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Article

A constrained problem of a nonlinear functional integral equation subject to the pantograph problem

EL-Sayed A.M. [1,*], EL-Alem M M [1], Israa Samy [1]

[1] *Department Mathematics and Computer Science, Faculty of Science, Alexandria University, 21321 Alexandria, Egypt.*
[*] *Correspondence Address:*
EL-Sayed A.M.: Department Mathematics and Computer Science, Faculty of Science, Alexandria University, 21321 Alexandria, Egypt.
Email: amasayed@alexu.edu.eg.

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ABSTRACT: Here we study the existence of solution and its continuous dependence of a constrained problem of a nonlinear functional integral equation subject to a constraint of an initial value problem of a pantograph differential equation. The Hyers-Ulam stability of the problem will be proved.

1. INTRODCTION

Differential and integral equations are crucial in nonlinear analysis. Many fundamental laws of physics and chemistry can be formulated as differential and integral equations. In biology and economics, differential equations are used to model the behavior of complex systems. Many authors are concerned with the study of this kind of equations see [2-5-6-9-10-16]. Pantograph equation is a delay differential equation (DDE) arising in electrodynamics. This type of equations have numerous applications in most fields, see [15-17-18-21-22-23].

Constrained problems are essential in the mathematical depiction of real-world situations, where such problems are transformed into mathematical models. The relevance of handling constraints or control variables arises from the unanticipated elements that persistently disrupt biological systems in the real world; biological traits like survival rates might change as a result. The question of whether an ecosystem can survive those erratic, disruptive occurrences that happen for a short while is of practical significance to ecology, see [1-3-4-7-8-11-12-13-14-15-19-20].

Now let γ , $\beta \in (0,\ 1)$, $\lambda > 0$. Let C[0,T] be the class of continuous function defined on [0,T], the norm of $x \in C[0,T]$ is given by.

$$\|x\| = \sup_{t\in[0,T]} |x(t)|$$

Consider the nonlinear functional integral equation

$$y(t) = f_1\left(t, \lambda \int_0^{\beta t} g\big(s, y(s), u(s)\big)\, ds\right),\ \ t \in [0,T]. \qquad (1)$$

$$\frac{du}{dt} = f_2\big(t, u(t), u(\gamma t)\big),\ a.e.\ t \in (0,T] \text{ and } u(0) = u_0\,. \qquad (2)$$

Here, Firstly, we prove the existence of a unique solution $u \in C\,[0,T]$ of the problem (2) and study the continuous dependence of the solution u on γ and u_0. Secondly, we prove the existence of a unique solution of the integral equation (1) and study the continuous dependence of y on u, β, λ. Finally, we study Hyers-Ulam stability of our problem (1), (2).

2. Existence of solution

Consider the following assumptions

1) $\boldsymbol{f_1}$: [0,T]×R→ R is continuous and satisfies the Lipchitz condition.

$$|f_1(t, x) - f_1(t, \bar{x})| \le k_1 |x - \bar{x}| \qquad (3)$$

2) f_2 : [0,T]×R×R → R is measurable in t ∈ [0,T] for all u ∈ R and satisfies the Lipchitz condition

$$|f_2(t, u_1, u_2) - f_2(t, \bar{u}_1, \bar{u}_2)| \le k_2 (|u_1 - \bar{u}_1| + |u_2 - \bar{u}_2|) \qquad (4)$$

3) g:[0,T] ×R×R → R is measurable in t ϵ [0,T] for all y and u ϵ R and satisfies the lipschitz condition

$$|g(t, y, u) - g(t, \bar{y}, \bar{u})| \le k_3 (|y - \bar{y}| + |u - \bar{u}|) \qquad (5)$$

4) Let k= max $\{k_1, k_2, k_3\}$.

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Remark

From (3) we have

(i) $|f_1(t,x)| - |f_1(t,0)| \leq |f_1(t,x) - f_1(t,0)| \leq k\,|x|$

and $|f_1(t,x(t)| \leq k\,|x(t)| + {f_1}^*$, ${f_1}^* = sup_{t\epsilon[0,T]}\,|f_1(t,0)|$.

Also, from (4) and (5) we can get

(ii) $|f_2(t,u,\bar{u})| \leq k(|u(t)| + |\bar{u}(t)|) + {f_2}^*$,
${f_2}^* = sup_{t\epsilon[0,T]}\,|f_2(t,0,0)|$.

