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Different Approaches for Outlier Detection in Life Testing Scenarios

Hager Ahmad Ibrahim [1] **Mahmoud Riad Mahmoud** [2] **Moshera A. M. Ahmad** [3*] **Rasha M. Mandouh** [2]

[1] Higher Institute of Information Technology, Badr City, Cairo, Egtpt.
[2] Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt.
[3] El Gazeera High Institute for Computer and Management Information System, Egypt.

* **Correspondence:** moshera1999@yahoo.com

Abstract: Sometimes the data to be analyzed is not complete, and this may be due to censoring. There are two type of censoring, namely Type I and Type II. Whether the censoring was intentional or accidental, it is no guarantee that the data does not include suspected observations (too small or too large). Theses suspected observations might invalidate the estimate of the parameters of the model. One way to remedy this is to use trimming or Winsorizing. Traditional methods of estimation such as maximum likelihood, least squares, Bayesian, and moments methods, usually, work well for ordinary cases. However, these methods of estimations get affected seriously with outlier observations. This suggest using methods of estimation that utilize trimmed data, such as L-moment and TL-moment. In this paper, we used six versions of L-moment and Trimmed L-moments with censored data for estimating the parameters of the Lomax distribution. The performance of the presented methods were compared with each other through a simulation study besides two real data sets. The results show that, for some cases, the use of Type-BD method is a better option than other methods. Our approach is similar to that of [28] which utilized the L-moments for the estimation of the parameters of a statistical distribution in the presence of censored observations.

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1. Introduction

In life testing applications, the Lomax distribution finds utility in modeling various real-world problems, spanning disciplines such as engineering sciences, medical sciences, actuarial science, economics, wealth, and income. Typically, the analysis of a dataset requires the estimation of distribution parameters. Various methods, including least squares (LS), Bayesian, maximum likelihood estimation (MLE) method, and method of moments, are employed for parameter estimation. Classical techniques like LS, MLEM, and Bayesian methods can be affected by extreme values. Therefore, there is a need for a robust estimation method to mitigate the impact of extreme observations.

The Lomax distribution, introduced by [15]. Lomax employed this distribution for analyzing data related to the lifespan of businesses. This distribution, fundamentally a Pareto distribution with a shifted starting point, is extensively applied in the field of survival analysis and is behind numerous practical uses in actuarial science, economics, and the business domain. Lomax distribution is characterized by two parameters: a scale parameter (λ) and a shape parameter (α). The notation $X \sim \text{Lomax}(\alpha, \lambda)$ indicates that the random variable X follows a Lomax distribution with these specific parameters. The probability density function (pdf) of the Lomax distribution is expressed as:

$$f(x; \alpha, \lambda) = \frac{\alpha}{\lambda}(1 + \frac{x}{\lambda})^{-(\alpha+1)}. \tag{1.1}$$

The cumulative distribution function (CDF) is expressed as:

$$F(x; \alpha, \lambda) = 1 - (1 + \frac{x}{\lambda})^{-\alpha}. \tag{1.2}$$

The associated quantile function is given by:

$$q(u) = \lambda\left[(1-u)^{-\frac{1}{\alpha}} - 1\right]. \tag{1.3}$$

L-moments, introduced by [10], offer numerous advantages over traditional product moments. They excel in identifying the best-fitted distribution, enabling more accurate parameter estimation compared to conventional moments. Consequently, L-moments serve as efficient estimators for parameter estimation (see, [10]). However, a notable limitation is their inapplicability to distributions with extremely heavy tails, such as the Cauchy distribution, where a finite mean does not exist, as noted by [21]. To address this problem, researchers have proposed generalizations of L-moments. One such extension is the trimmed L-moment (TL-MOM) method, introduced by [7]. This method, by assigning a weight of zero to extreme observations, claims to be more robust against extreme values compared to traditional L-moment methods.

Many statisticians utilize various methods, including moments, L-moments, and trimmed L-moments, for parameter distribution estimation. Examples of studies employing these methods include [23], [17], [2], [14], [22], [26], [27], [31], [32], [29], [9], [24], [12], and [1].

In many instances, the data being analyzed is often incomplete due to censorship. Censored data is typically categorized as either censored Type I or censored Type II. [28] pioneered the attempt to estimate right-censored data using L-moments. Several researchers have explored this method, as evidenced by studies such as [5], [6], [25], [30], and [33].

Numerous statisticians employ various methods, including maximum likelihood, Bayesian, and method moments, to estimate Lomax parameters. These estimations cover a range of sample types, including complete samples and various types of censored samples such as Type I and II censoring, hyper-censoring Type I and II, Generalized hyper-censoring Type I and II, and progressive Type I and Type II censoring. Examples of studies in this domain include those conducted by [4], [34], [3], [18], [20], and [19].

In this paper, our focus is on estimating the parameters of the Lomax distribution using six methods with censored data. To the best of our knowledge, this particular research point has not been previously addressed by other researchers.

This manuscript is arranged as follows: Sections 2 and 3 concentrate on delineating four distinct methodologies tailored for handling outliers. Section 4 presents the r^{th} population L-moments and TL-moments within the framework of the Lomax distribution for all the cases outlined. Section 5 is dedicated to the analysis of real-world data. The simulation study and subsequent results are elaborated upon in Section 6. Finally, Section 7 encapsulates the summary and conclusions drawn from the study. The equations and proofs were presented in the appendix, while the simulation tables, which were instrumental in showcasing the results, were provided at the end of the article.

2. L-Moment with Censored Data

Assuming X is a real-valued random variable with a cumulative distribution function denoted by $F(x)$ and a quantile function denoted by $q(u)$, where $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ represents the ordered statistics of a random sample of size n drawn from the distribution of X, [10] defined the L-moments of X as follows:

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E(X_{r-k:r}), \quad r = 1, 2, \ldots \tag{2.1}$$

The term "L" in "L-moments" highlights that λ_r exhibits linearity as a function of the anticipated order statistics. The expectation of an order statistic can be expressed as:

$$E(X_{i:r}) = \frac{r!}{(i-1)!(r-i)!}\int_0^1 q(u)u^{i-1}(1-u)^{r-i}du. \tag{2.2}$$

[16] introduced two method variations applicable for both right and left censored data. Consider a Type-I censored random sample of size n, denoted as $x_1, x_2, ..., x_n$, extracted from a distribution function $F(x)$ with a corresponding quantile function $q(u)$. Here, the censoring time T fulfills the condition $F(T) = c$, where c represents the fraction of observed data.

$$\underbrace{x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{m:n}}_{\text{m (observed)}} \leq T \leq \underbrace{x_{m+1:n} \leq \cdots \leq x_{n-1:n} \leq x_{n:n}}_{\text{n-m (censored)}}$$

[16] introduced two distinct variations of Direct L-moments, termed Type-AD and Type-BD, specifically designed for application in L-moments with right-censored data.

