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# On a Mixture of Bivariate Chen Distribution

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**Abstract:**

The univariate Chen distribution has a number of appealing characteristics. It behaves similarly to Weibull, gamma, and generalized exponential distributions with two parameters. In this paper, we consider the bivariate mixture models starting with two independent univariate Chen models. Several useful structural properties of such a mixture model are discussed. Inferential aspects under the classical method are considered to estimate the model parameters. For illustrative purposes, a well- known motor data is re-analyzed to exhibit the flexibility of this proposed bivariate mixture model.

## 1 Introduction

In the last two decades, a major point of interest for statisticians and practitioners was to study populations that exhibit similar behaviors with respect to some pre-determined criteria. The earliest evidence regarding the study of heterogeneous populations was mostly due to Newcomb (1886) and Pearson (1894) who utilized/developed an approach, commonly known as finite mixture distributions. With modern days and advance of computation facilities, studies focusing on heterogeneous populations became more popular, for some useful references, see Titterington et al. (1985), Everitt and Hand (1981), McLachlan and Basford (1988), and the references cited there in. In recent times, there is a growing trend to study and explore the application of finite mixture models, for more details, see Al-Hussaini and Sultan (2001).

In several studies concerning heterogeneous population a two-parameter Chen distribution probability model appears to be really useful. The behavior of the probability
density function and the hazard function of the Chen distribution are quite close to the behavior of the pdf and the hazard function of the Weibull model, for more information in this topic see Chen and Gui (2020),. The two parameter Chen distribution, as many lifetime distributions, have been proposed in the literature to analyze data with bathtub-shaped failure rates (Chen (2000)), where distributions hazard functions provide an appropriate conceptual model for some electronic and mechanical products. In this paper, we exclusively focus on the

two-parameter bathtub-shaped lifetime model introduced by Chen (2000). Chen (2000) introduced a probability distribution that can give different shapes of hazard functions, including decreasing, increasing with bathtub shapes. Some studies obtained a mixture of bivariate inverse Weibull and gamma models, for more details, see Jones et al. (2000) and AL-Moisheer et al. (2020).

The main objective of this paper is to develop and study the mixture of a new bivariate absolutely continuous distribution via a mixture of two independent two parameter Chen distributions. We call this new model as the Bivariate Mixture of Chen Distribution (BMC).

Let X and Y be two random variables where each variable is independent and distributed as Chen distribution with parameters $\lambda$ and $\theta$; respectively. In this paper, several useful mathematical properties of the proposed model are derived; classical estimation methods are discussed; in addition, the performance of the suggested BMC model is examined using a real data set. The rest of the paper is organized as follows. Section 2, introduces the BMC distribution and discusses its construction via two independent variable and provides some contour plots. Section 3, provides some useful mathematical properties and obtains expressions for the bivariate survival function, hazard rate function, bivariate moment generating function, conditional moments, joint moments, stochastic ordering, etc., for the BMC distribution. Section 4 discusses the estimation strategy of the model parameters via EM algorithm. In Section 5, a well-known Motor data set has been re-analyzed to exhibit the efficiency of the proposed BMC model. Finally, some concluding remarks are presented in Section 6.

## 2 Mixture Bivariate Independent Chen Model

Given two independent univariate Chen distributions with parameters $(\lambda_1, \theta_1)$ and $(\lambda_2, \theta_2)$ respectively. The central idea of compounding is to consider that $\theta_1$ and $\theta_2$ as random variables ,and the observed (marginal) distribution of X and Y can be obtained from the joint distribution of $\theta_1$ and $\theta_2$ :

$$h_{x,y}(x, y) = \iint h(x, y, \theta_1, \theta_2)\ d\theta_1\ d\theta_2. \tag{1}$$

To construct a bivariate of Chen mixture distribution, assume that X and Y are independent Chen distributions with the scale parameters having a generalized bivariate Bernoulli distribution. A random variable with a Chen distribution has a cumulative distribution function (cdf) and a probability density function (pdf) for $x > 0$, given by

$$F(x; \theta, \lambda) = 1 - e^{\lambda\left(1-e^{x^{\theta}}\right)+x^{\theta}}, \qquad x > 0\ , \theta, \lambda > 0, \tag{2}$$

$$f(x; \theta, \lambda) = \theta\lambda x^{\theta-1} e^{\lambda\left(1-e^{x^{\theta}}\right)+x^{\theta}}, \qquad x > 0, \theta, \lambda > 0, \tag{3}$$

where $\theta$, and $\lambda$ are the shape and scale parameters, respectively.

