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Fixed point approaches fo an orthogonal (Π, ξ)-weak contraction in orthogonal Branciari metric spaces with applications

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Abstract:

The target of this manuscript is to obtain some fixed point results for generalized orthogonal (Π, ξ)-weak contraction mappings in the setting of orthogonal Branciari metric spaces. Also, axillary functions are given to help prove our results. Moreover, Some of the consequences that can be obtained from the main theorem are presented in the form of corollaries. Ultimately, the theoretical result is applied to obtain the solution of a differential equation as reinforcement and support for the results shown. **keywords:** Orthogonal Branciari metric spaces, lower limit, fixed point technique, existence solution, differential equation.

1 Introduction

M. Frechet fundamentally defined the manifest evolution of a metric space (shortly, MS) in 1906. The concept of identifying the fixed point (shortly, f.p.) of self-map was first proposed by Stefen Banach (1892–1945) in 1922. Many academics have generalized and expanded this approach in recent years, spurred on by this modern notion. Later, the notion of f.p.s was used to solve integral and differential equations with unique solutions. Following these, the literature saw the introduction of numerous MSs and f.p. theorems. Following the development of Banach's fixed theory, Branciari [1] worked on Banach's f.p. theory and one of the requirements for the theory's continuation. Azam, Arshad and Kannan [2] introduced a novel concept of f.p. result in generalized MSs (shortly, **gms**) in 2008. Interested academics may consult the works of the following writers, who used single-valued mappings and multi-valued mappings to arrive at this novel idea and propose a wide variety of f.p. theorems with contractive conditions. For more details, see [3,4,5,6,7,8,9,10,11,12,13]. Although f.p. theory has various uses, its main purpose was to demonstrate the establishment and, in certain cases, the uniqueness of a specific class of points that obeyed a specified criterion. It shows how an equation, which may take the form of an integral equation, a differential equation, a matrix equation, and so on. Since they need to be connected to an operator, these elements are known as f.p.s. A f.p. problem must be given in a basic space that has an abstract metric context, or a mapping that determines the separation between two random points. Since only MSs satisfy the prerequisites of non-negativity, the identity of indiscernible, symmetry, and the triangle inequality, these were initially the only ones that were explored. By introducing the idea of orthogonality and establishing the f.p. result, Gordji et al. [14] recently added to the body of knowledge on MS. This innovative notion of an orthogonal set, as well as many different kinds of orthogonality, has many applications. According to Eshaghi Gordji and Habibi [15], the f.p. in generalized orthogonal MS and associated findings in orthogonal MSs(shortly, OMS) are established. In addition, we suggest the papers [16,17,18,19], to the reader for more information. In the framework of orthogonal Branciri type MS, we establish novel f.p. theorem for orthogonal (Π, ξ)-weak contractions. Finally, an application of these findings to the proof of conditions for f.p. theorem of differential type equations is also provided.

2 Preliminaries

We will use the notation $[0,\infty)$ by $\Re_0^+$ throughout this paper. Gordji et al. [14] initiated the notion of an orthogonal set (or O-set) as follows:

Definition 21[14] *Let $\mathbb{U} \neq \emptyset$ and if a binary relation $\wedge \subseteq \mathbb{U} \times \mathbb{U}$ satisfies the following condition:*

$$\exists\ \mathfrak{c}_0 \in \mathbb{U} : (\forall \mathfrak{c} \in \mathbb{U}, \mathfrak{c} \wedge \mathfrak{c}_0) \quad or \quad (\forall \mathfrak{c} \in \mathbb{U}, \mathfrak{c}_0 \wedge \mathfrak{c}),$$

then it is known orthogonal set (shortly O-set) and the O-set is denoted by $(\mathbb{U}, \wedge)$.

Example 1.[14] Let the world's population, $\mathbb{U}$, be the set. If u is capable of giving blood to $\mathfrak{c}$, define the binary relation ρ on $\mathbb{U}$ by $\rho \wedge \mathfrak{c}$. According to Table 1, if $\mathfrak{c}_0$ is a person whose blood type is $\mathbb{O}^-$, then we have $\mathfrak{c}_0 \wedge \mathfrak{c}$ $\forall\ \mathfrak{c} \in \mathbb{U}$. In other words, $(\mathbb{U}, \wedge)$ is an O-set. The $\mathfrak{c}_0$ value from Definition 21 is not unique in this O-set. Notably, $\mathfrak{c}_0$ in this instance might be an individual of blood type $\mathbb{AB}^+$. In this situation, we get $\mathfrak{c} \wedge \mathfrak{c}_0$ $\forall\ \mathfrak{c} \in \mathbb{U}$.

Type	You can receive blood from	You can give blood to
$\mathbb{A}^+$	$\mathbb{A}^+\mathbb{A}^-\mathbb{O}^+\mathbb{O}^-$	$\mathbb{A}^+\mathbb{AB}^+$
$\mathbb{O}^+$	$\mathbb{O}^+\mathbb{O}^-$	$\mathbb{O}^-\mathbb{A}^+\mathbb{B}^+\varsigma\mathbb{B}^+$
$\mathbb{B}^+$	$\mathbb{B}^+\mathbb{B}^-\mathbb{O}^+\mathbb{O}^-$	$\mathbb{B}^+\mathbb{AB}^+$
$\mathbb{AB}^+$	Everyone	$\mathbb{AB}^+$
$\mathbb{A}^-$	$\mathbb{A}^-\mathbb{O}^-$	$\mathbb{A}+\mathbb{A}^-\mathbb{AB}^+\mathbb{AB}^-$
$\mathbb{O}^-$	$\mathbb{O}^-$	Everyone
$\mathbb{B}^-$	$\mathbb{B}^-\mathbb{O}^-$	$\mathbb{B}^+\mathbb{B}^-\mathbb{AB}^+\mathbb{AB}^-$
$\mathbb{AB}^-$	$\mathbb{AB}^-\mathbb{B}^-\mathbb{O}^-\mathbb{A}^-$	$\mathbb{AB}^+\mathbb{AB}^-$

Now, in this section recalls some classical and definitions of an O-sequence, properties, and preliminary notions of an $\wedge$-continuous mapping, an O-complete Branciari MSs, a $\wedge$-preserving.