2.1. Type-AD Direct L-moments

The r^{th} Direct L-moment for Type-AD in the population is:

$$\mu_r^A = \frac{(r-1)!}{c^r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r-k-1)!k!} \binom{r-1}{k} \int_0^c u^{r-k-1}(c-u)^k q(u)du. \tag{2.3}$$

Type-AD L-moment estimators are derived utilizing the method of expectations as follows:

$$M_r^A = \frac{1}{r\binom{m}{r}} \sum_{i=1}^{m} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r-k-1} \binom{m-i}{k} X_{i:n}. \tag{2.4}$$

2.2. *Type-BD Direct L-moments*

The r^{th} Direct L-moment for Type-BD in the population is:

$$\begin{aligned}\mu_r^B = (r-1)! \sum_{k=0}^{r-1} \frac{(-1)^k}{(r-k-1)!k!} \binom{r-1}{k} \quad * \\ \left[\beta^c(r-k, k+1)q(c) + \int_0^c u^{r-k-1}(1-u)^k q(u)du\right].\end{aligned} \tag{2.5}$$

Type-BD L-moment estimators are obtained through the method of expectations as follows:

$$\begin{aligned}M_r^B = \frac{1}{r\binom{n}{r}} \Bigg[\sum_{i=1}^{m} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r-k-1} \binom{n-i}{k} X_{i:n} \\ + \sum_{i=m+1}^{n} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r-k-1} \binom{n-i}{k} T \Bigg].\end{aligned} \tag{2.6}$$

3. TL-Moment with Censored Data

[8] introduced TL-moments, where the core concept involves substituting the expected value $E(X_{r-k:r})$ with $E(X_{r+t_1-k:r+t_1+t_2})$. Consequently, for each r, the sample size of a random sample is increased from the original r to $r+t_1+t_2$. This augmentation is achieved by exclusively manipulating the anticipated values of these r adapted order statistics, specifically $X_{t_1+1:r+t_1+t_2}, X_{t_1+2:r+t_1+t_2}, ..., X_{t_1+r:r+t_1+t_2}$ through the process of trimming the smallest t_1 and largest t_2 elements from the theoretical random sample.

The TL-moment pertaining to the r^{th} order of the random variable X is characterized as:

$$\lambda_r^{(t_1,t_2)} = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r+t_1-k:r+t_1+t_2}), \quad r = 1, 2, \ldots. \tag{3.1}$$

The anticipated values of the order statistics are expressed as:

$$E(X_{i:r}) = \frac{r!}{(i-1)!(r-i)!} \int_0^1 q(u)u^{i-1}(1-u)^{r-i}du. \tag{3.2}$$

[8] introduced the subsequent estimator for sample TL-moments:

$$l_r^{(t_1,t_2)} = \frac{1}{r\binom{n}{r+t_1+t_2}} \sum_{i=t_1+1}^{n-t_2} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r+t_1-k-1} \binom{n-i}{k+t_2} X_{i:n}. \tag{3.3}$$

The fundamental concept behind the method of expectation (Analogy principle) involves deriving the expected values of certain functions associated with the random variable under consideration. These expected values are then extrapolated to a sample, wherein the resultant outcomes are equated. By resolving these corresponding results, the unknown parameters can be determined.

Assume a Type-I censored random sample of size n, denoted as $x_1, x_2, ..., x_n$, drawn from a population characterized by a distribution function $F(x)$ and quantile function $q(u)$. As per the formula for TL-moments (3.1), it is established that TL-moments are defined as:

$$\lambda_r^{(t_1,t_2)} = \frac{(r+t_1+t_2)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!(t_2+k)!} \binom{r-1}{k} \int_0^1 q(u) u^{r+t_1-k-1} (1-u)^{t_2+k} du. \tag{3.4}$$

When considering the left trim as t_1, wherein $t_2 = 0$, the utilization of formula (3.4) yields:

$$\lambda_r^{(t_1,0)} = \frac{(r+t_1)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!k!} \binom{r-1}{k} \int_0^1 u^{r+t_1-k-1} (1-u)^k q(u) du. \tag{3.5}$$

In this scenario, let the censoring time T adhere to the condition $F(T) = c$, where c denotes the fraction of observed data. Consequently, the random sample assumes the structure $x_{t_1+1}, x_{t_1+2}, ..., x_n$.

$$\underbrace{x_{1:n} \le x_{2:n} \le \cdots \le x_{t_1:n}}_{t_1 \text{ (trimmed)}} \le \underbrace{x_{t_1+1:n} \le x_{t_1+2:n} \le \cdots \le x_{m:n}}_{\text{m (observed)}} \le T \le \underbrace{x_{m+1:n} \le \cdots \le x_{n-1:n} \le x_{n:n}}_{n-t_1-m \text{ (censored)}}$$

3.1. *Type-AT TL-moments*

The quantile function related to Type-AT TL-moments can be expressed as:

$$y^A(u) = q(uc), \quad 0 < u < 1. \tag{3.6}$$

Replacing the variables into equation (3.5) results in the formulation of Type-AT TL-moments, wherein:

$$\begin{aligned} \mu_r^{A(t_1,0)} &= \frac{(r+t_1)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!k!} \binom{r-1}{k} \int_0^1 u^{r+t_1-k-1} (1-u)^k y^A(u) du \\ &= \frac{(r+t_1)!}{rc^{r+t_1}} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!k!} \binom{r-1}{k} \int_0^c u^{r+t_1-k-1} (c-u)^k q(u) du. \end{aligned} \tag{3.7}$$

If we consider the smallest trim value to be one, denoted as $t_1 = 1$, then the application of (3.7) yields:

$$\mu_r^{A(1,0)} = \frac{(r+1)!}{rc^{r+1}} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r-k)!k!} \binom{r-1}{k} \int_0^c u^{r-k} (c-u)^k q(u) du. \tag{3.8}$$

If we assume that the minimum trim value is two, i.e., $t_1 = 2$, then employing (3.7) leads to:

$$\mu_r^{A(2,0)} = \frac{(r+2)!}{rc^{r+2}} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r-k+1)!k!} \binom{r-1}{k} \int_0^c u^{r-k+1} (c-u)^k q(u) du. \tag{3.9}$$

The estimators for Type-AT TL-moments, derived through the method of expectations, are expressed as:

$$M_r^{A(t_1,0)} = \frac{1}{r\binom{m}{r+t_1}} \sum_{i=t_1+1}^{m} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r+t_1-k-1} \binom{m-i}{k} X_{i:n} \tag{3.10}$$

3.2. *Type-BT TL-moments*

The quantile function associated with Type-BT TL-moments can be described as:

$$y^B(u) = \begin{cases} q(u), & 0 < u < c \\ q(c), & c \le u < 1 \end{cases}$$

Upon substituting into the left trimming formula in (3.5), the expression for Type-BT TL-moments is derived as:

$$\begin{aligned}
\mu_r^{B(t_1,0)} &= \frac{(r+t_1)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!k!} \binom{r-1}{k} \int_0^1 u^{r+t_1-k-1}(1-u)^k y^B(u)du \\
&= \frac{(r+t_1)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!k!} \binom{r-1}{k} * \\
&\quad \left[\int_0^c u^{r+t_1-k-1}(1-u)^k q(u)du + q(c) \int_c^1 u^{r+t_1-k-1}(1-u)^k du\right].
\end{aligned}$$

Upon substituting into the left trimming formula in (3.5), the expression for Type-BT TL-moments is derived as:

$$\begin{aligned}
\mu_r^{B(t_1,0)} &= \frac{(r+t_1)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!k!} \binom{r-1}{k} * \\
&\quad \left[\beta^c(r+t_1-k, k+1)q(c) + \int_0^c u^{r+t_1-k-1}(1-u)^k q(u)du\right],
\end{aligned} \tag{3.11}$$

where $\beta^c(a, b)$ represents the upper incomplete beta function.

If we assume the smallest trim value to be one, represented as $t_1 = 1$, then by utilizing (3.11), we obtain:

$$\begin{aligned}
\mu_r^{B(1,0)} &= \frac{(r+1)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r-k)!k!} \binom{r-1}{k} * \\
&\quad \left[\beta^c(r-k+1, k+1)q(c) + \int_0^c u^{r-k}(1-u)^k q(u)du\right].
\end{aligned} \tag{3.12}$$

If we consider the smallest trim value to be two, denoted as $t_1 = 2$, then applying (3.11) yields:

$$\begin{aligned}
\mu_r^{B(2,0)} &= \frac{(r+2)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r-k+1)!k!} \binom{r-1}{k} * \\
&\quad \left[\beta^c(r-k+2, k+1)q(c) + \int_0^c q(u)u^{r-k+1}(1-u)^k du\right].
\end{aligned} \tag{3.13}$$

The estimators for Type-BT TL-moments, obtained through the method of expectations, are expressed as:

$$M_r^{B(t_1,0)} = \frac{1}{r\binom{n}{r+t_1}} \left[\sum_{i=t_1+1}^{m} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r+t_1-k-1} \binom{n-i}{k} X_{i:n} + \left(\sum_{i=m+1}^{n} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r+t_1-k-1} \binom{n-i}{k} \right) T \right]. \tag{3.14}$$

4. Parameter Estimation based on Censored Data

In cases where outliers are identified on the right side of the data, we conduct comparisons across different scenarios. This includes the maximum likelihood method, examining L-moments with Type-AD and Type-BD within the censored method, along with TL-moments that exclude either one or two of the largest values. This section introduces the r^{th} population L-moments and TL-moments for all the cases presented in the context of the Lomax distribution.