In the bivariate case, let X and Y be two random variables with parameters $\theta_1$, and $\theta_2$ respectively. For given fixed values of $\theta_1$, $\theta_2$, X and Y are independent. The pdf of BMC distribution is defined as

$$f(x; \theta) = \sum_{i=1}^{2} p_i\ f_i\ ((x)_i, \theta_i), \quad x > 0, \theta > 0 \tag{4}$$

where $p_i$ are the mixing proportions which must satisfy $\sum_{i=1}^{2} p_i = 1$, and $p_i \geq 0$, and all parameters are unknowns. The pdf of the first component of Chen is given by (3), with fixed shape parameter θ, and a random scale parameter $\lambda > 0$ that takes two distinct values $\lambda_1$ and $\lambda_2$. Likewise, for fixed shape parameter $\theta_i$ , let Y have a Chen mixture density and the pdf of the second component (Chen) is given by

$$g(y; \varphi, \beta) = \varphi\beta y^{\varphi-1} e^{\beta\left(1-e^{y^{\varphi}}\right)+y^{\varphi}}, \quad y > 0, \theta, \lambda > 0. \qquad (5)$$

where β is a random scale parameter ($\beta > 0$ ) that takes two distinct values $\beta_1$ and $\beta_2$. For given values of $(\lambda, \beta)$, assuming that X and Y are independent, but λ, β are correlated through their generalized bivariate distribution with the following mixture component matrix:

$$P = \begin{array}{c c} & \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} & \begin{bmatrix} P_{\lambda_1\beta_1} & P_{\lambda_1\beta_2} \\ P_{\lambda_2\beta_1} & P_{\lambda_2\beta_2} \end{bmatrix} \end{array}, \qquad (6)$$

where P is the mixture components and $P_{\lambda_1\beta_1} + P_{\lambda_1\beta_2} + P_{\lambda_2\beta_1} + P_{\lambda_2\beta_2} = 1$.

Let $h_{x,y}(x, y)$ be the joint pdf of (X,Y), then

$$h_{x,y}(x, y) = f(x|\theta, \lambda_1)g(y|\varphi, \beta_1)P_{\lambda_1\beta_1} + f(x|\theta, \lambda_1)g(y|\varphi, \beta_2)P_{\lambda_1\beta_2} \qquad (7)$$
$$+f(x|\theta, \lambda_2)g(y|\varphi, \beta_1)P_{\lambda_2\beta_1} + f(x|\theta, \lambda_2)g(y|\varphi, \beta_2)P_{\lambda_2\beta_2}$$

$$\begin{aligned} h(x, y) = {} & P_{\lambda_1\beta_1}\theta\lambda_1 x^{\theta-1} e^{\lambda_1\left(1-e^{x^{\theta}}\right)+x^{\theta}} \varphi\beta_1 y^{\varphi-1} e^{\beta_1\left(1-e^{y^{\varphi}}\right)+y^{\varphi}} + \\ & P_{\lambda_1\beta_2}\theta\lambda_1 x^{\theta-1} e^{\lambda_1\left(1-e^{x^{\theta}}\right)+x^{\theta}} \varphi\beta_2 y^{\varphi-1} e^{\beta_2\left(1-e^{y^{\varphi}}\right)+y^{\varphi}} + \\ & P_{\lambda_2\beta_1}\theta\lambda_2 x^{\theta-1} e^{\lambda_2\left(1-e^{x^{\theta}}\right)+x^{\theta}} \varphi\beta_1 y^{\varphi-1} e^{\beta_1\left(1-e^{y^{\varphi}}\right)+y^{\varphi}} + \\ & P_{\lambda_2\beta_2}\theta\lambda_2 x^{\theta-1} e^{\lambda_2\left(1-e^{x^{\theta}}\right)+x^{\theta}} \varphi\beta_2 y^{\varphi-1} e^{\beta_2\left(1-e^{y^{\varphi}}\right)+y^{\varphi}} \end{aligned} \qquad (8)$$

Figure 1 shows the different contour curves of pdf at different values of parameters. The various contour curves are closed, left-skewed, right skewed and approximate unimodal. It is clear from the different contour curves that the pdf is flexible and can be applied to several fields in real life.