Definition 22[14] *A sequence $\{\mathfrak{c}_\mathfrak{o}\}$ in O-set $(\mathbb{U}, \wedge)$ is known an orthogonal sequence (shortly, O-sequence) if*

$$(\forall \mathfrak{o} \in \mathbb{N}, \mathfrak{c}_\mathfrak{o} \wedge \mathfrak{c}_{\mathfrak{o}+1}) \quad or \quad (\forall \mathfrak{o} \in \mathbb{N}, \mathfrak{c}_{\mathfrak{o}+1} \wedge \mathfrak{c}_\mathfrak{o}).$$

Definition 23*An orthogonal partial b-metric on $\mathbb{U} \neq \emptyset$ is a mapping $\mathscr{O}_{\mathscr{P}_\mathfrak{b}} : \mathbb{U} \times \mathbb{U} \to \Re_0^+$ satisfy the following requirements $\forall\ \mathfrak{c}, \rho, \mathfrak{z} \in \mathbb{U}$ with $\mathfrak{c} \wedge \rho, \mathfrak{c} \wedge \mathfrak{z}, \rho \wedge \mathfrak{z}$:*

1. $\mathfrak{c} = \rho$ *iff* $\mathscr{O}_{\mathscr{P}_\mathfrak{b}}(\mathfrak{c},\mathfrak{c}) = \mathscr{O}_{\mathscr{P}_\mathfrak{b}}(\mathfrak{c},\rho) = \mathscr{O}_{\mathscr{P}_\mathfrak{b}}(\rho,\rho)$,
2. $\mathscr{O}_{\mathscr{P}_\mathfrak{b}}(\mathfrak{c},\mathfrak{c}) \leq \mathscr{O}_{\mathscr{P}_\mathfrak{b}}(\mathfrak{c},\rho)$,
3. $\mathscr{O}_{\mathscr{P}_\mathfrak{b}}(\mathfrak{c},\rho) = \mathscr{O}_{\mathscr{P}_\mathfrak{b}}(\rho,\mathfrak{c})$,
4. $\mathscr{O}_{\mathscr{P}_\mathfrak{b}}(\mathfrak{c},\rho) \leq \flat[\mathscr{O}_{\mathscr{P}_\mathfrak{b}}(\mathfrak{c},\mathfrak{z}) + \mathscr{O}_{\mathscr{P}_\mathfrak{b}}(\mathfrak{z},\rho)] - \mathscr{O}_{\mathscr{P}_\mathfrak{b}}(\mathfrak{z},\mathfrak{z})$.

An orthogonal partial b-MS is a pair $(\mathbb{U}, \mathscr{O}_{\mathscr{P}_\mathfrak{b}})$ s.t. (shortly, s.t.) $\mathbb{U}$ is a nonempty O-set and $\mathscr{O}_{\mathscr{P}_\mathfrak{b}}$ is an orthogonal partial b-MS on $\mathbb{U}$. The number $\flat \geq 1$ is called the coefficient of $(\mathbb{U}, \mathscr{O}_{\mathscr{P}_\mathfrak{b}})$.

Definition 24*Let $\mathbb{U} \neq \emptyset$ be an O-set and a function $\mho : \mathbb{U} \times \mathbb{U} \to \Re_0^+$ s.t. $\forall\ \mathfrak{c}, \rho \in \mathbb{U}$ with $\mathfrak{c} \wedge \rho$ and $\forall$ distinct points $\nu, \delta \in \mathbb{U}$ with $\nu \wedge \delta$ each of them different from $\mathfrak{c}$ and ρ satisfy the following requirements:*

$(\mho_1)$ $\mho(\mathfrak{c},\rho) = 0$ *iff* $\mathfrak{c} = \rho$,
$(\mho_2)$ $\mho(\mathfrak{c},\rho) = \mho(\rho,\mathfrak{c})$,
$(\mho_3)$ $\mho(\mathfrak{c},\rho) \leq \mho(\mathfrak{c},\nu) + \mho(\nu,\delta) + \mho(\delta,\rho)$ *(the rectangular inequality).*

Then $(\mathbb{U}, \mho)$ is known orthogonal Branciari type MS (shortly, OBMS).

Every OMS is an OBMS, but the converse is not true.

Definition 25[14] *Let $(\mathbb{U}, \wedge, \mho)$ be an OBMS. Then, a function $\triangledown : \mathbb{U} \to \mathbb{U}$ is known to be orthogonally continuous (or $\wedge$-continuous) in $\mathfrak{c} \in \mathbb{U}$ if for each O-sequence $\{\mathfrak{c}_\mathfrak{o}\}$ in $\mathbb{U}$ with $\mathfrak{c}_\mathfrak{o} \to \mathfrak{c}$ as $\mathfrak{o} \to \infty$, we have $\triangledown(\mathfrak{c}_\mathfrak{o}) \to \triangledown(\mathfrak{c})$ as $\mathfrak{o} \to \infty$. Also, $\triangledown$ is known to be $\wedge$-continuous on $\mathbb{U}$ if $\triangledown$ is $\wedge$-continuous in each $\mathfrak{c} \in \mathbb{U}$.*

Definition 26*Let $(\mathbb{U}, \wedge, \mho)$ be an OBMS and $\{\mathfrak{c}_\mathfrak{o}\}$ be an O-sequence in $\mathbb{U}$ and $\mathfrak{c} \in \mathbb{U}$. We call that*

(i) $\{\mathfrak{c}_\mathfrak{o}\}$ *is converge to $\mathfrak{c}$ iff $\mho(\mathfrak{c}_\mathfrak{o}, \mathfrak{c}) \to 0$ as $\mathfrak{o} \to \infty$(denoted by $\mathfrak{c}_\mathfrak{o} \to \mathfrak{c}$).*
(ii) $\{\mathfrak{c}_\mathfrak{o}\}$ *is a Cauchy O-sequence iff for each $\gamma > 0$ $\exists$ a natural number $\mathbb{N}$ s.t.$\mho(\mathfrak{c}_\mathfrak{o}, \mathfrak{c}_\aleph) < \gamma$ $\forall$ $\aleph, \mathfrak{o} > \mathbb{N}$.*
(iii) $\mathbb{U}$ *is an orthogonally complete (briefly, O-complete) iff every Cauchy O-sequence is convergent in $\mathbb{U}$.*

Definition 27[14] *Let $(\mathbb{U}, \wedge)$ be an O-set. A mapping $\triangledown : \mathbb{U} \to \mathbb{U}$ is said to be $\wedge$-preserving if $\triangledown\mathfrak{c} \wedge \triangledown\rho$ whenever $\mathfrak{c} \wedge \rho$.*

Lakzian and Samet proved a f.p. theorem of the **gms** in 2012.

Theorem 21*Let $(\mathbb{U}, \mho)$ be a Hausdorff and complete **gms**, and let $\triangledown : \mathbb{U} \to \mathbb{U}$ be a self-map satisfying*

$$\Pi(\mho(\triangledown\mathfrak{c}, \triangledown\rho)) \leq \Pi(\mho(\mathfrak{c},\rho)) - \xi(\mho(\mathfrak{c},\rho)) \qquad (1)$$

$\forall$ $\mathfrak{c}, \rho \in \mathbb{U}$, where

(i) $\Pi : \Re_0^+ \to \Re_0^+$ *is a continuous and monotone non decreasing function with $\Pi(\varphi) = 0$ iff $\varphi = 0$,*
(ii) $\xi : \Re_0^+ \to \Re_0^+$ *is a continuous function with $\xi(\varphi) = 0$ iff $\varphi = 0$.*

*Then $\triangledown$ has a unique f.p. (shortly, **ufp**).*

Liu and Chai obtained a generalization of f.p. Theorem 1.1 in 2013.