4.1. Maximum Likelihood method

In real life, it can be difficult to obtain a complete data set; often, the data is censored. Because of time constraints or a lack of funds, scientific experiments may have to be halted (or stopped) before all items fail. Type-I and Type-II censoring are the most fundamental of the various censoring schemes. When the experimental time T is fixed but the number of failures is random, Type-I censoring occurs on the right (The failure time of some items is considered to be left censored if it is less than the value observed). When the number of failures r is fixed but the experimental time is random, Type-II censoring occurs. We will focus on Type-I right censoring in this paper.

Suppose we have a random sample $x_1, x_2, ..., x_n$ from Lomax distribution, defined in Equation (1.1). The Likelihood function of the parameters α and λ based on Type I right censored is given by:

$$L(\alpha, \lambda; x) \propto \prod_{i=1}^{m} f(x_i)(1 - F(T))^{n-m}.$$

Then the log-likelihood function can be expressed by

$$l = logL(\alpha, \lambda; x) \propto \sum_{i=1}^{m} logf(x_i) + (n - m)log(1 - F(T))$$

,

$$\begin{aligned} l = logL(\alpha, \lambda; x) &\propto \sum_{i=1}^{m} logf(x_i) + (n - m)log(1 - F(T)) \\ &\propto \sum_{i=1}^{m} log(\frac{\alpha}{\lambda}(1 + \frac{x_i}{\lambda})^{-(\alpha+1)}) + (n - m)log(1 - (1 - (1 + \frac{T}{\lambda})^{-\alpha})) \end{aligned}$$

$$\propto \quad mlog(\alpha) - mlog(\lambda) - (\alpha + 1) \sum_{i=1}^{m} log(1 + \frac{x_i}{\lambda}) + (n-m)log(1 + \frac{T}{\lambda})^{-\alpha} \tag{4.1}$$

Differentiating Equation (4.1) with respect to α and λ and equating each to zero and solving them numerically, we get the estimates ofα and λ.
On the other hand the likelihood function of the parameters α and λ based on type I left censored takes the forms

$$L(\alpha, \lambda; x) \propto (F(T))^s \prod_{i=s+1}^{n} f(x_i).$$

and the log-likelihood function can be expressed by

$$l = logL(\alpha, \lambda; x) \propto +slog(F(T)) + \sum_{i=s+1}^{n} logf(x_i)$$

,

$$\begin{aligned} l = logL(\alpha, \lambda; x) &\propto +slog(F(T)) + \sum_{i=s+1}^{n} logf(x_i) \\ &\propto slog(1 - (1 + \frac{T}{\lambda})^{-\alpha}) + \sum_{i=s+1}^{n} log(\frac{\alpha}{\lambda}(1 + \frac{x_i}{\lambda})^{-(\alpha+1)}) \\ &\propto slog(1 - (1 + \frac{T}{\lambda})^{-\alpha}) + (n-s)log(\alpha) - (n-s)log(\lambda) - (\alpha + 1) \sum_{i=s+1}^{n} log(1 + \frac{x_i}{\lambda}) \end{aligned} \tag{4.2}$$

Differentiating Equation (4.2) with respect to α and λ and equating each to zero and solving them numerically, we get the estimates ofα and λ.

4.2. Type-AD

The initial four Type-AD moments for the Lomax distribution are as follows:

$$\mu_1^A = \frac{\lambda}{c} Beta_c(1, 1 - \frac{1}{\alpha}) - \lambda, \tag{4.3}$$

$$\mu_2^A = \frac{-\lambda}{c} Beta_c(1, 1 - \frac{1}{\alpha}) + \frac{2\lambda}{c^2} Beta_c(2, 1 - \frac{1}{\alpha}), \tag{4.4}$$

$$\mu_3^A = \frac{\lambda}{c} Beta_c(1, 1 - \frac{1}{\alpha}) - \frac{6\lambda}{c^2} Beta_c(2, 1 - \frac{1}{\alpha}) + \frac{6\lambda}{c^3} Beta_c(3, 1 - \frac{1}{\alpha}), \tag{4.5}$$

$$\mu_4^A = \frac{-\lambda}{c} Beta_c(1, 1 - \frac{1}{\alpha}) + \frac{12\lambda}{c^2} Beta_c(2, 1 - \frac{1}{\alpha}) - \frac{30\lambda}{c^3} Beta_c(3, 1 - \frac{1}{\alpha}) + \frac{20\lambda}{c^4} Beta_c(4, 1 - \frac{1}{\alpha}). \tag{4.6}$$

4.3. Type-BD

The first four Type-BD moments for the Lomax distribution are as follows:

$$\mu_1^B = \lambda(1-c)^{1-\frac{1}{\alpha}} + \lambda[Beta_c(1, 1-\frac{1}{\alpha}) - 1], \tag{4.7}$$

$$\mu_2^B = \lambda c(1-c)^{1-\frac{1}{\alpha}} - \lambda Beta_c(1, 1-\frac{1}{\alpha}) + 2\lambda Beta_c(2, 1-\frac{1}{\alpha}), \tag{4.8}$$

$$\begin{aligned}\mu_3^B =&[-c + c^2 - \frac{2}{3}c^3 + 4Beta_c(2,2)]\lambda(1-c)^{-\frac{1}{\alpha}}\\ &- [2c^2 - \frac{4}{3}c^3 + 4Beta_c(2,2)]\lambda + \lambda Beta_c(1, 1-\frac{1}{\alpha})\\ &- 6\lambda Beta_c(2, 1-\frac{1}{\alpha}) + 6\lambda Beta_c(3, 1-\frac{1}{\alpha}),\end{aligned} \tag{4.9}$$

$$\begin{aligned}\mu_4^B =&[\frac{1}{4}(1-(1-c)^4) - \frac{c^4}{4} + 9Beta_c(3,2) - 9Beta_c(2,3)][\lambda((1-c)^{-\frac{1}{\alpha}} - 1)]\\ &+ \lambda c - 6\lambda c^2 + 10\lambda c^3 - 5\lambda c^4 - \lambda Beta_c(1, 1-\frac{1}{\alpha})\\ &+ 12\lambda Beta_c(2, 1-\frac{1}{\alpha}) - 30\lambda Beta_c(3, 1-\frac{1}{\alpha}).\end{aligned} \tag{4.10}$$

4.4. Type-AT ($t_1 = 1$)

The initial four Type-AT TL-moments with censored data, considering the exclusion of the largest value, are as follows:

$$\mu_1^{A(1,0)} = \frac{2\lambda}{c^2} Beta_c(2, 1-\frac{1}{\alpha}) - \lambda, \tag{4.11}$$

$$\mu_2^{A(1,0)} = \frac{3\lambda}{c^3}\left[-cBeta_c(2, 1-\frac{1}{\alpha}) + \frac{3}{2}Beta_c(3, 1-\frac{1}{\alpha})\right], \tag{4.12}$$

$$\mu_3^{A(1,0)} = \frac{4\lambda}{c^4}\left[c^2Beta_c(2, 1-\frac{1}{\alpha}) - 4cBeta_c(3, 1-\frac{1}{\alpha}) + \frac{10}{3}Beta_c(4, 1-\frac{1}{\alpha})\right], \tag{4.13}$$

$$\mu_4^{A(1,0)} = \frac{5\lambda}{c^5}\left[-c^3Beta_c(2, 1-\frac{1}{\alpha}) + \frac{15}{2}c^2Beta_c(3, 1-\frac{1}{\alpha}) - 15cBeta_c(4, 1-\frac{1}{\alpha}) + \frac{35}{4}Beta_c(5, 1-\frac{1}{\alpha})\right]. \tag{4.14}$$