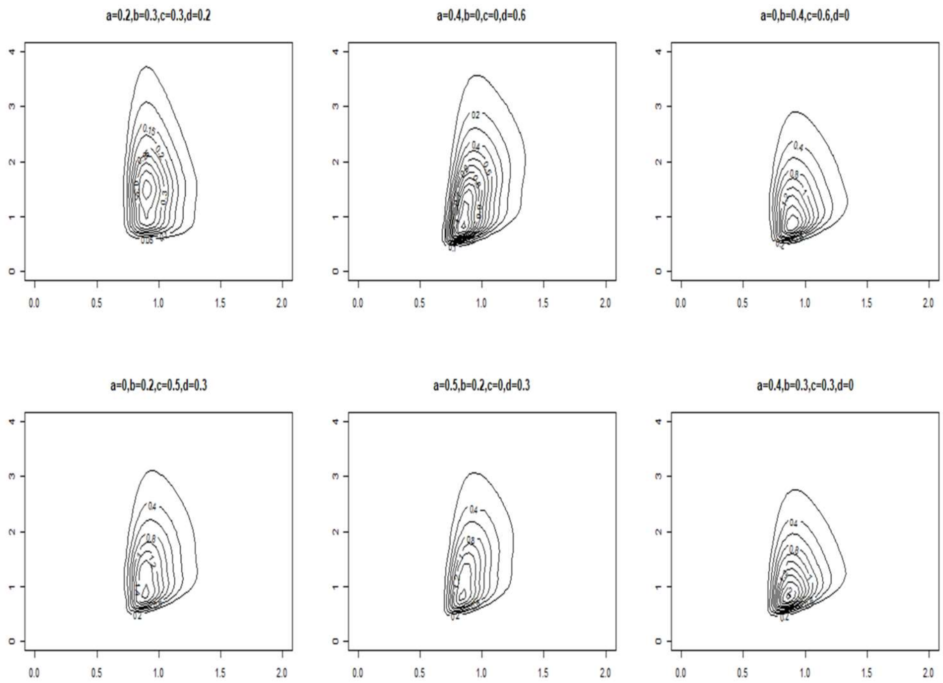



**Figure 1:** The contour plots of BMC distribution for varying parameters

For simplification, let $a = p_{\lambda_1\beta_1}$, $b = p_{\lambda_1\beta_2}$ , $c = p_{\lambda_2\beta_1}$ , $d = p_{\lambda_2\beta_2}$. From Figure 1, it is evident that the joint pdf in (8) can produce various shapes corresponding to several parameter choices. The joint pdf mixture of four univariate Chen mixture distributions, that involves 9 parameters for its specification. In application to real life data sets, not all four components might be necessary. Consequently, if we put some restrictions, such as b= c =0, a =d = 0 or a = b =c= 0 and substitute in equation 8 as follows:

- At b=c=0, then

$$h(x,y) = a\theta\lambda_1 x^{\theta-1} e^{\lambda_1\left(1-e^{x^\theta}\right)+x^\theta} \varphi\beta_1 y^{\varphi-1} e^{\beta_1\left(1-e^{y^\varphi}\right)+y^\varphi} + d\theta\lambda_2 x^{\theta-1} e^{\lambda_2\left(1-e^{x^\theta}\right)+x^\theta} \varphi\beta_2 y^{\varphi-1} e^{\beta_2\left(1-e^{y^\varphi}\right)+y^\varphi},$$

- At a =d = 0, then

$$h(x,y) = b\theta\lambda_1 x^{\theta-1} e^{\lambda_1\left(1-e^{x^\theta}\right)+x^\theta} \varphi\beta_2 y^{\varphi-1} e^{\beta_2\left(1-e^{y^\varphi}\right)+y^\varphi} + c\theta\lambda_2 x^{\theta-1} e^{\lambda_2\left(1-e^{x^\theta}\right)+x^\theta} \varphi\beta_1 y^{\varphi-1} e^{\beta_1\left(1-e^{y^\varphi}\right)+y^\varphi},$$

- At a = b =c= 0, then

$$h(x,y) = d\theta\lambda_2 x^{\theta-1} e^{\lambda_2\left(1-e^{x^\theta}\right)+x^\theta} \varphi\beta_2 y^{\varphi-1} e^{\beta_2\left(1-e^{y^\varphi}\right)+y^\varphi}.$$

These restrictions result in a correlation values (among scale parameters) +1, -1 respectively.

The marginal densities of X and Y respectively, are given as follows:

$$h_x(x) = \pi_1 f_1(x) + (1-\pi_1) f_2(x), \qquad (9)$$

where $\pi_1 = a + b$

$$h_y(Y) = \pi_2 g_1(Y) + (1-\pi_2) g_2(Y), \qquad (10)$$

where $\pi_2 = a + c$.

The joint cdf will be

$$F(x,y) = \int_0^y \int_0^x h(t,s)\,dt\,ds$$

$$= \int_0^y \int_0^x \Big[ P_{\lambda_1\beta_1} \theta\lambda_1 x^{\theta-1} e^{\lambda_1\left(1-e^{x^{\theta}}\right)+x^{\theta}} \varphi\beta_1 y^{\varphi-1} e^{\beta_1\left(1-e^{y^{\varphi}}\right)+y^{\varphi}} +$$
$$P_{\lambda_1\beta_2} \theta\lambda_1 x^{\theta-1} e^{\lambda_1\left(1-e^{x^{\theta}}\right)+x^{\theta}} \varphi\beta_2 y^{\varphi-1} e^{\beta_2\left(1-e^{y^{\varphi}}\right)+y^{\varphi}} +$$
$$P_{\lambda_2\beta_1} \theta\lambda_2 x^{\theta-1} e^{\lambda_2\left(1-e^{x^{\theta}}\right)+x^{\theta}} \varphi\beta_1 y^{\varphi-1} e^{\beta_1\left(1-e^{y^{\varphi}}\right)+y^{\varphi}} +$$
$$P_{\lambda_2\beta_2} \theta\lambda_2 x^{\theta-1} e^{\lambda_2\left(1-e^{x^{\theta}}\right)+x^{\theta}} \varphi\beta_2 y^{\varphi-1} e^{\beta_2\left(1-e^{y^{\varphi}}\right)+y^{\varphi}} \Big] dt\, ds$$