Theorem 22*Let $(\mathbb{U}, \mho)$ be a Hausdorff and complete **gms**, and let $\triangledown : \mathbb{U} \to \mathbb{U}$ be a self-map satisfying*

$$\Pi(\mho(\triangledown\mathfrak{c}, \triangledown\rho)) \leq \Pi(\varsigma_1\mho(\mathfrak{c},\rho) + \varsigma_2\mho(\mathfrak{c},\triangledown\mathfrak{c}) + \varsigma_3\mho(\rho,\triangledown\rho)) - \theta(\varsigma_1\mho(\mathfrak{c},\rho) + \varsigma_2\mho(\mathfrak{c},\triangledown\mathfrak{c}) + \varsigma_3\mho(\rho,\triangledown\rho)) \qquad (2)$$

$\forall$ $\mathfrak{c}, \rho \in \mathbb{U}$, where

$(i)\Pi : \Re_0^+ \to \Re_0^+$ *is a continuous and monotone non decreasing function with* $\Pi(\varphi) = 0$ *iff* $\varphi = 0$,
$(ii)\theta : \Re_0^+ \to \Re_0^+$ *satisfies* $\lim_{\varphi\to\tau} \theta(\varphi) > 0$ *for* $\tau > 0$ *and* $\lim_{\varphi\to\tau} \theta(\varphi) = 0$ *iff* $\tau = 0$,
$(iii)\varsigma_\vee \geq 0(\vee = 1,2,3)$ *with* $\varsigma_1 + \varsigma_2 + \varsigma_3 \leq 1$.

Then $\triangledown$ *has a* ***ufp***.

It is important to note that $\Pi : \Re_0^+ \to \Re_0^+$ is a continuous and monotone nondecreasing function, but we cannot acquire that $\varphi_1 \leq \varphi_2$ if $\Pi(\varphi_1) \leq \Pi(\varphi_2)$. The incorrect conclusion has been used extensively in the above theorems proofs. The weaken the theorems criteria and to present the right results for the previously mentioned theorems, read this work.

3 Main results

Now, we propose the new estimates of f.p. result for an orthogonal (Π,ξ)-weak Contraction on an OBMS.
Let Π be the collection of all functions $\Pi : \Re_0^+ \to \Re_0^+$ satisfy the requirements:

$(\varsigma_1)\Pi$ is monotone nondecreasing,
$(\varsigma_2)\lim_{\varphi\to\tau} \Pi(\varphi) > 0$ for $\tau > 0$ and $\lim_{\varphi\to o^+} \Pi(\varphi) = 0$,
$(\varsigma_3)\Pi(\varphi) = 0$ iff $\varphi = 0$.

Let ξ be the set of functions $\xi : \Re_0^+ \to \Re_0^+$ satisfy the requirements:

$(\mathfrak{b}_1)\liminf_{\varphi\to\tau} \xi(\varphi) > 0$ for each $\tau > 0$,
$(\mathfrak{b}_2)\xi(\varphi) \to 0$ implies that $\varphi \to 0$,
$(\mathfrak{b}_3)\xi(\varphi) = 0$ iff $\varphi = 0$.

Theorem 31*Let* $(\mathbb{U},\wedge,\mho)$ *be an O-complete Braniciari type MS, and let* $\triangledown : \mathbb{U} \to \mathbb{U}$ *be a self-map satisfying*

(i) $\forall\ \mathfrak{c},\rho \in \mathbb{U}$ *with* $\mathfrak{c}\wedge\rho$,

$$\begin{aligned}&\mho(\triangledown\mathfrak{c},\triangledown\rho) > 0\ [\Pi(\mho(\triangledown\mathfrak{c},\triangledown\rho))\\&\leq \Pi(\varsigma_1\mho(\mathfrak{c},\rho) + \varsigma_2\mho(\mathfrak{c},\triangledown\mathfrak{c}) + \varsigma_3\mho(\rho,\triangledown\rho))\\&- \xi(\varsigma_1\mho(\mathfrak{c},\rho) + \varsigma_2\mho(\mathfrak{c},\triangledown\mathfrak{c}) + \varsigma_3\mho(\rho,\triangledown\rho))]\end{aligned}$$

where $\Pi \in \Psi, \xi \in \Phi$ *and* $\varsigma_\vee \geq 0(\vee = 1,2,3)$ *with* $\varsigma_1 + \varsigma_2 + \varsigma_3 \leq 1$,
(ii)$\wedge$*-continuous,*
(iii)$\wedge$*-preserving.*

Then $\triangledown$ *has a* ***ufp***.

*Proof.*Proof of this theorem consists of the two steps.
Step 1. $\triangledown$ has the f.p. in $\mathbb{U}$.
By orthogonality, $\exists\ \mathfrak{c}_0 \in \mathbb{U}$ s.t.

$$(\forall\ \rho \in \mathbb{U}, \mathfrak{c}_0 \wedge \rho) \text{ or } (\forall\ \rho \in \mathbb{U}, \rho \wedge \mathfrak{c}_0).$$

It follows that $\mathfrak{c}_0 \wedge \triangledown(\mathfrak{c}_0)$ or $\triangledown(\mathfrak{c}_0) \wedge \mathfrak{c}_0$.
Let

$$\begin{aligned}&\mathfrak{c}_1 = \triangledown(\mathfrak{c}_0), \mathfrak{c}_2 = \triangledown(\mathfrak{c}_1) = \triangledown^2(\mathfrak{c}_0), \ldots, \mathfrak{c}_{\mathfrak{o}+1}\\&= \triangledown(\mathfrak{c}_\mathfrak{o}) = \triangledown^{\mathfrak{c}_{\mathfrak{o}+1}}(\mathfrak{c}_0)\end{aligned}$$