4.5. Type-AT ($t_1 = 2$)

The initial four Type-AT TL-moments with censored data, where the two largest values have been removed, are as follows:

$$\mu_1^{A(2,0)} = \frac{3\lambda}{c^3} Beta_c(3, 1-\frac{1}{\alpha}) - \lambda, \tag{4.15}$$

$$\mu_2^{A(2,0)} = \frac{2\lambda}{c^4}\left[-3cBeta_c(3, 1-\frac{1}{\alpha}) + 4Beta_c(4, 1-\frac{1}{\alpha})\right], \tag{4.16}$$

$$\mu_3^{A(2,0)} = \frac{10\lambda}{c^3}Beta_c(3, 1-\frac{1}{\alpha}) - \frac{100\lambda}{3c^4}Beta_c(4, 1-\frac{1}{\alpha}) + \frac{25\lambda}{c^5}Beta_c(5, 1-\frac{1}{\alpha}), \tag{4.17}$$

$$\mu_4^{A(2,0)} = \frac{-15\lambda}{c^3}Beta_c(3, 1-\frac{1}{\alpha}) + \frac{90\lambda}{c^4}Beta_c(4, 1-\frac{1}{\alpha}) - \frac{315\lambda}{2c^5}Beta_c(5, 1-\frac{1}{\alpha}) + \frac{84\lambda}{c^6}Beta_c(6, 1-\frac{1}{\alpha}). \tag{4.18}$$

4.6. Type-BT ($t_1 = 1$)

The first four Type-BT TL-moments with censored data, where one largest value has been trimmed, can be described as follows:

$$\mu_1^{B(1,0)} = \lambda\left[(1-c^2)(1-c)^{-\frac{1}{\alpha}} - 1\right] + 2\lambda Beta_c(2, 1-\frac{1}{\alpha}), \tag{4.19}$$

$$\begin{aligned}\mu_2^{B(1,0)} =& \lambda\left[3Beta_c(2,2) - \frac{c^3}{2}\right]\left[(1-c)^{-\frac{1}{\alpha}} - 1\right] \\ &+ \frac{3\lambda}{2}c^2(1-c) - 3\lambda Beta_c(2, 1-\frac{1}{\alpha}) + \frac{9}{2}\lambda Beta_c(3, 1-\frac{1}{\alpha}),\end{aligned} \tag{4.20}$$

$$\begin{aligned}\mu_3^{B(1,0)} =& \lambda\left[8Beta_c(3,2) - 4Beta_c(2,3) - \frac{c^4}{3}\right]\left[(1-c)^{-\frac{1}{\alpha}} - 1\right] \\ &+ 4\lambda Beta_c(2, 1-\frac{1}{\alpha}) - 2\lambda c^2 - 16\lambda Beta_c(3, 1-\frac{1}{\alpha}) + \frac{16}{3}\lambda c^3 \\ &+ \frac{40}{3}\lambda Beta_c(4, 1-\frac{1}{\alpha}) - \frac{10}{3}\lambda c^4,\end{aligned} \tag{4.21}$$

$$\begin{aligned}\mu_4^{B(1,0)} =& \lambda\left[15Beta_c(4,2) - \frac{45}{2}Beta_c(3,3) + 5Beta_c(2,4) - \frac{c^5}{4}\right]\left[(1-c)^{-\frac{1}{\alpha}} - 1\right] \\ &- 5\lambda Beta_c(2, 1-\frac{1}{\alpha}) + \frac{5}{2}\lambda c^2 + \frac{75}{2}\lambda Beta_c(3, 1-\frac{1}{\alpha}) - \frac{75}{6}\lambda c^3 \\ &- 75\lambda Beta_c(4, 1-\frac{1}{\alpha}) + \frac{75}{4}\lambda c^4 + \frac{175}{4}\lambda Beta_c(5, 1-\frac{1}{\alpha}) - \frac{35}{4}\lambda c^5.\end{aligned} \tag{4.22}$$

4.7. Type-BT ($t_1 = 2$)

The first four Type-BT TL-moments with data that has been censored, where two of the largest values have been omitted, are as follows:

$$\mu_1^{B(2,0)} = \lambda\left[(1-c^3)(1-c)^{-\frac{1}{\alpha}} - 1\right] + 3\lambda Beta_c(3, 1-\frac{1}{\alpha}), \tag{4.23}$$

$$\begin{aligned}\mu_2^{B(2,0)} =& \lambda\left[6Beta_c(3,2) - \frac{c^4}{2}\right]\left[(1-c)^{-\frac{1}{\alpha}} - 1\right] \\ &- 6\lambda Beta_c(3, 1-\frac{1}{\alpha}) + 8\lambda Beta_c(4, 1-\frac{1}{\alpha}) + 2\lambda c^3 - 2\lambda c^4,\end{aligned} \tag{4.24}$$

$$\mu_3^{B(2,0)} =\lambda\left[\frac{40}{3}Beta_c(4,2) - 10Beta_c(3,3) - \frac{c^5}{3}\right]\left[(1-c)^{-\frac{1}{\alpha}} - 1\right]$$
$$+ 10\lambda\left[Beta_c(3, 1-\frac{1}{\alpha}) - \frac{c^3}{3}\right] - \frac{100}{3}\lambda\left[Beta_c(4, 1-\frac{1}{\alpha}) - \frac{c^4}{4}\right]$$
$$+ 25\lambda\left[Beta_c(5, 1-\frac{1}{\alpha}) - \frac{c^5}{5}\right], \tag{4.25}$$

$$\mu_4^{B(2,0)} =\lambda\left[\frac{45}{2}Beta_c(5,2) - 45Beta_c(4,3) + 15Beta_c(3,4) - \frac{c^6}{4}\right]\left[(1-c)^{-\frac{1}{\alpha}} - 1\right]$$
$$- 15\lambda Beta_c(3, 1-\frac{1}{\alpha}) + 5\lambda c^3 + 90\lambda Beta_c(4, 1-\frac{1}{\alpha}) - \frac{45}{2}\lambda c^4$$
$$- \frac{315}{2}\lambda Beta_c(5, 1-\frac{1}{\alpha}) + \frac{63}{2}\lambda c^5 + 84\lambda Beta_c(6, 1-\frac{1}{\alpha}) - 14\lambda c^6. \tag{4.26}$$

5. Real Data Analysis

In this segment, we examine actual data to demonstrate that the Lomax distribution can serve as a suitable model for the lifetime data.

Data set 1

The dataset is sourced from [13], comprising 128 observations representing remission times (in months) for a random sample of 128 bladder cancer patients. Table (1) provides a descriptive summary of this dataset.

Table 1. Descriptive statistics

Minimum	0.08
Maximum	79.05
Mean	9.36
Median	6.39
Standard deviation	10.50
Kurtosis	18.48

Figure 1 depicts the graphical comparison between the empirical distribution of the dataset and the theoretical distribution of the Lomax. We can observe easily from this figure that the Lomax distribution is a good fit for this data.

The numerical estimation of the sample parameter is conducted using equations (4.3) through (4.26). The Moments estimate is obtained, and Table 2 presents the parameter Moments of the Lomax distribution.