$$= a e^{\lambda_1\left(1-e^{x^{\theta}}\right)+x^{\theta}} e^{\beta_1\left(1-e^{y^{\varphi}}\right)+y^{\varphi}} + b e^{\lambda_1\left(1-e^{x^{\theta}}\right)+x^{\theta}} e^{\beta_2\left(1-e^{y^{\varphi}}\right)+y^{\varphi}} +$$
$$c e^{\lambda_2\left(1-e^{x^{\theta}}\right)+x^{\theta}} e^{\beta_1\left(1-e^{y^{\varphi}}\right)+y^{\varphi}} +$$
$$d e^{\lambda_2\left(1-e^{x^{\theta}}\right)+x^{\theta}} e^{\beta_2\left(1-e^{y^{\varphi}}\right)+y^{\varphi}}. \qquad (11)$$

The associated survival function of BMC distribution will be

$$R(x,y) = P(X > x, Y > y) = 1 - P(X < x) - P(Y < y) + P(X < x, Y < y)$$
$$= 1 - F_X(x) - F_Y(y) + F_{X,Y}(x,y), \qquad (12)$$

where $F_X(x)$, $F_Y(y)$ is the marginal cumulative function of X and Y respectively. The hazard rate function (hrf) is given as follow:

$$hrf(x,y) = {}^{A}\!/_{B}, \qquad (13)$$

where A is given in (8) and B is given in (11). The conditional pdf of X given Y for each fixed Y= y will be

$$h(x|y) = {}^{\mathrm{h}(\mathrm{x},\mathrm{y})}\!/_{h_y(\mathrm{y})}. \qquad (14\ )$$

The conditional pdf of Y given X will be

$$(h(y|x) = {}^{\mathrm{h}(\mathrm{x},\mathrm{y})}\!/_{h_x(\mathrm{x})}. \qquad (15\ )$$

## 3 Structural Properties

Some properties of the BMC distribution are derived as follows:

***Property 1.*** Let $(X, Y) \sim BMC(a, b, c, d, \lambda_1, \lambda_2, \beta_1, \beta_2, \theta, \phi)$. Then, $(X, Y)$ has a total positivity of order 2 Positive association ($TP_2$ property).

*Proof.* Observe that an absolute continuous bivariate random vector, say $(U_1, U_2)$ has $TP_2$ property, if and only if for any $u_{11}, u_{12}, u_{21}, u_{22}$ whenever $u_{11} < u_{12}$ and $u_{21} > u_{22}$ we have

$$\boldsymbol{f}_{U_1,U_2}(u_{11}, u_{21})\boldsymbol{f}_{U_1,U_2}(u_{11}, u_{12}) - \boldsymbol{f}_{U_1,U_2}(u_{12}, u_{21})\boldsymbol{f}_{U_1,U_2}(u_{11}, u_{22}) \geq \mathbf{0},$$

where $\boldsymbol{f}_{U_1,U_2}(,)$ is the joint pdf of $(U_1, U_2)$. Observe that by taking different ordered $u_{11}$, $u_{12}, u_{21}, u_{22}$, such that $u_{11} < u_{12}$, and $u_{21} > u_{22}$, our result immediately follows:

As a consequence, the positive quadrant dependence property (alternatively, the $TP_2$ Property) will indicate several other non-increasing properties related to conditional survival, conditional cdf of X given Y and Y given X and including this result:

$g_1(.)$ and $g_2(.)$, $Cov(g_1(X), g_2(Y)) \geq 0$.

***Property 2. Moments***

The product moments, $\acute{\mu}_{x,y}$, about zero can be computed as follows:

$$\acute{\mu}_{X,Y} = \int_0^\infty \int_0^\infty x\, y\, h(x,y) dx\, dy =$$
$$\int_0^\infty \int_0^\infty x\, y \Big[ \mathrm{P}_{\lambda_1\beta_1} \theta\lambda_1 x^{\theta-1} \mathrm{e}^{\lambda_1\left(1-e^{x^\theta}\right)+x^\theta} \varphi\beta_1 y^{\varphi-1} \mathrm{e}^{\beta_1\left(1-e^{y^\varphi}\right)+y^\varphi} +$$
$$P_{\lambda_1\beta_2} \theta\lambda_1 x^{\theta-1} \mathrm{e}^{\lambda_1\left(1-e^{x^\theta}\right)+x^\theta} \varphi\beta_2 y^{\varphi-1} \mathrm{e}^{\beta_2\left(1-e^{y^\varphi}\right)+y^\varphi} +$$
$$P_{\lambda_2\beta_1} \theta\lambda_2 x^{\theta-1} \mathrm{e}^{\lambda_2\left(1-e^{x^\theta}\right)+x^\theta} \varphi\beta_1 y^{\varphi-1} \mathrm{e}^{\beta_1\left(1-e^{y^\varphi}\right)+y^\varphi} +$$
$$P_{\lambda_2\beta_2} \theta\lambda_2 x^{\theta-1} \mathrm{e}^{\lambda_2\left(1-e^{x^\theta}\right)+x^\theta} \varphi\beta_2 y^{\varphi-1} \mathrm{e}^{\beta_2\left(1-e^{y^\varphi}\right)+y^\varphi} \Big] dxdy \qquad (16)$$

Using the joint pdf of X and Y, and/or using the MGF expression given above. The different product moments of the order $X^m X^n, (m,n) \geq 1$, can be obtained from (16).