$\forall\ \ \mathfrak{o} \geq 0$. Since $\triangledown$ is $\wedge$-preserving, $\{\mathfrak{c}_\mathfrak{o}\}_\mathfrak{o} \geq 0$ is an O-sequence.
Case 1. $\triangledown$ has a periodic point.
Case 1.1. If $\mathfrak{c}_{\mathfrak{o}+1} = \mathfrak{c}_\mathfrak{o}$ for some $\mathfrak{o}$, then $\mathfrak{c}_\mathfrak{o}$ is a f.p. of $\triangledown$.
The remainder, we presume that $\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_\mathfrak{o}) \neq 0\ \forall\ \mathfrak{o}$.
Case 1.2. If $\mathfrak{c}_{\mathfrak{o}+2} = \mathfrak{c}_\mathfrak{o}$ fro some $\mathfrak{o}$, then $\triangledown\mathfrak{c}_\mathfrak{o}$ is a f.p. of $\triangledown$. On contrary, assume that $\triangledown\mathfrak{c}_\mathfrak{o} \neq \triangledown^2\mathfrak{c}_\mathfrak{o}$, i.e., $\mho(\triangledown\mathfrak{c}_\mathfrak{o},\triangledown^2\mathfrak{c}_\mathfrak{o}) > 0$, which implies that $\xi(\mho(\triangledown\mathfrak{c}_\mathfrak{o},\triangledown^2\mathfrak{c}_\mathfrak{o})) > 0$. By contraction (3), we have

$$\begin{aligned}&\Pi(\mho(\mathfrak{c}_\mathfrak{o},\mathfrak{c}_{\mathfrak{o}+1})) = \Pi(\mho(\triangledown^2\mathfrak{c}_\mathfrak{o},\triangledown\mathfrak{c}_\mathfrak{o}))\\&\leq \Pi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_\mathfrak{o}) + \varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}+2},\mathfrak{c}_{\mathfrak{o}+1}) + \varsigma_3\mho(\mathfrak{c}_\mathfrak{o},\mathfrak{c}_{\mathfrak{o}+1}))\\&- \xi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_\mathfrak{o}) + \varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}+2},\mathfrak{c}_{\mathfrak{o}+1}) + \varsigma_3\mho(\mathfrak{c}_\mathfrak{o},\mathfrak{c}_{\mathfrak{o}+1}))\\&= \Pi((\varsigma_1+\varsigma_2+\varsigma_3)\mho(\mathfrak{c}_\mathfrak{o},\mathfrak{c}_{\mathfrak{o}+1})) - \xi((\varsigma_1+\varsigma_2+\varsigma_3)\mho(\mathfrak{c}_\mathfrak{o},\mathfrak{c}_{\mathfrak{o}+1}))\\&= \Pi(\mho(\mathfrak{c}_\mathfrak{o},\mathfrak{c}_{\mathfrak{o}+1})) - \xi((\varsigma_1+\varsigma_2+\varsigma_3)\mho(\mathfrak{c}_\mathfrak{o},\mathfrak{c}_{\mathfrak{o}+1})), \qquad (3)\end{aligned}$$

i.e., $\xi((\varsigma_1+\varsigma_2+\varsigma_3)\mho(\mathfrak{c}_\mathfrak{o},\mathfrak{c}_{\mathfrak{o}+1})) = 0$. If $\sum_{\vee=1}^3 \varsigma_\vee \neq 0$, we obtain $\mho(\mathfrak{c}_\mathfrak{o},\mathfrak{c}_{\mathfrak{o}+1}) = 0$, a contradiction. If $\sum_{\vee=1}^3 \varsigma_\vee = 0$, by (3) we have $\Pi(\mho(\mathfrak{c}_\mathfrak{o},\mathfrak{c}_{\mathfrak{o}+1})) = 0$, i.e., $\mho(\mathfrak{c}_\mathfrak{o},\mathfrak{c}_{\mathfrak{o}+1}) = 0$ is a contradiction to the hypothesis, and so $\triangledown\mathfrak{c}_\mathfrak{o}$ is a f.p. of $\triangledown$.
Case 1-3. If $\aleph,\mathfrak{o} \in \mathbb{N}$ s.t.$\mathfrak{c}_\aleph = \mathfrak{c}_\mathfrak{o}$ with $\aleph - \mathfrak{o} > 2$ and $\mathfrak{c}_\vee \neq \mathfrak{c}_\varepsilon\ \forall\ \mathfrak{o} \leq \vee \neq \varepsilon < \aleph$, we claim that $\triangledown^{\aleph-\mathfrak{o}-1}\mathfrak{c}_\mathfrak{o}$ is a f.p. of $\triangledown$. Contrary assume that it is not hold, then

$$\begin{aligned}&\triangledown^{\aleph-\mathfrak{o}-1}\mathfrak{c}_\mathfrak{o} \neq \triangledown^{\aleph-\mathfrak{o}}\mathfrak{c}_\mathfrak{o} \iff \mho(\triangledown^{\aleph-\mathfrak{o}-1}\mathfrak{c}_\mathfrak{o},\\&\triangledown^{\aleph-\mathfrak{o}}\mathfrak{c}_\mathfrak{o}) > 0 \iff \mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_\aleph) > 0,\end{aligned}$$

conclude that

$$\xi(\mho(\triangledown^{\aleph-\mathfrak{o}-1}\mathfrak{c}_\mathfrak{o},\triangledown^{\aleph-\mathfrak{o}}\mathfrak{c}_\mathfrak{o})) > 0.$$

Again using (3), we get

$$\begin{aligned}&\Pi(\mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_\aleph)) = \Pi(\mho(\triangledown\mathfrak{c}_\aleph,\triangledown\mathfrak{c}_{\aleph-1}))\\&\leq \Pi(\varsigma_1\mho(\mathfrak{c}_\aleph,\mathfrak{c}_{\aleph-1}) + \varsigma_2\mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_\aleph) + \varsigma_3\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_\aleph))\\&- \xi(\varsigma_1\mho(\mathfrak{c}_\aleph,\mathfrak{c}_{\aleph-1}) + \varsigma_2\mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_\aleph) + \varsigma_3\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_\aleph)). \qquad (4)\end{aligned}$$

If $\mho(\mathfrak{c}_\aleph,\mathfrak{c}_{\aleph-1}) < \mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_\aleph)$, then

$$\begin{aligned}&\Pi(\mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_\aleph)) \leq \Pi((\varsigma_1+\varsigma_2+\varsigma_3)\mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_\aleph))\\&- \xi(\varsigma_1\mho(\mathfrak{c}_\aleph,\mathfrak{c}_{\aleph-1}) + \varsigma_2\mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_\aleph) + \varsigma_3\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_\aleph))\\&\leq \Pi(\mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_\aleph))\\&- \xi(\varsigma_1\mho(\mathfrak{c}_\aleph,\mathfrak{c}_{\aleph-1}) + \varsigma_2\mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_\aleph) + \varsigma_3\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_\aleph)), \qquad (5)\end{aligned}$$

i.e.,

$$\xi(\varsigma_1\mho(\mathfrak{c}_\aleph,\mathfrak{c}_{\aleph-1}) + \varsigma_2\mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_\aleph) + \varsigma_3\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_\aleph)) = 0,$$

that is,

$$(\varsigma_1+\varsigma_3)\mho(\mathfrak{c}_\aleph,\mathfrak{c}_{\aleph-1}) + \varsigma_2\mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_\aleph) = 0,$$