Data set 2

Here, Arset data has been considered which was reported by [18], [18] and some references therein. This data represents the failure times of 50 electronic devices in weeks.The data are:

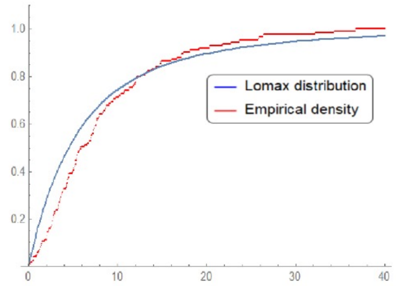

Figure 1. The empirical distribution of the bladder cancer data and the estimated Lomax distribution.

Table 2. Parameter estimates in the case of the Lomax distribution using different estimation methods based on real data set 1

Method	alpha	lamda	skew	kur.
Type-AD	27	0.0163	0.2563	0.0938
Type-BD	75	0.0633	-217.7	0.0962

1.7,2.2,14.4,1.1,0.4,20.6,5.3,0.7,1.9,13.0,12.0,9.3,1.4, 18.7,8.5,25.5,11.6,14.1,22.1,1.1,2.5,14.4, 1.7,37.6, 0.6, 2.2,39.0,0.3,15.0,11.0,7.3,22.9,1.7,0.1,1.1,0.6,9.0,1.7, 7.0,20.1,0.4,2.8,14.1,9.9, 10.4, 10.7,30.0, 3.6,5.6, 30.8, 13.3,4.2,5.5,3.4,11.9,21.5,27.6,36.4,2.7,64.0,1.5,2.5,27.4,1.0, 27.1,20.2,16.8, 53,9.7,27.5,2.5,27.0.

The maximum likelihood estimates are obtained for the Lomax distribution as $\hat{\alpha} = 2.39$ and $\hat{\lambda} = 19.790$.

The Kolomogorov-Smirno(KS) goodness of-fit test was applied to verify the Lomax distribution fits this data set.The test was carried out at 5% level of significance and gave test statistics value as 0.1205 and p-value as 0.2467. Using this data set, the estimates of different estimation methods for Lomax parameters based on Type I left and right censored are displayed in Table 3.

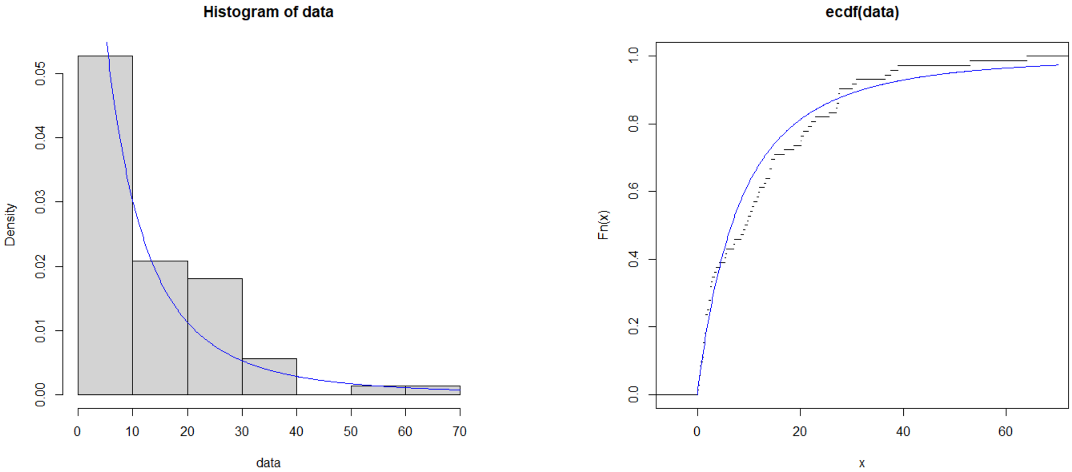

Figure 2. Plots of fitted pdf and cdf of the Lomax distribution for the data set

Table 3. Parameter estimates in the case of the Lomax distribution using different estimation methods based on real data set 2

Type I Right Censored, 10%			
Parameter	mle	Direct L-Moments	
		Type-AD	Type-BD
α	3.6491	1.9541	3.5662
λ	0.4000	17.1872	34.9653

6. Simulation and Results

We conducted a simulation study to compare four distinct estimation approaches with the MLE. These methods are: Type-AD, Type-BD, Type-AT1, Type-AT2, Type-BT1, and Type-BT2. The study involved generating random samples in sizes of 50, 100, and 200 from the Lomax distribution. These samples were initialized using parameter values of $\lambda = 0.5, 0.9, 2$ and $\alpha = 0.5, 0.9$. Following the generation process, the resultant data was methodically arranged in ascending order. Employing censoring levels fixed at 10% and 25%, the simulation iterated 10,000 times, computing statistical measures including means, root mean square errors (RMSEs), and relative absolute biases (RBs) for each combination of sample size and parameter values. The outcomes were systematically organized into Tables 6 to 17 to facilitate a comprehensive and comparative evaluation of the performance of the six estimation methods under various scenarios and conditions.

At a 25% censoring level, the Type-BD method consistently exhibited the lowest values in both RAB and RMSE.

At a 10% censoring level, the Type-BD method was typically the least values in both RAB and RMSE, followed by the Type-BT1 method.

At $n = 200$, with a censoring level of 25% and $\alpha = 0.5$, as the value of λ increases, the RAB value decreases, and the RMSE value also decreases, for all methods (the same effect did not occur at a 10% censoring level).

When $n = 50$, with a 25% censoring level and $\alpha = 0.5$, it is observed that, in most cases, an increase in the value of λ leads to an elevation in the RAB value.

At $n = 50$, with a 10% censoring level and $\alpha = 0.5$, it is consistently observed that an increase in the value of λ results in a reduction of both RAB and RMSE values.

When $n = 100$, the trend exhibited instability, yet noteworthy observations surfaced when $\alpha = 0.5$ and $\lambda = 0.9$, as the results consistently exhibited superiority.

When $n = 200$ and at a 25% censoring level, it is observed that the RMSE value for α increases as the α value itself increases.

In the Type-AD method, it is noted that an increase in the α value leads to an increase in both RAB for (α and λ) and RMSE for α.

At a 10% censoring level, it is consistently observed that an increase in the α value leads to an increase in both RAB and RMSE in all methodologies.

In the majority of instances, the Type-AD method provides the lowest RMSE.

The difference between alpha and lambda is significant, with alpha being small, and when lambda is large, the results are better.

In small samples, we observe poor results, and when we attempted to reduce the sample size to 50, the program ceased to function.

In the case of the MLE, We generate different sample sizes (50, 100 and 200) from the Lomax distribution with various combinations of the values of the two parameters (α , λ) such as (0.5, 0.5); (0.5, 2); (2,0.5) and (2,2). The estimated values, biases, absolute value of relative bias (RAB) and root mean squared error (RMSE) are computed based on Type I censoring and reported in Tables 4 and 5.

From Tables 4 and 5, as it was expected, the RAB and RMSE decrease when sample sizes increase in most cases.

7. Summary and Conclusions

This paper focuses on utilizing six methods suitable for censored data within the context of the Lomax distribution. The performance of these methods is thoroughly compared through simulation studies and examination of two real datasets. The simulation study shows that, for some cases, the use of Type- BD method is usually a better option than other methods and therefore it is recommended.