(iii) $|g(t,y,u)| \leq k\,(|y(t)| + |u(t)|) + g^*$,
$g^* = sup_{t\epsilon[0,T]}\,|g(t,0,0)|$.

Now, we study the problem (2).

2.1 The problem (2)

Here we study the initial value problem (2)

Theorem 1

Let the assumption (2) be satisfied, if 2kT<1, then there exists a unique solution $u \,\epsilon\, C[0,T]$ of the problem (2).

Proof.

Integrating (2), we obtain

$$u(t) = u_0 + \int_0^t f_2\,(s,u(s),u(\Upsilon s))ds\,,\quad t \in [0,T]. \qquad (6)$$

Differentiating (6) we obtain (2) and from (2) we deduce $u(0) = u_0$.

Define the operator F by

$$Fu(t) = u_0 + \int_0^t f_2\left(s,u(s),u(\Upsilon s)\right)ds. \qquad (7)$$

Let u∈ C[0,T], let t_1 , t_2 ϵ [0,T] and $|t_2 - t_1| < \delta$, then

$$|Fu(t_2) - Fu(t_1)| =$$
$$|\int_0^{t_2} f_2\left(\mathrm{s},\mathrm{u}(\mathrm{s}),\mathrm{u}(\Upsilon\mathrm{s})\right)\mathrm{ds} - \int_0^{t_1} f_2\,(\mathrm{s},\mathrm{u}(\mathrm{s}),\mathrm{u}(\Upsilon\mathrm{s}))\,\mathrm{ds}\,|$$
$$= |\int_{t1}^{t2} f_2\,(s,u(s),u(\Upsilon s))\,ds\,|$$
$$\leq \int_{t1}^{t2} k\,(|u(s)| + |u(\Upsilon s)|)\,ds + |t_2 - t_1|{f_2}^*$$
$$\leq |t_2 - t_1|(2k||u||) + |t_2 - t_1|{f_2}^*.$$

Then we obtain

$$|\mathrm{Fu}(t_2)\text{-}\mathrm{Fu}(t_1)| \leq \varepsilon.$$

This proves that F: C[0,T] → C[0,T].

Now, let u , $\bar{u}$ ∈ C[0,T] be two solutions of equation (6), then

$$|Fu(t) - F\bar{u}(t)| = |\int_0^t f_2\,(s,u(s),u(\Upsilon s))\,ds - \int_0^t f_2(s,\bar{u}(s),\bar{u}(\Upsilon s))ds|$$
$$\leq k|\int_0^t |u(s) - \bar{u}(s)|ds + \int_0^t |u(\Upsilon s) - \bar{u}(\Upsilon s)|ds\,|$$
$$\leq k\,T\,||\,u - \bar{u}|| + k\,T\,||\,u - \bar{u}||$$
$$\leq 2\,k\,T\,||\,u - \bar{u}||.$$

Then F is contraction [5] and (2) has a unique solution $u\,\epsilon\,C\,[0,T]$.

Definition 1

The solution u ϵ C [0,T] of (2) depends continuously on the functions ϒ , u_0 if $\forall\, \epsilon > 0\,, \exists\, \delta(\epsilon)$>0 , such that

$$max\,\{|u_0 - {u_0}^*|\,, |\Upsilon - \Upsilon^*\,|\} < \delta.$$

Then

$$||u - u^*|| < \epsilon\,,$$

Where

$$u^*(t) = {u_0}^* + \int_0^t f_2\,(t,u^*(s),u^*(\Upsilon^* s)ds.$$

Theorem 2 Let the assumptions of Theorem 1 be satisfied, then the solution u ∈ C[0,T] of (2) depends continuously on u_0 and ϒ.

Proof.