which shows that $\varsigma_1 = \varsigma_2 = \varsigma_3 = 0$. From (4), we have $\Pi(\mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_\aleph)) = 0 \iff \mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_\aleph) = 0$, a contradiction and so $\mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_{\aleph}) \leq \mho(\mathfrak{c}_{\aleph},\mathfrak{c}_{\aleph-1})$. We obtain

$$\begin{aligned}
&\Pi(\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})) = \Pi(\mho(\mathfrak{c}_{\aleph+1},\mathfrak{c}_{\aleph}))\\
&= \Pi(\mho(\triangledown\mathfrak{c}_{\aleph},\triangledown\mathfrak{c}_{\aleph-1}))\\
&\leq \Pi(\varsigma_1\mho(\mathfrak{c}_{\aleph},\mathfrak{c}_{\aleph-1})+\varsigma_2\mho(\mathfrak{c}_{\aleph},\mathfrak{c}_{\aleph+1})+\varsigma_3\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_{\aleph}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\aleph},\mathfrak{c}_{\aleph-1})+\varsigma_2\mho(\mathfrak{c}_{\aleph},\mathfrak{c}_{\aleph+1})+\varsigma_3\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_{\aleph}))\\
&\leq \Pi((\varsigma_1+\varsigma_2+\varsigma_3)\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_{\aleph}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\aleph},\mathfrak{c}_{\aleph-1})+\varsigma_2\mho(\mathfrak{c}_{\aleph},\mathfrak{c}_{\aleph+1})+\varsigma_3\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_{\aleph}))\\
&\leq \Pi(\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_{\aleph}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\aleph},\mathfrak{c}_{\aleph-1})+\varsigma_2\mho(\mathfrak{c}_{\aleph},\mathfrak{c}_{\aleph+1})+\varsigma_3\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_{\aleph})),
\end{aligned} \tag{6}$$

than $\varsigma_1\mho(\mathfrak{c}_{\aleph},\mathfrak{c}_{\aleph-1}) + \varsigma_2\mho(\mathfrak{c}_{\aleph},\mathfrak{c}_{\aleph+1}) + \varsigma_3\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_{\aleph}) > 0$. Otherwise, $\varsigma_1 = \varsigma_2 = \varsigma_3 = 0$, we get a contradiction. Therefore, (6) implies

$$\begin{aligned}
&\Pi(\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})) \leq \Pi(\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_{\aleph}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\aleph},\mathfrak{c}_{\aleph-1})+\varsigma_2\mho(\mathfrak{c}_{\aleph},\mathfrak{c}_{\aleph+1})+\varsigma_3\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_{\aleph}))\\
&< \Pi(\mho(\mathfrak{c}_{\aleph-1},\mathfrak{c}_{\aleph}))\\
&\ldots\\
&\leq \Pi(\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}}))
\end{aligned} \tag{7}$$

a contradiction. Hence, the assumptions are hold.
Case 2. $\triangledown$ has no periodic point, i.e., $\mathfrak{c}_{\aleph} \neq \mathfrak{c}_{\mathfrak{o}}\ \forall\ \aleph \neq \mathfrak{o}$.
Step 1-1. Prove that $\lim_{\mathfrak{o}\to\infty}\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}}) = 0$. Taking $\mathfrak{c} = \mathfrak{c}_{\mathfrak{o}}, \rho = \mathfrak{c}_{\mathfrak{o}-1})$ in (3), we have

$$\begin{aligned}
&\Pi(\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})) = \Pi(\mho(\triangledown\mathfrak{c}_{\mathfrak{o}},\triangledown\mathfrak{c}_{\mathfrak{o}-1}))\\
&\leq \Pi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}-1})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}+1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}-1},\mathfrak{c}_{\mathfrak{o}}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}-1})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}+1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}-1},\mathfrak{c}_{\mathfrak{o}})).
\end{aligned} \tag{8}$$

If $\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}-1}) < \mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})$, then

$$\begin{aligned}
&\Pi(\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})) \leq \Pi(\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}-1})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}+1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}-1},\mathfrak{c}_{\mathfrak{o}})),
\end{aligned} \tag{9}$$

it implies that

$$\xi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}-1})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}+1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}-1},\mathfrak{c}_{\mathfrak{o}})) = 0, \tag{10}$$

than $\varsigma_1 = \varsigma_2 = \varsigma_3 = 0$. Thus $\Pi(\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})) = 0 \iff \mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}}) = 0$, a contradiction. Hence

$$\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}}) \leq \mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}-1}) \tag{11}$$

$\forall\ \mathfrak{o}$. Since Π is monotonically nondecreasing, then

$$\Pi(\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})) \leq \Pi(\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}-1})).$$

There exist numbers τ and τ^* s.t.

$$\lim_{\mathfrak{o}\to\infty}\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}}) = \tau,\ \lim_{\mathfrak{o}\to\infty}\Pi(\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})) = \tau^*.$$

If $\tau > 0$, we get

$$\begin{aligned}
&\lim_{\mathfrak{o}\to\infty}[\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}-1})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}},\\
&\mathfrak{c}_{\mathfrak{o}+1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}-1},\mathfrak{c}_{\mathfrak{o}})] = (\varsigma_1+\varsigma_2+\varsigma_3)\tau > 0,
\end{aligned} \tag{12}$$

then

$$\liminf_{\mathfrak{o}\to\infty}\xi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}-1})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}+1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}-1},\mathfrak{c}_{\mathfrak{o}})) > 0.$$

By (8), we have

$$\begin{aligned}
&\Pi(\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})) \leq \Pi(\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}-1}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}-1})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}+1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}-1},\mathfrak{c}_{\mathfrak{o}})).
\end{aligned} \tag{13}$$

Letting $\mathfrak{o} \to \infty$ in (13), applying lower limits on both sides of the above inequality, we obtain

$$\liminf_{\mathfrak{o}\to\infty}\xi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}-1})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}+1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}-1},\mathfrak{c}_{\mathfrak{o}})) \leq 0,$$

a contradiction, and so $\lim_{\mathfrak{o}\to\infty}\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}}) = 0$.
Step 1-2. Prove that $\lim_{\mathfrak{o}\to\infty}\mho(\mathfrak{c}_{\mathfrak{o}+2},\mathfrak{c}_{\mathfrak{o}}) = 0$. Again letting $\mathfrak{c} = \mathfrak{c}_{\mathfrak{o}+1}, \rho = \mathfrak{c}_{\mathfrak{o}-1}$ in (3), then we have

$$\begin{aligned}
&\Pi(\mho(\mathfrak{c}_{\mathfrak{o}+2},\mathfrak{c}_{\mathfrak{o}})) = \Pi(\mho(\triangledown\mathfrak{c}_{\mathfrak{o}+1},\triangledown\mathfrak{c}_{\mathfrak{o}-1}))\\
&\leq \Pi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}-1})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}+2},\mathfrak{c}_{\mathfrak{o}+1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}-1},\mathfrak{c}_{\mathfrak{o}}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}-1})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}+2},\mathfrak{c}_{\mathfrak{o}+1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}-1},\mathfrak{c}_{\mathfrak{o}}))\\
&\leq \Pi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}-1})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}-1},\mathfrak{c}_{\mathfrak{o}})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}-1},\mathfrak{c}_{\mathfrak{o}}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}-1})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}+2},\mathfrak{c}_{\mathfrak{o}+1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}-1},\mathfrak{c}_{\mathfrak{o}})).
\end{aligned} \tag{14}$$