APPENDIX

When we seek to integrate the quantile function of the Lomax distribution over the range from zero to c, as follows:

$$\int_0^c \lambda\left[(1-u)^{-\frac{1}{\alpha}} - 1\right] du$$

Knowing that, the lower incomplete beta function known as follows:

$$Beta_c(a, b) = \int_0^c t^{a-1}(1-t)^{b-1} dt$$

Therefore,

$$\int_0^c q(u)du = \lambda Beta_c(1, 1 - \frac{1}{\alpha}) - \lambda c. \tag{A. 1}$$

Furthermore,

$$\int_0^c uq(u)du$$

utilizing the quantile function of the Lomax distribution,

$$q(u) = \lambda\left[(1-u)^{-\frac{1}{\alpha}} - 1\right].$$

$$\lambda \int_0^c u\left[(1-u)^{-\frac{1}{\alpha}} - 1\right] du$$
$$= \lambda \int_0^c u(1-u)^{-\frac{1}{\alpha}} du - \lambda \int_0^c u du$$

Thus,

$$\int_0^c uq(u)du = \lambda Beta_c(2, 1 - \frac{1}{\alpha}) - \frac{\lambda}{2}c^2. \tag{A. 2}$$

Also,

$$\int_0^c u^2 q(u)du$$
$$= \lambda \int_0^c u^2\left[(1-u)^{-\frac{1}{\alpha}} - 1\right] du$$
$$= \lambda \int_0^c u^2(1-u)^{-\frac{1}{\alpha}} du - \lambda \int_0^c u^2 du$$

This implies,

$$\int_0^c u^2 q(u)du = \lambda Beta_c(3, 1 - \frac{1}{\alpha}) - \frac{\lambda}{3}c^3. \tag{A. 3}$$

Similarly,

$$\int_0^c u^3 q(u)du = \lambda Beta_c(4, 1 - \frac{1}{\alpha}) - \frac{\lambda}{4}c^4. \tag{A. 4}$$

Too,

$$\int_0^c u^4 q(u)du = \lambda Beta_c(5, 1 - \frac{1}{\alpha}) - \frac{\lambda}{5}c^5. \tag{A. 5}$$

Table 4. MLE estimates in the case of the Lomax distribution based on censoring type I

alpha=0.5 , lambda=0.5							
		α			λ		
10%	n	Est.	RAB	Rmse	Est.	RAB	Rmse
Right	50	0.5641	0.1281	0.1546	0.6608	0.3215	0.3279
	100	0.5385	0.0770	0.9845	0.6000	0.2002	0.2245
	200	0.5349	0.06983	0.6737	0.6002	0.2003	0.1686
alpha=0.5 , lambda=2							
		alpha			lambda		
10%	n	Est.	RAB	Rmse	Est.	RAB	Rmse
Right	50	0.5040	0.0080	0.0967	1.9243	0.1514	0.3811
	100	0.4894	0.0212	0.0626	1.8687	0.2626	0.2552
	200	0.4870	0.0260	0.0380	1.8528	0.2945	0.1500

Table 5. MLE estimates in the case of the Lomax distribution based on censoring type I

alpha=2 , lambda=0.5							
		α			λ		
10%	n	Est.	RAB	Rmse	Est.	RAB	Rmse
Right	50	1.9723	0.0139	0.2544	0.4937	0.0032	0.1547
	100	1.9545	0.0241	0.2552	0.4759	0.0121	0.1100
	200	1.9628	0.0186	0.2540	0.4802	0.0099	0.0936
alpha=2 , lambda=2							
		alpha			lambda		
10%	n	Est.	RAB	Rmse	Est.	RAB	Rmse
Right	50	2.2223	0.1111	0.8918	2.2319	0.1160	1.2010
	100	2.1431	0.0716	0.7531	2.1683	0.0841	1.0195
	200	2.1243	0.0621	0.6135	2.1702	0.0851	0.8363

Table 6. The estimations of two parameters in the Lomax distribution, namely, the root mean square error (RMSE) and the relative absolute biases (RAB), were obtained using all proposed methods employing right censoring with parameters set at alpha=0.5 and lambda=0.5, and a sample size of n=50

C		alpha			lamda			L-c	Skew	Kur.
		Est.	RAB	RMSE	Est.	RAB	RMSE			
25%	Type-AD	0.548	0.097	0.590	0.607	0.214	0.774	0.582	0.407	0.165
	Type-BD	0.516	0.032	0.553	0.540	0.081	0.699	0.562	-0.054	-0.037
	Type-AT1	0.550	0.100	0.595	0.620	0.241	0.809	0.387	0.360	0.137
	Type-AT2	0.555	0.111	0.608	0.647	0.295	0.894	0.311	0.328	0.118
	Type-BT1	0.524	0.048	0.572	0.581	0.162	0.830	0.342	0.162	-0.083
	Type-BT2	0.541	0.083	0.607	0.654	0.308	1.060	0.246	0.070	-0.112
10%	Type-AD	0.526	0.052	0.550	0.586	0.172	0.703	0.726	0.614	0.371
	Type-BD	0.470	0.059	0.485	0.457	0.084	0.568	0.774	0.565	0.330
	Type-AT1	0.532	0.064	0.561	0.632	0.264	0.811	0.507	0.539	0.303
	Type-AT2	0.535	0.071	0.568	0.679	0.358	0.928	0.424	0.495	0.264
	Type-BT1	0.463	0.073	0.482	0.465	0.068	0.619	0.536	0.527	0.235
	Type-BT2	0.458	0.082	0.481	0.479	0.041	0.670	0.443	0.460	0.175

Table 7. The estimations of two parameters in the Lomax distribution, namely, the root mean square error (RMSE) and the relative absolute biases (RAB), were obtained using all proposed methods employing right censoring with parameters set at alpha=0.5 and lambda=0.5, and a sample size of n=100

C		alpha			lamda			L-c	Skew	Kur.
		Est.	RAB	RMSE	Est.	RAB	RMSE			
25%	Type-AD	0.519	0.038	0.535	0.544	0.088	0.605	0.581	0.403	0.158
	Type-BD	0.507	0.014	0.522	0.518	0.037	0.583	0.549	-0.096	-0.070
	Type-AT1	0.521	0.042	0.539	0.553	0.106	0.627	0.386	0.354	0.130
	Type-AT2	0.524	0.048	0.545	0.565	0.131	0.656	0.310	0.322	0.111
	Type-BT1	0.511	0.022	0.529	0.536	0.072	0.623	0.329	0.121	-0.115
	Type-BT2	0.518	0.036	0.540	0.565	0.131	0.682	0.234	0.021	-0.143
10%	Type-AD	0.522	0.044	0.532	0.557	0.114	0.614	0.738	0.629	0.389
	Type-BD	0.509	0.019	0.523	0.526	0.053	0.607	0.765	0.535	0.327
	Type-AT1	0.524	0.048	0.536	0.575	0.150	0.655	0.518	0.554	0.320
	Type-AT2	0.526	0.052	0.540	0.594	0.189	0.700	0.435	0.510	0.280
	Type-BT1	0.509	0.018	0.525	0.541	0.082	0.660	0.528	0.521	0.232
	Type-BT2	0.509	0.019	0.528	0.562	0.124	0.724	0.434	0.543	0.173

Table 8. The estimations of two parameters in the Lomax distribution, namely, the root mean square error (RMSE) and the relative absolute biases (RAB), were obtained using all proposed methods employing right censoring with parameters set at alpha=0.5 and lambda=0.5, and a sample size of n=200

C		alpha			lamda			L-c	Skew	Kur.
		Est.	RAB	RMSE	Est.	RAB	RMSE			
25%	Type-AD	0.520	0.041	0.536	0.542	0.084	0.599	0.581	0.403	0.158
	Type-BD	0.506	0.012	0.522	0.513	0.027	0.580	0.551	-0.090	-0.069
	Type-AT1	0.522	0.045	0.540	0.551	0.103	0.621	0.386	0.355	0.130
	Type-AT2	0.130	0.050	0.545	0.563	0.126	0.648	0.310	0.322	0.111
	Type-BT1	0.509	0.019	0.529	0.531	0.063	0.623	0.330	0.123	-0.114
	Type-BT2	0.517	0.034	0.541	0.562	0.124	0.685	0.235	0.024	-0.142
10%	Type-AD	0.510	0.021	0.515	0.523	0.046	0.548	0.732	0.617	0.371
	Type-BD	0.504	0.009	0.510	0.506	0.012	0.540	0.754	0.510	0.290
	Type-AT1	0.511	0.022	0.517	0.529	0.058	0.562	0.512	0.543	0.304
	Type-AT2	0.511	0.023	0.518	0.535	0.070	0.576	0.429	0.498	0.264
	Type-BT1	0.503	0.007	0.510	0.510	0.020	0.559	0.517	0.495	0.194
	Type-BT2	0.503	0.007	0.511	0.517	0.034	0.581	0.422	0.424	0.133