$$|u(t) - u^*(t)| = \left|u_0 + \int_0^t f_2\,(s,u(s),u(\Upsilon s))\,ds - u_0^* - \int_0^t f_2(s,u^*(s),u^*(\Upsilon^* s))ds\right|$$
$$\leq \delta_1 + |\int_0^t f_2\,(s,u(s),u(\Upsilon s))ds - \int_0^t f_2\,(s,u^*(s),u^*(\Upsilon^* s\,)ds|$$
$$\leq \delta_1 + |\,k\int_0^t |u(s) - u^*(s)|\,ds + k\int_0^t |u(\Upsilon s) - u^*(\Upsilon^* s)|\,ds$$
$$\leq \delta_1 + kT\,||u - u^*|| + k\int_0^t\left[|u(\Upsilon s) - u\left(\Upsilon^*(s)\right) + u\left(\Upsilon^*(s)\right) - u^*\left(\Upsilon^*(s)\right)|\right]ds$$
$$\leq \delta_1 + kT\,||u - u^*|| + k\int_0^t\left|u(\Upsilon s) - u\left(\Upsilon^*(s)\right)\right|ds + k\int_0^t\left|u\left(\Upsilon^*(s)\right) - u^*\left(\Upsilon^*(s)\right)\right|ds$$
$$\leq \delta_1 + kT\,||u - u^*|| + k\int_0^t(\varepsilon + \|u - u^*\|)\,ds$$
$$\leq \delta_1 + kT\,||u - u^*|| + k\varepsilon T + kT\|u - u^*\|$$
$$\|u - u^*\| \leq 2kT\|u - u^*\| + \delta_1 + k\varepsilon T$$

And

$$(1 - 2kT)\|u - u^*\| \leq \delta_1 + kT\varepsilon$$

Then

$$||u - u^*|| \leq \frac{\delta_1 + kT\,\varepsilon}{(1 - 2kT)}\,.$$

Definition 2

Let the solution of (2) be exists, then approximate problem (2) is Hyers-Ulam stable if $\forall\, \epsilon > 0\,, \exists\, \delta(\epsilon)$>0 and any solution u_s of (2) satisfying.

$$|\frac{du_s}{dt} - f_2(t,u_s(t),u_s(\Upsilon t))\,| < \delta. \qquad (8)$$

Then $||u - u_s|| < \varepsilon.$

Theorem 3 Let the assumptions of Theorem 1 be satisfied, then the problem (2) is Hyers-Ulam stable.

Proof.

Integrating both sides of (8), we obtain

$$-\delta T < u_s(t) - u_0 - \int_0^t f_2(\theta,u_s(\theta),u_s(\Upsilon\theta))d\theta < \delta T$$

Now,

$$|u(t) - u_s(t)| = |u_0 + \int_0^t f_2\,(\theta),u(\theta),u(\Upsilon\theta) - u_s\,(t)|$$
$$\leq \left|u_0 + \int_0^t f_2(\theta,u(\theta),u(\Upsilon\theta)\,d\theta - \int_0^t f_2\left(\theta,u_s(\theta),u_s(\Upsilon\theta)\right)d\theta + \int_0^t f_2\left(\theta,u_s(\theta),u_s(\Upsilon\theta)\right)d\theta - u_s(t)\right|$$
$$\leq \delta T + |\int_0^t f_2\left(\theta,u(\theta),u(\Upsilon\theta)\right)d\theta - \int_0^t f_2\left(\theta,u_s(\theta),u_s(\Upsilon\theta)\right)d\theta|$$
$$\leq \delta T + k\int_0^t |u(\theta) - u_s(\theta)|\,d\theta + k\int_0^t |u(\Upsilon\theta) - u_s(\Upsilon\theta)|\,d\theta$$
$$\leq \delta T + kT||u - u_s|| + kT||u - u_s\,||$$
$$\leq \delta T + 2\,kT||u - u_s||.$$

Then

$$(1 - 2kT)||u - u_s|| \leq \delta T,$$

and

$$||u-u_s|| \le \frac{\delta T}{1-2kT} = \varepsilon.$$

2.2 The initial value problem (1)

Theorem 4

Let the assumptions 1, 2 and 4 be satisfied, Let u be the solution of (2), if $\lambda k^2 \beta T < 1$, then the problem (1) has a unique solution x∈ C[0,T].

Proof.

Define the operator F by

$$F_1 y(t) = f_1(t, \lambda\int_0^{\beta t} g(s,y(s),u(s))ds \quad (9)$$

Let y ∈ C[0,T], and for t_2,t_1 ∈ [0,T] such that $|t_2-t_1|<\delta$, then we have

$$|F_1 y(t_2)- F_1 y(t_1)|= |f_1(t_2,\lambda\int_0^{\beta t_2} g(s,y(s),u(s))\,ds) - f_1(t_1,\lambda\int_0^{\beta t_1} g(s,y(s),u(s))\,ds)|$$

$$\le |f_1(t_2,\lambda\int_0^{\beta t_2} g(s,y(s),u(s))\,ds) - f_1\left(t_2,\lambda\int_0^{\beta t_1} g(s,y(s),u(s))\,ds\right)$$