***Property 3.*** **Shape of the Distribution**

A critical point of a function with two variables is a point where the partial derivatives of first order are equal to zero. The most two reasons to study the critical points for bivariate distributions are:

(1) A real-life data set(s) can have several different shapes. The flexibility of any proposed model can well be determined from such a study.

(2) In dealing with bivariate distributions, quite often it is imperative to study the tails of the joint pdf as well as the point of inflection. A knowledge on critical point(s) will help to better understand these properties.

This Property is helpful for data fitting because, a distribution shape one of the properties that can fitting model data sets with larger tails and/or smaller sizes

Let us examine the shape of the BMC distribution. From the joint density in (8), we have

$$\frac{\partial f(x,y)}{\partial x} = P_{\lambda_1\beta_1}\theta\lambda_1\Bigg((\theta-1)x^{\theta-2}e^{\lambda_1\left(1-e^{x^\theta}\right)+x^\theta} +$$

$$x^{\theta-1}e^{\lambda_1\left(1-e^{x^\theta}\right)+x^\theta}\left(-\lambda_1 e^{x^\theta}x^\theta log(x)\left(1-e^{x^\theta}\right)+\right.$$

$$\left.\left.x^\theta log(x)\right)\right)\varphi\beta_1 y^{\varphi-1}e^{\beta_1\left(1-e^{y^\varphi}\right)+y^\varphi} +$$

$$P_{\lambda_1\beta_2}\theta\lambda_1 x^{\theta-1}e^{\lambda_1\left(1-e^{x^\theta}\right)+x^\theta}\Bigg((\theta-1)x^{\theta-2}e^{\lambda_1\left(1-e^{x^\theta}\right)+x^\theta} +$$

$$x^{\theta-1}e^{\lambda_1\left(1-e^{x^\theta}\right)+x^\theta}\left(-\lambda_1 e^{x^\theta}x^\theta log(x)\left(1-e^{x^\theta}\right)+\right.$$

$$\left.\left.x^\theta log(x)\right)\right)\varphi\beta_2 y^{\varphi-1}e^{\beta_2\left(1-e^{y^\varphi}\right)+y^\varphi} +$$

$$P_{\lambda_2\beta_1}\theta\lambda_2 x^{\theta-1}e^{\lambda_2\left(1-e^{x^\theta}\right)+x^\theta}\left((\theta-1)x^{\theta-2}e^{\lambda_2\left(1-e^{x^\theta}\right)+x^\theta}+\right.$$
$$x^{\theta-1}e^{\lambda_2\left(1-e^{x^\theta}\right)+x^\theta}\left(-\lambda_2 e^{x^\theta}x^\theta log(x)\left(1-e^{x^\theta}\right)+\right.$$
$$\left.\left.x^\theta log(x)\right)\right)\varphi\beta_1 y^{\varphi-1}e^{\beta_1\left(1-e^{y^\varphi}\right)+y^\varphi}+$$
$$P_{\lambda_2\beta_2}\theta\lambda_2 x^{\theta-1}e^{\lambda_2\left(1-e^{x^\theta}\right)+x^\theta}\left((\theta-1)x^{\theta-2}e^{\lambda_2\left(1-e^{x^\theta}\right)+x^\theta}+\right.$$
$$x^{\theta-1}e^{\lambda_2\left(1-e^{x^\theta}\right)+x^\theta}\left(-\lambda_2 e^{x^\theta}x^\theta log(x)\left(1-e^{x^\theta}\right)+\right.$$
$$\left.\left.x^\theta log(x)\right)\right)\varphi\beta_2 y^{\varphi-1}e^{\beta_2\left(1-e^{y^\varphi}\right)+y^\varphi}. \qquad (17)$$