Table 9. The estimations of two parameters in the Lomax distribution, namely, the root mean square error (RMSE) and the relative absolute biases (RAB), were obtained using all proposed methods employing right censoring with parameters set at alpha=0.5 and lambda=0.9, and a sample size of n=50

C		alpha			lamda			L-c	Skew	Kur.
		Est.	RAB	RMSE	Est.	RAB	RMSE			
25%	Type-AD	0.551	0.102	0.602	0.131	0.257	1.543	0.579	0.403	0.163
	Type-BD	0.512	0.024	0.548	0.978	0.087	1.245	0.561	-0.052	-0.036
	Type-AT1	0.558	0.117	0.639	1.191	0.324	2.032	0.384	0.356	0.134
	Type-AT2	0.712	0.424	4.877	2.206	1.451	31.705	0.280	0.284	0.092
	Type-BT1	0.520	0.040	0.566	1.049	0.166	1.437	0.341	0.162	-0.080
	Type-BT2	0.536	0.073	0.596	1.176	0.306	1.728	0.246	0.071	-0.108
10%	Type-AD	0.531	0.063	0.552	1.067	0.186	1.253	0.742	0.639	0.406
	Type-BD	0.492	0.014	0.517	0.901	0.002	1.139	0.784	0.586	0.378
	Type-AT1	0.541	0.082	0.568	1.162	0.291	1.472	0.519	0.561	0.332
	Type-AT2	0.549	0.099	0.583	1.262	0.402	1.741	0.435	0.515	0.289
	Type-BT1	0.490	0.019	0.520	0.940	0.044	1.295	0.547	0.557	0.283
	Type-BT2	0.489	0.020	0.524	0.990	0.100	1.459	0.455	0.494	0.226

Table 10. The estimations of two parameters in the Lomax distribution, namely, the root mean square error (RMSE) and the relative absolute biases (RAB), were obtained using all proposed methods employing right censoring with parameters set at alpha=0.5 and lambda=0.9, and a sample size of n=100

C		alpha			lamda			L-c	Skew	Kur.
		Est.	RAB	RMSE	Est.	RAB	RMSE			
25%	Type-AD	0.513	0.026	0.528	0.969	0.077	1.069	0.581	0.402	0.157
	Type-BD	0.499	0.001	0.513	0.918	0.020	1.021	0.548	-0.090	-0.074
	Type-AT1	0.515	0.030	0.531	0.989	0.099	1.106	0.386	0.354	0.129
	Type-AT2	0.518	0.037	0.536	1.013	0.126	1.158	0.310	0.321	0.110
	Type-BT1	0.503	0.007	0.521	0.956	0.062	1.104	0.328	0.116	-0.118
	Type-BT2	0.511	0.023	0.533	1.015	0.128	1.221	0.232	0.015	-0.145
10%	Type-AD	0.518	0.036	0.528	0.995	0.105	1.093	0.738	0.629	0.388
	Type-BD	0.506	0.013	0.519	0.944	0.049	1.084	0.764	0.533	0.323
	Type-AT1	0.521	0.042	0.533	1.034	0.149	1.176	0.517	0.553	0.318
	Type-AT2	0.524	0.048	0.538	1.077	0.196	1.269	0.434	0.508	0.278
	Type-BT1	0.507	0.014	0.523	0.978	0.086	1.189	0.527	0.518	0.228
	Type-BT2	0.508	0.016	0.526	1.020	0.134	1.309	0.432	0.450	0.168

Table 11. The estimations of two parameters in the Lomax distribution, namely, the root mean square error (RMSE) and the relative absolute biases (RAB), were obtained using all proposed methods employing right censoring with parameters set at alpha=0.5 and lambda=0.9, and a sample size of n=200

C		alpha			lamda			L-c	Skew	Kur.
		Est.	RAB	RMSE	Est.	RAB	RMSE			
25%	Type-AD	0.507	0.015	0.514	0.929	0.033	0.979	0.581	0.400	0.154
	Type-BD	0.501	0.002	0.508	0.907	0.008	0.965	0.544	-0.110	-0.085
	Type-AT1	0.508	0.016	0.516	0.936	0.041	0.994	0.385	0.352	0.126
	Type-AT2	0.509	0.018	0.518	0.945	0.050	1.015	0.310	0.320	0.108
	Type-BT1	0.503	0.006	0.512	0.923	0.026	1.004	0.324	0.102	-0.129
	Type-BT2	0.506	0.013	0.518	0.949	0.055	1.058	0.229	-0.001	-0.156
10%	Type-AD	0.506	0.012	0.510	0.927	0.031	0.972	0.731	0.616	0.369
	Type-BD	0.499	0.000	0.505	0.896	0.003	0.963	0.753	0.509	0.284
	Type-AT1	0.506	0.013	0.511	0.939	0.043	0.998	0.512	0.541	0.302
	Type-AT2	0.507	0.014	0.513	0.951	0.056	1.026	0.429	0.497	0.262
	Type-BT1	0.498	0.002	0.505	0.905	0.006	1.001	0.516	0.491	0.188
	Type-BT2	0.498	0.002	0.506	0.920	0.023	1.047	0.420	0.419	0.126

Table 12. The estimations of two parameters in the Lomax distribution, namely, the root mean square error (RMSE) and the relative absolute biases (RAB), were obtained using all proposed methods employing right censoring with parameters set at alpha=0.5 and lambda=2, and a sample size of n=50

C		alpha			lamda			L-c	Skew	Kur.
		Est.	RAB	RMSE	Est.	RAB	RMSE			
25%	Type-AD	0.534	0.069	0.572	2.363	0.181	2.952	0.585	0.411	0.169
	Type-BD	0.505	0.010	0.542	2.129	0.064	2.751	0.563	-0.037	-0.035
	Type-AT1	0.539	0.079	0.591	2.459	0.229	3.309	0.389	0.363	0.140
	Type-AT2	0.549	0.098	0.633	2.621	0.310	4.009	0.312	0.330	0.119
	Type-BT1	0.515	0.030	0.564	2.321	0.160	3.302	0.342	0.164	-0.079
	Type-BT2	0.535	0.071	0.611	2.666	0.333	4.375	0.246	0.072	-0.107
10%	Type-AD	0.522	0.045	0.545	2.147	0.073	2.701	0.769	0.680	0.462
	Type-BD	0.499	0.001	0.531	2.002	0.001	2.833	0.805	0.631	0.443
	Type-AT1	0.523	0.047	0.550	2.254	0.127	3.029	0.547	0.603	0.386
	Type-AT2	0.524	0.049	0.554	2.350	0.175	3.302	0.465	0.560	0.343
	Type-BT1	0.498	0.002	0.539	2.169	0.084	3.349	0.567	0.597	0.345
	Type-BT2	0.500	0.000	0.548	2.367	0.183	3.839	0.476	0.538	0.288

Table 13. The estimations of two parameters in the Lomax distribution, namely, the root mean square error (RMSE) and the relative absolute biases (RAB), were obtained using all proposed methods employing right censoring with parameters set at alpha=0.5 and lambda=2, and a sample size of n=100