If $\sum_{\vee=1}^{3}\varsigma_{\vee} = 0$, then $\varsigma_{\vee} = 0$ for $\vee = 1,2,3$. Thus, $\Pi(\mho(\mathfrak{c}_{\mathfrak{o}+2},\mathfrak{c}_{\mathfrak{o}})) = 0$, a contradiction. If $\sum_{\vee=1}^{3}\varsigma_{\vee} \neq 0$, we arise the two cases.
Case 1-2-1. If $\exists$ an orthogonal infinite subsequence $\{\mathfrak{c}_{\mathfrak{o}(\vee)}\}$ of $\mathfrak{c}_{\mathfrak{o}}$ s.t. $\mho(\mathfrak{c}_{\mathfrak{o}(\vee)},\mathfrak{c}_{\mathfrak{o}(\vee)-1}) < \mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)-1})$ $\forall\ \vee$. Without loss of generality, we have

$$\begin{aligned}
\mho(\mathfrak{c}_{\mathfrak{o}(\vee)},\mathfrak{c}_{\mathfrak{o}(\vee)-2}) &\leq \mho(\mathfrak{c}_{\mathfrak{o}(\vee)-1},\mathfrak{c}_{\mathfrak{o}(\vee)-2})\\
&\leq \mho(\mathfrak{c}_{\mathfrak{o}(\vee)-2},\mathfrak{c}_{\mathfrak{o}(\vee)-3})\\
&\leq \ldots\\
&\leq \mho(\mathfrak{c}_{\mathfrak{o}(\vee-1)},\mathfrak{c}_{\mathfrak{o}(\vee-1)-1})\\
&< \mho(\mathfrak{c}_{\mathfrak{o}(\vee-1)+1},\mathfrak{c}_{\mathfrak{o}(\vee-1)-1})
\end{aligned} \tag{15}$$

$\forall\ \vee \geq 1$. Again by (3), we get

$$
\begin{aligned}
&\Pi(\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)-1})) = \Pi(\mho(\triangledown\mathfrak{c}_{\mathfrak{o}(\vee)},\triangledown\mathfrak{c}_{\mathfrak{o}(\vee)-2}))\\
&\leq \Pi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}(\vee)},\mathfrak{c}_{\mathfrak{o}(\vee)-2})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)})\\
&+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}(\vee)-1},\mathfrak{c}_{\mathfrak{o}(\vee)-2}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}(\vee)},\mathfrak{c}_{\mathfrak{o}(\vee)-2})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)})\\
&+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}(\vee)-1},\mathfrak{c}_{\mathfrak{o}(\vee)-2}))\\
&\leq \Pi((\varsigma_1+\varsigma_2+\varsigma_3)\mho(\mathfrak{c}_{\mathfrak{o}(\vee)-1},\mathfrak{c}_{\mathfrak{o}(\vee)-2}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}(\vee)},\mathfrak{c}_{\mathfrak{o}(\vee)-2})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)})\\
&+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}(\vee)-1},\mathfrak{c}_{\mathfrak{o}(\vee)-2}))\\
&\leq \Pi(\mho(\mathfrak{c}_{\mathfrak{o}(\vee)-1},\mathfrak{c}_{\mathfrak{o}(\vee)-2}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}(\vee)},\mathfrak{c}_{\mathfrak{o}(\vee)-2})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)})\\
&+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}(\vee)-1},\mathfrak{c}_{\mathfrak{o}(\vee)-2})),
\end{aligned}
$$

$\forall\ \vee$. If $\sum_{\vee=1}^{3}\varsigma_\vee = 0$, then $\varsigma_\vee = 0$ for $\vee = 1,2,3$. Therefore, we obtain $\Pi(\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)-1})) = 0$, i.e., $\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)-1}) = 0$ is a contradiction. If $\sum_{\vee=1}^{3}\varsigma_\vee \neq 0$, then we get from (16) and (15) that

$$
\begin{aligned}
&\Pi(\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)-1})) \leq \Pi(\mho(\mathfrak{c}_{\mathfrak{o}(\vee)-1},\mathfrak{c}_{\mathfrak{o}(\vee)-2}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\mathfrak{o}(\vee)},\mathfrak{c}_{\mathfrak{o}(\vee)-2})+\varsigma_2\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)}) \qquad (16)\\
&+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}(\vee)-1},\mathfrak{c}_{\mathfrak{o}(\vee)-2}))\\
&< \Pi(\mho(\mathfrak{c}_{\mathfrak{o}(\vee)-1},\mathfrak{c}_{\mathfrak{o}(\vee)-2}))\\
&\ldots\\
&\leq \mho(\mathfrak{c}_{\mathfrak{o}(\vee-1)},\mathfrak{c}_{\mathfrak{o}(\vee-1)-1})\\
&\leq \mho(\mathfrak{c}_{\mathfrak{o}(\vee-1)+1},\mathfrak{c}_{\mathfrak{o}(\vee-1)-1}). \qquad (17)
\end{aligned}
$$

It concludes from (17) and by the Step 1-1 that

$$\Pi(\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)-1})) < \Pi(\mho(\mathfrak{c}_{\mathfrak{o}(\vee)-1},\mathfrak{c}_{\mathfrak{o}(\vee)-2})) \to 0$$

as $\vee \to \infty$, that is,

$$\lim_{\vee\to\infty}\Pi(\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)-1})) = 0. \qquad (18)$$

And we also obtain from (17) that

$$\Pi(\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)-1})) < \Pi(\mho(\mathfrak{c}_{\mathfrak{o}(\mathfrak{i}-1)+1},\mathfrak{c}_{\mathfrak{o}(\mathfrak{i}-1)-1})),$$

which shows that

$$\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)-1}) < \mho(\mathfrak{c}_{\mathfrak{o}(\mathfrak{i}-1)+1},\mathfrak{c}_{\mathfrak{o}(\mathfrak{i}-1)-1}),$$

so the O-sequence $\{\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)-1})\}$ is monotone decreasing and bounded below, $\exists\ \mathfrak{R} \geq 0$ s.t.

$$\lim_{\vee\to\infty}\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)-1}) = \mathfrak{R}.$$

If $\mathfrak{R} > 0$, then

$$\lim_{\vee\to\infty}\Pi(\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)-1})) > 0$$

contradicts (18). Thus $\lim_{\vee\to\infty}\mho(\mathfrak{c}_{\mathfrak{o}(\vee)+1},\mathfrak{c}_{\mathfrak{o}(\vee)-1}) \to 0$ as $\vee \to \infty$.

