0 + \int_0^t y(s)ds - x_0 - \int_0^t y^*(s)ds| \le \int_0^t |y(s) - y^*(s)|ds \le \| y - y^* \|_1$$

But

$$|y(t) - y^*(t)| = |f(t, y(t), \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)\, ds) - f(t, y^*(t), \int_0^{\phi(t)} g*(s, x_0 + \int_0^s y^*(\theta)d\theta)\, ds)|$$

$$\le |f(t, y(t), \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)\, ds) - f\left(t, y(t), \int_0^{\phi(t)} g*(s, x_0 \int_0^s y(\theta)d\theta)\, ds\right) + f\left(t, y(t), \int_0^{\phi(t)} g*(s, x_0 \int_0^s y(\theta)d\theta)\, ds\right) - f(t, y^*(t), \int_0^{\phi(t)} g*(s, x_0 + \int_0^s y^*(\theta)d\theta)\, ds)|$$

$$\le \delta\, b_1 + b_1|y(t) - y*(t)| + b_1 b_2 \int_0^{\phi(t)} |y(t) - y*(t)|dt,$$

Then

$$\int_0^1 |y(t) - y^*(t)|\, dt \le b_1\delta \int_0^1 dt + b_1 \int_0^1 |y(t) - y*(t)|\, dt + b_1 b_2 \| y - y^* \|_1 \int_0^1 dt$$

and

$$\| y - y^* \|_1 \le \delta\, b_1 + b_1 \| y - y^* \|_1 + b_1 b_2 \| y - y^* \|_1.$$

Hence

$$\| y - y^* \|_1 (1 - (b_1 + b_1 b_2)) \le \delta\, b_1,$$

Then

$$\| y - y^* \|_1 \le \frac{\delta\, b_1}{(1 - (b_1 + b_1 b_2))} = \epsilon$$

and

$$\| x - x^* \|_C \le \epsilon.$$

Definition 3. The solution x of the initial value problem (1)-(2) depends continuously on the function φ, if

$\forall\epsilon > 0,\ \exists\, \delta(\epsilon) > 0\ s.t\ |\varphi(t) - \varphi*(t)| < \delta \Rightarrow \| x - x^* \| < \epsilon.$

Theorem 6. Let the assumptions of Theorem 3 be satisfied, then the unique solution of the problem (1)-(2) depends continuously on the delay function φ.

Proof. Let δ > 0 be given such that |φ(t) − φ*(t)| ≤ δ and let x^* be the solution of (1)-(2) corresponding to $\varphi*(t)$, then

$$|x(t) - x^*(t)| = |x_0 + \int_0^t y(s)ds - x_0 - \int_0^t y^*(s)ds \le \int_0^t |y(s) - y^*(s)ds| \le \| y - y^* \|_1$$

But

$$|y(t) - y^*(t)| = |f\left(t, y(t), \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)\, ds\right) - f(t, y^*(t), \int_0^{\phi*(t)} g(s, x_0 + \int_0^s y^*(\theta)d\theta)\, ds)|$$

$$\le |f\left(t, y(t), \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)\, ds\right) - f\left(t, y(t), \int_0^{\phi*(t)} g(s, x_0 \int_0^s y*(\theta)d\theta)\, ds\right) + f\left(t, y(t), \int_0^{\phi*(t)} g(s, x_0 \int_0^s y*(\theta)d\theta)\, ds\right) \quad + b_1 |y(t) - y*(t)| - f(t, y^*(t), \int_0^{\phi*(t)} g(s, x_0 + \int_0^s y^*(\theta)d\theta)\, ds|$$

$$\le b_1 \left| \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)\, ds - \int_0^{\phi*(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)\, ds \right|$$

$$\le b_1 \left| \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)\, ds - \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)\, ds + \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)\, ds - \int_0^{\phi*(t)} g(s, x_0 + \int_0^s y*(\theta)d\theta)\, ds \right| + b_1 |y(t) - y*(t)|$$

$$\le b_1 b_2 \int_0^{\phi(t)} \int_0^s |y(\theta) - y^*(\theta)|\, d\theta ds + b_1 \int_{\phi*(t)}^{\phi(t)} |g(s, x_0 + \int_0^s y*(\theta)d\theta)\, ds + b_1 |y(t) - y*(t)|,$$

$$\le b_1 b_2 \| y - y^* \|_1 + b_1(\| m_2 \| + b_2 |x_0| + b_2 r)|\phi(t) - \phi*(t)| + b_1|y(t) - y^*(t)| \le b_1 b_2 \| y - y^* \|_1 + b_1(\| m_2 \| 1 + b_2 |x_0| + b_2 r)\delta + b_1 |y(t) - y^*(t)|,$$

Then

$$\int_0^1 |y(t) - y^*(t)|\, dt \le b_1 b_2 \| y - y^* \|_1 \int_0^1 dt + b_1(\| m_2 \| + b_2 |x_0| + b_2 r)\delta \int_0^1 dt + b_1 \int_0^1 |y(t) - y*(t)|dt$$

and

$$\| y - y^* \| L_1(I) \le b_1 b_2 \| y - y^* \|_1 + b_1(\| m_2 \| + b_2 |x_0| + b_2 r)\delta + b_1 \| y - y^* \|_1.$$

Hence

$$\| y - y^* \|_1 \left(1 - (b_1 + b_1 b_2)\right) \le b_1(\| m_2 \| + b_2 |x_0| + b_2 r)\delta,$$

Then

$$\| y - y^* \|_1 \le \frac{b_1(\| m_2 \| + b_2 |x_0| + b_2 r)\delta}{(1 - (b_1 + b_1 b_2))} = \epsilon$$

and

$$\| x - x^* \|_C \le \epsilon.$$

Example 1. Consider the following initial value problem

$$\frac{dx}{dt} = \frac{3t}{40} + \frac{1}{4}\frac{dx}{dt} + \frac{1}{4}\int_0^{t\beta} \left(\frac{s}{3} + \frac{1}{2} |x(s)|\right) ds.\ t \in (0,1] \quad (5)$$

with initial data

$$x(0) = 1. \quad (6)$$

Then

$$\frac{dx}{dt} = f\left(t, \frac{dx}{dt}, \int_0^{\phi(t)} g(s, x(s))ds\right) = \frac{3t}{40} + \frac{1}{4}\frac{dx}{dt} + \frac{1}{4}\int_0^{t\beta} \left(\frac{s}{3} + \frac{1}{2}|x(s)|\right) ds.\ t \in I,\ \beta \ge 1$$

$$g(t, x(t)) = \frac{t}{3} + \frac{1}{2} x(t).$$

It is clear that all assumptions of Theorem 2 are verified, where t = 1, then

$\| m_1 \|_1 = \frac{3}{40},\ \| m_2 \|_1 = \frac{1}{3},\ b_1 = \frac{1}{4}$ and $b_2 = \frac{1}{2}$

and r satisfied

$$r = \frac{\| m_2 \|_1 b_1 + b_1 b_2 |x_0| + \| m_1 \|_1}{1 - (b_1 + b_1 b_2)}$$

r = 0.45. Then the initial value problem (5)-(6) has at least one solution.

4. Conclusions

In this paper, we have studied a delay implicit functional integro-differential equation. We have prove the existence of solutions for an initial value problem of a delay implicit functional integro-differential equation. Then we have established the sufficient conditions for the uniqueness of solution and continuous dependence of solution on some initial data and the functions g, ϕ are studied. An example is given to illustrate our results.

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