C		alpha			lamda			L-c	Skew	Kur.
		Est.	RAB	RMSE	Est.	RAB	RMSE			
25%	Type-AD	0.513	0.027	0.530	2.177	0.088	2.441	0.582	0.403	0.158
	Type-BD	0.503	0.007	0.520	2.096	0.048	2.396	0.548	-0.097	-0.073
	Type-AT1	0.514	0.029	0.533	2.209	0.104	2.529	0.386	0.355	0.130
	Type-AT2	0.517	0.034	0.538	2.254	0.127	2.651	0.311	0.323	0.111
	Type-BT1	0.508	0.016	0.528	2.179	0.089	2.591	0.327	0.117	-0.117
	Type-BT2	0.516	0.033	0.541	2.311	0.155	2.857	0.232	0.017	-0.144
10%	Type-AD	0.521	0.042	0.532	2.228	0.114	2.472	0.737	0.629	0.389
	Type-BD	0.508	0.016	0.521	2.009	0.049	2.421	0.765	0.535	0.326
	Type-AT1	0.521	0.043	0.534	2.278	0.139	2.612	0.518	0.555	0.321
	Type-AT2	0.522	0.045	0.537	2.337	0.168	2.766	0.436	0.511	0.281
	Type-BT1	0.507	0.014	0.523	2.145	0.072	2.611	0.528	0.521	0.232
	Type-BT2	0.506	0.013	0.525	2.216	0.108	2.838	0.435	0.454	0.173

Table 14. The estimations of two parameters in the Lomax distribution, namely, the root mean square error (RMSE) and the relative absolute biases (RAB), were obtained using all proposed methods employing right censoring with parameters set at alpha=0.5 and lambda=2, and a sample size of n=200

C		alpha			lamda			L-c	Skew	Kur.
		Est.	RAB	RMSE	Est.	RAB	RMSE			
25%	Type-AD	0.506	0.013	0.513	2.062	0.031	2.167	0.581	0.400	0.154
	Type-BD	0.500	0.000	0.507	2.009	0.004	2.130	0.544	-0.110	-0.085
	Type-AT1	0.507	0.015	0.515	2.078	0.039	2.0201	0.385	0.352	0.126
	Type-AT2	0.508	0.016	0.516	2.095	0.047	2.241	0.310	0.320	0.108
	Type-BT1	0.502	0.004	0.511	2.043	0.021	2.208	0.324	0.102	-0.129
	Type-BT2	0.505	0.011	0.516	2.099	0.049	2.316	0.229	-0.002	-0.156
10%	Type-AD	0.508	0.017	0.513	2.095	0.047	2.191	0.731	0.616	0.368
	Type-BD	0.503	0.006	0.508	2.038	0.019	2.175	0.753	0.506	0.285
	Type-AT1	0.509	0.018	0.514	2.119	0.059	2.249	0.511	0.541	0.301
	Type-AT2	0.509	0.019	0.515	2.146	0.073	2.315	0.429	0.496	0.262
	Type-BT1	0.502	0.005	0.509	0.061	0.030	2.264	0.515	0.492	0.189
	Type-BT2	0.502	0.005	0.510	2.098	0.049	2.372	0.420	0.419	0.127

Table 15. The estimations of two parameters in the Lomax distribution, namely, the root mean square error (RMSE) and the relative absolute biases (RAB), were obtained using all proposed methods employing right censoring with parameters set at alpha=0.9 and lambda=0.5, and a sample size of n=50

C		alpha			lamda			L-c	Skew	Kur.
		Est.	RAB	RMSE	Est.	RAB	RMSE			
25%	Type-AD	1.270	0.411	2.029	0.844	0.689	1.558	0.472	0.247	0.070
	Type-BD	0.978	0.087	1.124	0.620	0.240	0.899	0.464	-1.522	-0.077
	Type-AT1	1.268	0.409	1.764	0.871	0.742	1.391	0.300	0.226	0.061
	Type-AT2	1.336	0.485	1.941	0.963	0.926	1.676	0.231	0.205	0.053
	Type-BT1	1.052	0.169	1.370	0.737	0.475	1.452	0.265	0.039	-0.103
	Type-BT2	1.681	0.867	5.165	1.782	2.565	8.374	0.165	-0.077	-0.115
10%	Type-AD	1.238	0.376	1.727	0.881	0.762	1.386	0.563	0.403	0.185
	Type-BD	1.050	0.166	1.227	0.705	0.410	1.002	0.612	-0.547	0.174
	Type-AT1	1.216	0.351	1.636	0.873	0.746	1.361	0.381	0.367	0.167
	Type-AT2	1.212	0.346	1.647	0.871	0.742	1.357	0.312	0.356	0.154
	Type-BT1	1.041	0.156	1.222	0.705	0.411	1.034	0.412	0.382	0.131
	Type-BT2	1.031	0.146	1.203	0.705	0.410	1.029	0.334	0.339	0.101

Table 16. The estimations of two parameters in the Lomax distribution, namely, the root mean square error (RMSE) and the relative absolute biases (RAB), were obtained using all proposed methods employing right censoring with parameters set at alpha=0.9 and lambda=0.5, and a sample size of n=100

C		alpha			lamda			L-c	Skew	Kur.
		Est.	RAB	RMSE	Est.	RAB	RMSE			
25%	Type-AD	1.012	0.124	1.199	0.608	0.217	0.883	0.489	0.271	0.079
	Type-BD	0.950	0.056	1.083	0.552	0.105	0.752	0.465	-1.452	-0.092
	Type-AT1	1.006	0.118	1.182	0.605	0.210	0.857	0.313	0.246	0.068
	Type-AT2	1.017	0.130	1.216	0.618	0.236	0.899	0.245	0.225	0.060
	Type-BT1	0.973	0.081	1.142	0.578	0.157	0.828	0.268	0.028	-0.122
	Type-BT2	1.106	0.229	3.437	0.722	0.444	3.452	0.179	-0.070	-0.137
10%	Type-AD	0.959	0.065	1.010	0.563	0.127	0.655	0.595	0.444	0.211
	Type-BD	0.934	0.038	0.973	0.538	0.077	0.603	0.628	-0.299	0.169
	Type-AT1	0.965	0.072	1.013	0.574	0.148	0.659	0.403	0.402	0.181
	Type-AT2	0.973	0.081	1.025	0.588	0.177	0.682	0.329	0.374	0.162
	Type-BT1	0.938	0.042	0.982	0.547	0.094	0.626	0.420	0.380	0.112
	Type-BT2	0.943	0.048	00994	0.559	0.118	0.657	0.337	0.326	0.072

Table 17. The estimations of two parameters in the Lomax distribution, namely, the root mean square error (RMSE) and the relative absolute biases (RAB), were obtained using all proposed methods employing right censoring with parameters set at alpha=0.9 and lambda=0.5, and a sample size of n=200

C		alpha			lamda			L-c	Skew	Kur.
		Est.	RAB	RMSE	Est.	RAB	RMSE			
25%	Type-AD	0.946	0.051	0.988	0.543	0.086	0.596	0.492	0.275	0.080
	Type-BD	0.908	0.009	0.935	0.511	0.023	0.548	0.465	-1.351	-0.101
	Type-AT1	0.939	0.043	0.978	0.538	0.076	0.588	0.316	0.249	0.069
	Type-AT2	0.938	0.043	0.980	0.539	0.078	0.595	0.248	0.229	0.060
	Type-BT1	0.911	0.012	0.942	0.516	0.032	0.561	0.268	0.018	-0.132
	Type-BT2	0.921	0.023	0.958	0.528	0.056	0.584	0.184	-0.077	-0.149
10%	Type-AD	0.932	0.036	0.947	0.535	0.070	0.559	0.596	0.444	0.209
	Type-BD	0.928	0.032	0.944	0.531	0.063	0.558	0.626	-0.300	0.160
	Type-AT1	0.935	0.039	0.951	0.541	0.082	0.569	0.404	0.401	0.179
	Type-AT2	0.940	0.044	0.958	0.548	0.097	0.582	0.331	0.373	0.159
	Type-BT1	0.932	0.036	0.951	0.539	0.078	0.575	0.418	0.373	0.101
	Type-BT2	0.937	0.041	0.960	0.548	0.097	0.595	0.334	0.316	0.058

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