$$\frac{\partial f(x,y)}{\partial y}=P_{\lambda_1\beta_1}\theta\lambda_1 x^{\theta-1}e^{\lambda_1\left(1-e^{x^\theta}\right)+x^\theta}\varphi\beta_1\left((\varphi-\right.$$
$$1)y^{\varphi-2}e^{\beta_1\left(1-e^{y^\varphi}\right)+y^\varphi}+y^{\varphi-1}e^{\beta_1\left(1-e^{y^\varphi}\right)+y^\varphi}\left(-\beta_1 e^{y^\varphi}y^\varphi log(y)(1-\right.$$
$$\left.\left.e^{y^\varphi})+y^\varphi log(y)\right)\right)+P_{\lambda_1\beta_2}\theta\lambda_1 x^{\theta-1}e^{\lambda_1\left(1-e^{x^\theta}\right)+x^\theta}\left((\varphi-\right.$$
$$1)y^{\varphi-2}e^{\beta_2\left(1-e^{y^\varphi}\right)+y^\varphi}+\{\{\{y^{\varphi-1}e^{\beta_2\left(1-e^{y^\varphi}\right)+y^\varphi}\left(-\beta_2 e^{y^\varphi}y^\varphi log(y)(1-\right.$$
$$\left.\left.e^{y^\varphi})+y^\varphi log(y)\right)\right)\varphi\beta_2 y^{\varphi-1}e^{\beta_2\left(1-e^{y^\varphi}\right)+y^\varphi}+$$
$$P_{\lambda_2\beta_1}\theta\lambda_2 x^{\theta-1}e^{\lambda_2\left(1-e^{x^\theta}\right)+x^\theta}\left((\varphi-1)y^{\varphi-2}e^{\beta_1\left(1-e^{y^\varphi}\right)+y^\varphi}+\right.$$
$$y^{\varphi-1}e^{\beta_1\left(1-e^{y^\varphi}\right)+y^\varphi}\left(-\beta_1 e^{y^\varphi}y^\varphi log(y)(1-e^{y^\varphi})+\right.$$
$$\left.\left.y^\varphi log(y)\right)\right)\varphi\beta_1 y^{\varphi-1}e^{\beta_1\left(1-e^{y^\varphi}\right)+y^\varphi}+$$
$$P_{\lambda_2\beta_2}\theta\lambda_2 x^{\theta-1}e^{\lambda_2\left(1-e^{x^\theta}\right)+x^\theta}\left((\varphi-1)y^{\varphi-2}e^{\beta_2\left(1-e^{y^\varphi}\right)+y^\varphi}+\right.$$
$$y^{\varphi-1}e^{\beta_2\left(1-e^{y^\varphi}\right)+y^\varphi}\left(-\beta_2 e^{y^\varphi}y^\varphi log(y)(1-e^{y^\varphi})+\right.$$
$$\left.\left.y^\varphi log(y)\right)\right)\varphi\beta_2 y^{\varphi-1}e^{\beta_2\left(1-e^{y^\varphi}\right)+y^\varphi}. \qquad (18)$$

Consequently, to explain more about the shape of the BMC distribution, there may be several critical points we can obtain from the equation. (18) For specific choices of the model parameters, a numerical study can be made.

## 4. EM Algorithm

Mclachlan and Krishnan (1997) introduced the Expectation-Maximization (EM) algorithm as a method of estimation. To apply the EM algorithm we augment the data $(x_k, y_k)$, $k = 1, \dots, n$ with the group membership variables $(a_k, b_k, c_k)$ , k= 1,….,n where $a_k$ is one if the k[th] observation is in f(X, $\lambda_1$, $B_1$) and zero otherwise and similarly for $b_k, c_k$. we have four groups $G_{ij}, i, j = 1, 2$ for which the densities are:

$$f_{ij}(X,Y) = f_i(X)f_j(Y) =$$
$$\theta\lambda_i x^{\theta-1} e^{\lambda_i\left(1-e^{x^{\theta}}\right)+x^{\theta}} \varphi\beta_j y^{\varphi-1} e^{\beta_j\left(1-e^{y^{\varphi}}\right)+y^{\varphi}}. \qquad (19)$$

The mixing proportions are $P(G_{11}) = a$, $P(G_{12}) = b$, $P(G_{21}) = c$ and $P(G_{22}) = 1 - a - b - c$.

We define $\ell_{ij}(x, y) = log f_{ij}(x, y)$, then the EM algorithm as method of estimation is given by finding the complete log likelihood, $\ell$ as follows:

$$\ell = \sum_{k=1}^{n} a_k\, \ell_{11}(x_k, y_k) + \sum_{k=1}^{n} b_k\, \ell_{12}(x_k, y_k) +$$
$$\sum_{k=1}^{n} c_k\, \ell_{21}(x_k, y_k) + \sum_{k=1}^{n}(1 - a_k - b_k - c_k)\, \ell_{22}(x_k, y_k). \qquad (20)$$

This is linear in the group membership variables $(a_k, b_k, c_k)$ , so in the E–step we enter into (19) their expected values given the current estimates $(\hat{\theta}, \widehat{\varphi}, \hat{\lambda}_1, \hat{\lambda}_2, \widehat{B}_1, \widehat{B}_2, \hat{a}, \hat{b}, \hat{c})$ of the parameter, calculated as:

$$\hat{a}_k = \frac{\hat{a} f_{11}(x_k, y_k)}{\hat{a} f_{11}(x_k, y_k) + \hat{b} f_{12}(x_k, y_k) + \hat{c} f_{21}(x_k, y_k) + (1 - \hat{a} - \hat{b} - \hat{c}) f_{22}(x_k, y_k)}, \qquad (21)$$

and similarly, for $b_k, c_k$, for more details in this topic see Jones et al. (2000). It should be noted that algebraic simplification of the above may be necessary to avoid numerical problems.