Case 1-2-2. If $\exists$ an infinite orthogonal subsequence $\{\mathfrak{c}_{\mathfrak{o}(\varepsilon)}\}$ of $\{\mathfrak{c}_\mathfrak{o}\}$ s.t.

$$\mho(\mathfrak{c}_{\mathfrak{o}(\varepsilon)+1},\mathfrak{c}_{\mathfrak{o}(\varepsilon)-1}) \leq \mho(\mathfrak{c}_{\mathfrak{o}(\varepsilon)},\mathfrak{c}_{\mathfrak{o}(\varepsilon)-1}),$$

then $\mho(\mathfrak{c}_{\mathfrak{o}(\varepsilon)+1},\mathfrak{c}_{\mathfrak{o}(\varepsilon)-1}) \to 0$ as $\varepsilon \to \infty$. Hence, the two cases we obtained that $\lim_{\mathfrak{o}\to\infty}\mho(\mathfrak{c}_{\mathfrak{o}+2},\mathfrak{c}_\mathfrak{o}) = 0$.

Step 1-3. Prove that $\{\mathfrak{c}_\mathfrak{o}\}$ is a Cauchy O-sequence. On the contrary, assume that $\exists\ \gamma > 0$ for which an orthogonal subsequences $\{\mathfrak{c}_{\aleph(\mu)}\}$ and $\{\mathfrak{c}_{\mathfrak{o}(\mu)}\}$ of $\{\mathfrak{c}_\mathfrak{o}\}$ s.t.

$$\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)}) \geq \gamma$$

for $\mathfrak{o}(\mu) > \aleph(\mu) > \mu$ with $\mathfrak{o}(\mu)$ is the smallest index, and so we obtain

$$\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)-1}) < \gamma$$

$\forall\ \mu$. By the rectangular inequality, we have

$$
\begin{aligned}
&\gamma \leq \mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)})\\
&\leq \mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)-1})+\mho(\mathfrak{c}_{\mathfrak{o}(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-2})+\mho(\mathfrak{c}_{\mathfrak{o}(\mu)-2},\mathfrak{c}_{\mathfrak{o}(\mu)})\\
&< \gamma+\mho(\mathfrak{c}_{\mathfrak{o}(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-2})+\mho(\mathfrak{c}_{\mathfrak{o}(\mu)-2},\mathfrak{c}_{\mathfrak{o}(\mu)}),
\end{aligned}
$$

then $\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)}) \to \gamma$ as $\mu \to \infty$. Similarly,

$$
\begin{aligned}
&\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)})-\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\aleph(\mu)-1})-\mho(\mathfrak{c}_{\mathfrak{o}(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)})\\
&\leq \mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-1})\\
&\leq \mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\aleph(\mu)})+\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)})+\mho(\mathfrak{c}_{\mathfrak{o}(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)-1}),
\end{aligned}
$$

then $\mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-1}) \to \gamma$ as $\mu \to \infty$. Furthermore, $\exists\ \mathscr{K}$ s.t.

$$\mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-1}) > \frac{\gamma}{2}, \quad \mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\aleph(\mu)-1}) < \frac{\gamma}{2},$$
$$\mho(\mathfrak{c}_{\mathfrak{o}(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)-1}) < \frac{\gamma}{2}$$

for $\aleph(\mu), \aleph(\mu) > \mathscr{K}$. Again by using (3), then

$$
\begin{aligned}
&\Pi(\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)})) = \Pi(\mho(\triangledown\mathfrak{c}_{\aleph(\mu)-1},\triangledown\mathfrak{c}_{\mathfrak{o}(\mu)-1}))\\
&\leq \Pi(\varsigma_1\mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-1})+\varsigma_2\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\aleph(\mu)-1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)-1}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-1})+\varsigma_2\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\aleph(\mu)-1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)-1}))\\
&\leq \Pi((\varsigma_1+\varsigma_2+\varsigma_3)\mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-1}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-1})+\varsigma_2\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\aleph(\mu)-1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)-1}))\\
&\leq \Pi(\mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-1}))\\
&-\xi(\varsigma_1\mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-1})+\varsigma_2\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\aleph(\mu)-1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)-1})). \qquad (19)
\end{aligned}
$$

Letting the limit as $\mathfrak{o} \to \infty$ in the above equation (19) implies

$$
\begin{aligned}
&\liminf_{\mu\to\infty}\xi(\varsigma_1\mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-1})+\varsigma_2\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\aleph(\mu)-1})\\
&+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)-1})) \leq 0. \qquad (20)
\end{aligned}
$$

On the other hand,

$$\lim_{\mu\to\infty}[\varsigma_1\mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-1})+\varsigma_2\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\aleph(\mu)-1}) + \varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)-1})]=\varsigma_1\gamma. \quad (21)$$

If $\varsigma_1=0$, then we obtain from (19) that

$$\Pi(\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)}))\leq\Pi(\varsigma_1\mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-1}) +\varsigma_2\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\aleph(\mu)-1}) +\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)-1}))\to 0 \quad (22)$$

as $\mu\to\infty$, i.e., $\lim_{\mu\to\infty}\Pi(\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)}))=0$, a contradiction. If $\varsigma_1\neq 0$, then, by (21), we have

$$\lim_{\mu\to\infty}\inf\xi(\varsigma_1\mho(\mathfrak{c}_{\aleph(\mu)-1},\mathfrak{c}_{\mathfrak{o}(\mu)-1}) +\varsigma_2\mho(\mathfrak{c}_{\aleph(\mu)},\mathfrak{c}_{\aleph(\mu)-1})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}(\mu)},\mathfrak{c}_{\mathfrak{o}(\mu)-1}))>0, \quad (23)$$

a contradiction. Therefore, $\{\mathfrak{c}_{\mathfrak{o}}\}$ is a Cauchy O-sequence. Since $(\mathbb{U},\mho)$ is an O-complete, $\exists\ \mathfrak{w}\in\mathbb{U}$ s.t. $\lim_{\mathfrak{o}\to\infty}\mathfrak{c}_{\mathfrak{o}}=\mathfrak{w}$.
Step 1-4. Let us prove that $\mathfrak{w}$ is a f.p. of $\bigtriangledown$. On the contrary, assume that $\mathfrak{w}$ is not a f.p. of $\bigtriangledown$, i.e., $\mho(\mathfrak{w},\bigtriangledown\mathfrak{w})>0$. From

$$\mho(\mathfrak{w},\bigtriangledown\mathfrak{w})-\mho(\mathfrak{w},\mathfrak{c}_{\mathfrak{o}})-\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}+1})\leq\mho(\bigtriangledown\mathfrak{w},\bigtriangledown\mathfrak{c}_{\mathfrak{o}}) \leq\mho(\bigtriangledown\mathfrak{w},\mathfrak{w})+\mho(\mathfrak{w},\mathfrak{c}_{\mathfrak{o}})+\mho(\mathfrak{c}_{\mathfrak{o}},\mathfrak{c}_{\mathfrak{o}+1}),$$

then

$$\lim_{\mathfrak{o}\to\infty}\mho(\bigtriangledown\mathfrak{w},\mathfrak{c}_{\mathfrak{o}+1})=\mho(\bigtriangledown\mathfrak{w},\mathfrak{w})>0.$$