$$+f_1\left(t_2,\lambda\int_0^{\beta t_1} g(s,y(s),u(s))\,ds\right) - f_1(t_1,\lambda\int_0^{\beta t_1} g(s,y(s),u(s))\,ds)|$$

$$\le \lambda k\int_{\beta t_1}^{\beta t_2} |g(s,y(s),u(s))|ds + \delta$$

$$\le \delta + \lambda k^2 \int_{\beta t_1}^{\beta t_2}(|y(s)|+|u(s)|)ds + \lambda k\int_{\beta t_1}^{\beta t_2}|g^*|\,ds$$

$$\le \delta + \lambda k^2(||y||+||u||)\beta(t_2-t_1) + \lambda k\beta\,(t_2-t_1)g^* = \varepsilon$$

$$\le \delta + \lambda k^2(||y||+||u||)\beta\delta + \lambda k\beta\,\delta\, g^* = \varepsilon.$$

This proves that F: C[0,T]→C[0,T].

Now, we prove that F is contraction. Let y, $\bar{y}$ be two solutions of (1), then

$$|Fy(t)-F\bar{y}(t)| = |f_1(t, \lambda\int_0^{\beta t} g(s,y(s),u(s))ds) - f_1(t, \lambda\int_0^{\beta t} g(s,\bar{y}(s),u(s))\,ds)|$$

$$\le \lambda k \int_0^{\beta t} |g(s,y(s),u(s)) - g(s,\bar{y}(s),u(s))|ds$$

$$\le \lambda k^2 \int_0^{\beta t}|y(s)-\bar{y}(s)|ds$$

$$\le \lambda k^2\beta T\,||y-\bar{y}||.$$

Then F is Contraction [5] and (1) has a unique solution y∈C [0,T].

2.3 Continuous dependence

Definition 3

The solution y∈ C[0,T] of (1) depends continuously on λ , β and u if $\forall\, \epsilon > 0$, ∃ $\delta(\epsilon)>0$ such that

$$\max\{|u-u^*|, |\lambda-\lambda^*|, |\beta-\beta^*|\} < \delta.$$

Then

$$||y-y^*|| < \varepsilon,$$

Where

y^* is the solution of (1)

$$y^*(t) = f_1(t, \lambda^*\int_0^{\beta^* t} g(s,y^*(s),u^*(s))\,ds.$$

Theorem 5 Let the assumptions of Theorem 1,2,3 be satisfied, then the solution y ∈ C[0,T] of (1) depends continuously on λ , β , u^*.

Proof.

$$|y(t)-y^*(t)| = |f_1(t,\lambda\int_0^{\beta t} g(s,y(s),u(s))ds - f_1(t,\lambda^*\int_0^{\beta^* t} g(s,y^*(s),u^*(s))\,ds|$$

$$\le k|\lambda\int_0^{\beta t} g(s,y(s),u(s))ds - \lambda^*\int_0^{\beta^* t} g(s,y^*(s),u^*(s))ds|$$

$$\le |k\lambda\int_0^{\beta t} g(s,y(s),u(s))ds - k\lambda\int_0^{\beta^* t} g\,(s,y(s),u(s))ds + k\lambda\int_0^{\beta^* t} g\,(s,y(s),u(s))ds$$

$$- k\lambda^*\int_0^{\beta^* t} g(s,y^*(s),u^*(s))\,ds|$$

$$\le k\lambda|\int_{\beta^* t}^{\beta t} g\,(s,y(s),u(s))ds| + k^2|\lambda-\lambda^*|\int_0^{\beta^* t}|y(s)-y^*(s)|ds$$

$$+ k^2|\lambda-\lambda^*|\int_0^{\beta^* t}|u(s)-u^*(s)|ds$$

$$\le k\lambda\int_{\beta^* t}^{\beta t}|g\,(s,y(s),u(s))|ds + k^2\delta\,||y-y^*||\beta^* t + k^2\delta\,||u-u^*||\beta^* T$$

$$\le \lambda k\,\varepsilon + k^2\delta\,\beta^* T\,||y-y^*|| + k^2\,\delta\,\beta^* T\delta.$$

Now

$$\le k\,\lambda\,\varepsilon + k^2\delta\beta^* T||y-y^*|| + k^2\beta^* T\,\delta^2$$