For the M-step we need to maximize (20) over $(\theta, \varphi, \beta_1, \beta_2, \lambda_1, \lambda_2)$ for fixed values of $(\alpha_k, b_k, c_k)$ , this is achieved by the conditional of independence x and y given the group membership. We can essentially deal with the univariate parameter separately. Differentiating (20) gives

$$\frac{\partial \iota}{\partial \lambda_1} = \frac{n}{\lambda_1}\sum_{k=1}^{n}(a_k + b_k) + \sum_{k=1}^{n}(a_k + b_k)\left(1 - e^{x_k^{\theta}}\right),$$

$$\frac{\partial \iota}{\partial \lambda_2} = \frac{n}{\lambda_2}\sum_{k=1}^{n}(c_k + d_k) + \sum_{k=1}^{n}(c_k + d_k)\left(1 - e^{x_k^{\theta}}\right),$$

$$\frac{\partial \iota}{\partial \beta_1} = \frac{n}{\beta_1}\sum_{k=1}^{n}(a_k + c_k) + \sum_{k=1}^{n}(a_k + c_k)\left(1 - e^{y_k^{\varphi}}\right),$$

$$\frac{\partial \iota}{\partial \beta_2} = \frac{n}{\beta_2}\sum_{k=1}^{n}(b_k + d_k) - \sum_{k=1}^{n}(b_k + d_k)\left(1 - e^{y_k^{\varphi}}\right),$$

$$\frac{\partial \iota}{\partial \theta} = \frac{4n}{\theta} + 4\sum_{k=1}^{n} logx_k + 2\sum_{k=1}^{n}(\lambda_1 x_k^{\theta})logx_k + 2\sum_{k=1}^{n}(\lambda_2 x_k^{\theta})logx_k + 2(\theta - 1)\sum_{k=1}^{n} e^{\lambda_1\left(1-e^{x_k^{\theta}}\right)+x_k^{\theta}} x_k^{\theta-2} - \lambda_1 \sum_{k=1}^{n} e^{\lambda_1\left(1-e^{x_k^{\theta}}\right)+x_k^{\theta}} x_k^{\theta-1} e^{x_k^{\theta}+\theta x_k^{\theta-1}} + 2(\theta - 1)\sum_{k=1}^{n} e^{\lambda_2\left(1-e^{x_k^{\theta}}\right)+x_k^{\theta}} x_k^{\theta-2} - \lambda_2 \sum_{k=1}^{n} e^{\lambda_2\left(1-e^{x_k^{\theta}}\right)+x_k^{\theta}} x_k^{\theta-1} e^{x_k^{\theta}+\theta x_k^{\theta-1}},$$

$$\frac{\partial \iota}{\partial \varphi} = \frac{4n}{\varphi} + 4\sum_{k=1}^{n} logy_k + 2\sum_{k=1}^{n}(\beta_1 y_k^{\varphi})logy_k + 2\sum_{k=1}^{n}(\beta_2 y_k^{\varphi})logy_k + 2(\varphi - 1)\sum_{k=1}^{n} e^{\beta_1\left(1-e^{y_k^{\varphi}}\right)+y_k^{\varphi}} y_k^{\varphi-2} - \beta_1 \sum_{k=1}^{n} e^{\beta_1\left(1-e^{y_k^{\varphi}}\right)+y_k^{\varphi}} y_k^{\varphi-1} e^{y_k^{\varphi}+\theta y_k^{\varphi-1}} + 2(\varphi - 1)\sum_{k=1}^{n} e^{\beta_2\left(1-e^{y_k^{\varphi}}\right)+y_k^{\varphi}} y_k^{\varphi-2} - \beta_2 \sum_{k=1}^{n} e^{\beta_2\left(1-e^{y_k^{\varphi}}\right)+y_k^{\varphi}} y_k^{\varphi-1} e^{y_k^{\varphi}+\theta y_k^{\varphi-1}}.$$

The M-step is completed by setting:

$$\hat{a} = \frac{1}{n}\sum_{k=1}^{n} a_k, \ldots .$$

## 5. Motor Data Analysis

A genuine data set was presented by Alotaibi et al. (2021) is investigated to show the usefulness of the suggested approaches in an engineering phenomenon. The information in Table 1 shows the failure times (in days) of a parallel system made up of two identical motors, X and Y. The Chen model's applicability to the entire motor data sets is checked first. The Kolmogorov-Smirnov (KS) statistics and its P-value, as well as the maximum likelihood estimates (MLEs) and their standard errors (SEs), are obtained and supplied in Table 1 for this purpose for each pair of X and Y data. We used the log likelihood in Equation (20) to obtain the MLE for the Chen distribution as shown in Table 1. Table 1 lists the MLE of the parameters Kolmogorov-Smirnov distance (KS-d) and its p-value for the marginals. Initially, we fit the marginal of X and Y independently on the motor data in Equations in (3) and (5). It demonstrates how well the Chen distribution fits the motor data sets. Also developed are several graphical approaches for determining the goodness-of-fit for motor data sets. Figure 2-I indicates the box-Whisker plot, Figure 2-II shows the quartile plot. In addition, Figure 2-III indicates the frequency. Also, Figure 2-IV illustrates quantile-quantile (QQ) plot for the two variables X and Y. Furthermore ensuring that the data sets meet the Chen model. While Figure 3 shows that the bivariate density and fitted model with 95% prediction limits

**Table 1.** Two motors' failure rates

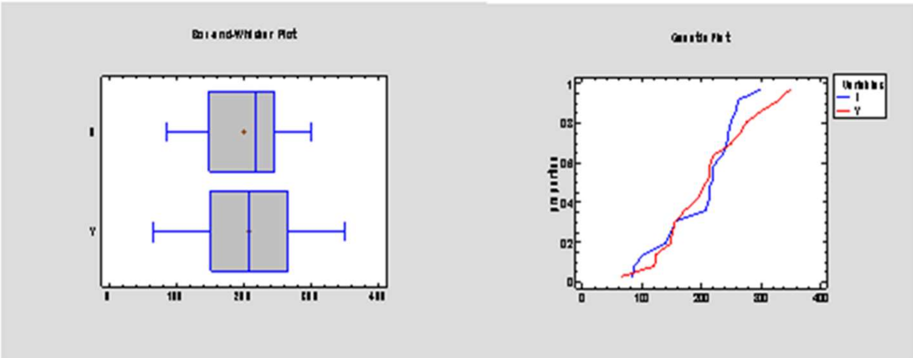
| Motor | Failure rates | MLE(SE) | | KS | |
|---|---|---|---|---|---|
| | | $\alpha$ | $\theta$ | Statistic | $P$-value |
| X | 84, 88, 102, 139, 148, 156, 207, 212, 213, 220, 220, 235, 243, 245, 250, 257, 263, 300 | 38.998(14.645) | 0.01186(0.0904) | 0.29017 | 0.0910 |
| Y | 65, 121, 123, 148, 150, 156, 172, 192, 202, 212, 214, 220, 248, 265, 275, 300, 330, 350 | 45.138(24.673) | 0.0096(0.0015) | 0.2619 | 0.3108 |

The MLEs, SE, 95% ACIs and length of $\theta$, $\varphi, \beta$, $\lambda$, are computed using the entire X and Y data sets. It shows that the maximum likelihood method as indicated in Equation (20), was used to estimate the unknown parameters. In conclusion, the numerical results of the provided estimation techniques based on motor data offer very good results (small SE small length).of the proposed model in Table 2.

**Table 2**. The estimates, SE, ACI and length of $\theta$, $\varphi$, $\mu$, $\lambda$ for motor data.

| Method | $\theta$ | $\varphi$ | $\beta$ | $\lambda$ |
|---|---|---|---|---|
| MLE | 3343.34 | 0.01589 | 3223.25 | 0.02012 |
| SE | (0.00667) | (0.00213) | (9.2154) | (0.09034) |
| ACI | (3487.5,3144.6) | (0.02163,0.01589) | (3286.9,3028.6) | (0.0154,0.0099) |
| Length | 342.9 | 0.00574 | 258.3 | 0.0055 |

I II



III IV

**Figure 2.** The Box-Whisker, Quartile, frequency and QQ plots under X and Y data

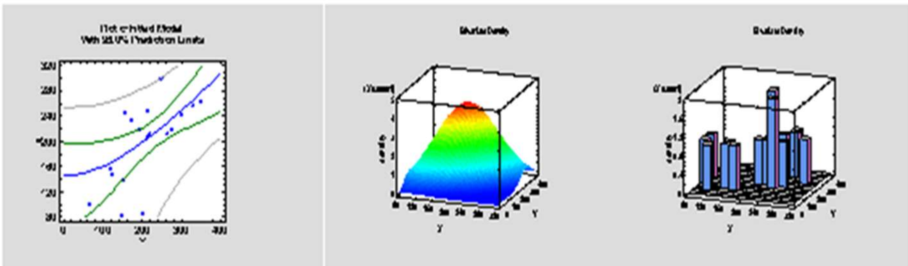


**Figure 3.** The bivariate density and fitted model with 95% interval limits

## 5. Concluding Remarks

In this paper, the BMC distribution was construed based on the independency of two variables. Some contour plots of the proposed model were illustrated figure 1. Some mathematical properties of the BMC were obtained as bivariate survival function equation (12), hazard rate function equation (8), the shape of the BMC was obtained equation(18), bivariate moment generating functions , joint moments (16)equation independent Chen. Also, the parameter estimation method was applied via EM algorithm. Finally, Motor data set were re-analyzed to indicate the efficiency of the suggested BMC model.

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