Then

$$(1- k^2\delta\beta^* T)||y-y^*|| \le k\,\lambda\,\varepsilon + k^2\beta^* T\,\delta^2,$$

$$||y-y^*|| \le \frac{k\,\lambda\varepsilon + k^2\beta^* T\delta^2}{1-k^2\delta\beta^* T}.$$

Definition 4

Let the solution of (1) be exists then the problem (1) is Hyers-Ulam stable if $\forall\, \epsilon > 0$, ∃ $\delta(\epsilon) > 0$ and any approximate solution y_s of (1) satisfying

$$|y_s(t) - f_1(t, \lambda\int_0^{\beta t} g(\theta,y_s(\theta),u(\theta))d\theta)| < \delta.$$

Then

$$||y-y_s|| < \varepsilon,$$

Where

$$-\delta < y_s(t) - f_1(t,\lambda\int_0^{\beta t} g\big(\theta,y_s(\theta),u(\theta)\big)d\theta) < \delta$$

Theorem 6 Let the assumptions of Theorem (4) be satisfied. Then the problem (1) is Hyres-Ulam stable.

Proof.

$$|y(t)-y_s(t)| = |\, f_1(t, \lambda\int_0^{\beta t} g(\theta,y(\theta),u(\theta))d\theta - y_s(t)\,|$$

$$= |\,f_1(t, \lambda\int_0^{\beta t} g(\theta,y(\theta),u(\theta))d\theta - f_1(t, \lambda\int_0^{\beta t} g(\theta,y_s(\theta),u(\theta))d\theta$$

$$+f_1(t, \lambda\int_0^{\beta t} g(\theta,y_s(\theta),u(\theta))\,d\theta) - y_s(t)|$$

$$\le \delta + |f_1(t, \lambda\int_0^{\beta t} g(\theta,y(\theta),u(\theta))d\theta) - f_1(t, \lambda\int_0^{\beta t} g(\theta,y_s(\theta),u(\theta))d\theta)\,|$$

$$\le \delta + \lambda k^2\int_0^{\beta t}|y(\theta)-y_s(\theta)|d\theta$$

$$\le \delta + \lambda k^2\beta T\,||y-y_s||.$$

Then

$$||y-y_s|| \le \delta + \lambda k^2\beta T\,||y-y_s||$$

$$(1-\lambda k^2 \beta T)\, \|y-y_s\| \leq \delta$$

$$\|y-y_s\| \leq \frac{\delta}{1-\lambda k^2 \beta T} = \varepsilon .$$

Example

Consider the following example

$$y(t)=\frac{1}{5} e^{-t} \cos^2 t + \frac{1}{8}\int_0^{\frac{1}{2}t} \left(\frac{e^{-s}}{7-s} + \frac{1}{3} y(s) + \frac{1}{3} u(s)\right) ds, \quad t \in [0,1], \quad (10)$$

$$\frac{du}{dt} = \frac{\ln(1+t)}{2} + \frac{e^{-t}}{5} u(t) + \frac{1}{5} u\left(\frac{1}{2}t\right) \text{ a.e., } u(0) = \frac{1}{6}. \quad t \in (0,1] \quad (11)$$

Here we have:

$f_1(t,x) = \frac{1}{5} e^{-t}\cos^2 t + \frac{1}{4}x$, thus $|f_1(t,x) - f_1(t,\bar{x})| \leq \frac{1}{4}|x - \bar{x}|$

$g(s,y,u) = \frac{e^{-s}}{7-s} + \frac{1}{3} y(s) + \frac{1}{3} u(s)$, thus $|g(s,y,u) - g(s,\bar{y},\bar{u})| \leq \frac{1}{3}(|y - \bar{y}| + |u - \bar{u}|)$

$f_2(t, u(t), u(\gamma t)) = \frac{\ln(1+t)}{2} + \frac{e^{-t}}{5} u(t) + \frac{1}{5} u\left(\frac{1}{2}t\right)$, thus

$|f_2(t,u_1,u_2) - f_2(t,\bar{u}_1,\bar{u}_2)| \leq \frac{1}{5}(|u_1(t) - \bar{u}_1(t)| + |u_2(t) - \overline{u_2}(t)|)$.

Here we obtain, $k_1=\frac{1}{4}$, $k_2=\frac{1}{5}$, $k_3=\frac{1}{3}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $\lambda = \frac{1}{2}$, $x_0=\frac{1}{6}$, and $2kT = \frac{2}{3} < 1$

Clear all assumptions of Theorem 1 is satisfied, thus problem (10)-(11) has unique solution.

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