Thus,

$$\lim_{\mathfrak{o}\to\infty}\Pi(\mho(\bigtriangledown\mathfrak{w},\mathfrak{c}_{\mathfrak{o}+1}))>0.$$

From (3), we obtain

$$\Pi(\mho(\bigtriangledown\mathfrak{w},\mathfrak{c}_{\mathfrak{o}+1}))=\Pi(\mho(\bigtriangledown\mathfrak{w},\bigtriangledown\mathfrak{c}_{\mathfrak{o}})) \leq\Pi(\varsigma_1\mho(\mathfrak{w},\mathfrak{c}_{\mathfrak{o}})+\varsigma_2\mho(\mathfrak{w},\bigtriangledown\mathfrak{w})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})) -\xi(\varsigma_1\mho(\mathfrak{w},\mathfrak{c}_{\mathfrak{o}})+\varsigma_2\mho(\mathfrak{w},\bigtriangledown\mathfrak{w})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})) \quad (24)$$

If $\varsigma_2=0$, then (24) yields

$$\Pi(\mho(\bigtriangledown\mathfrak{w},\mathfrak{c}_{\mathfrak{o}+1}))\leq\Pi(\varsigma_1\mho(\mathfrak{w},\mathfrak{c}_{\mathfrak{o}})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}}))\to 0 \quad (25)$$

as $\mathfrak{o}\to\infty$, i.e., $\lim_{\mathfrak{o}\to\infty}\Pi(\mho(\bigtriangledown\mathfrak{w},\mathfrak{c}_{\mathfrak{o}+1}))=0$, a contradiction. If $\varsigma_2\neq 0$, then, we get

$$\lim_{\mathfrak{o}\to\infty}\inf\xi(\varsigma_1\mho(\mathfrak{w},\mathfrak{c}_{\mathfrak{o}})+\varsigma_2\mho(\mathfrak{w},\bigtriangledown\mathfrak{w})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}}))>0.$$

And we get from (24) that

$$\Pi(\mho(\bigtriangledown\mathfrak{w},\mathfrak{c}_{\mathfrak{o}+1}))\leq\Pi(\varsigma_1\mho(\mathfrak{w},\mathfrak{c}_{\mathfrak{o}}) +\varsigma_2\mho(\mathfrak{w},\bigtriangledown\mathfrak{w})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})) -\xi(\varsigma_1\mho(\mathfrak{w},\mathfrak{c}_{\mathfrak{o}})+\varsigma_2\mho(\mathfrak{w},\bigtriangledown\mathfrak{w})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})).$$

Since

$$\lim_{\mathfrak{o}\to\infty}[\varsigma_1\mho(\mathfrak{w},\mathfrak{c}_{\mathfrak{o}})+\varsigma_2\mho(\mathfrak{w},\bigtriangledown\mathfrak{w})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}})]=\mho(\bigtriangledown\mathfrak{w},\mathfrak{w}),$$

then

$$\lim_{\mathfrak{o}\to\infty}\Pi(\varsigma_1\mho(\mathfrak{w},\mathfrak{c}_{\mathfrak{o}})+\varsigma_2\mho(\mathfrak{w},\bigtriangledown\mathfrak{w})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}+1}, \mathfrak{c}_{\mathfrak{o}}))=\lim_{\mathfrak{o}\to\infty}\Pi(\mho(\bigtriangledown\mathfrak{w},\mathfrak{c}_{\mathfrak{o}+1})).$$

Applying limits as $\lim_{\mathfrak{o}\to\infty}$ on both of (26), then

$$\lim_{\mathfrak{o}\to\infty}\inf\xi(\varsigma_1\mho(\mathfrak{w},\mathfrak{c}_{\mathfrak{o}})+\varsigma_2\mho(\mathfrak{w},\bigtriangledown\mathfrak{w})+\varsigma_3\mho(\mathfrak{c}_{\mathfrak{o}+1},\mathfrak{c}_{\mathfrak{o}}))=0,$$

which is a contradiction, and hence $\mathfrak{w}=\bigtriangledown\mathfrak{w}$.
step 2. If $\exists$ a f.p. of $\bigtriangledown$ is unique.
Assume that there exist two f.p.s $\bigtriangledown\mathfrak{z}=\mathfrak{z}\neq\mathfrak{w}=\bigtriangledown\mathfrak{w}$, it means that $\mho(\mathfrak{z},\mathfrak{w})=\mho(\bigtriangledown\mathfrak{z},\bigtriangledown\mathfrak{w})>0$. Taking $\mathfrak{c}=\mathfrak{z}$ and $\rho=\mathfrak{w}$ in (3), Since $\bigtriangledown$ is $\wedge$-preserving, we have

$$\begin{aligned}\Pi(\mho(\mathfrak{z},\mathfrak{w}))&=\Pi(\mho(\bigtriangledown\mathfrak{z},\bigtriangledown\mathfrak{w}))\leq\Pi(\mho(\bigtriangledown\mathfrak{z},\bigtriangledown\rho))\\ &\leq\Pi(\varsigma_1\mho(\mathfrak{z},\mathfrak{w})+\varsigma_2\mho(\mathfrak{z},\bigtriangledown\mathfrak{z})+\varsigma_3\mho(\mathfrak{w},\bigtriangledown\mathfrak{w}))\\ &\quad-\xi(\varsigma_1\mho(\mathfrak{z},\mathfrak{w})+\varsigma_2\mho(\mathfrak{z},\bigtriangledown\mathfrak{z})+\varsigma_3\mho(\mathfrak{w},\bigtriangledown\mathfrak{w}))\\ &=\Pi(\varsigma_1\mho(\mathfrak{z},\mathfrak{w}))-\xi(\varsigma_1\mho(\mathfrak{z},\mathfrak{w})). \end{aligned} \quad (26)$$

If $\varsigma_1=0$, then $\Pi(\mho(\mathfrak{z},\mathfrak{w}))=0$, i.e., $\mho(\mathfrak{z},\mathfrak{w})=0$, which contradicts $\mho(\mathfrak{z},\mathfrak{w})\neq 0$. If $\varsigma_1>0$, from (26) we obtain that

$$\Pi(\mho(\mathfrak{z},\mathfrak{w}))<\Pi(\varsigma_1\mho(\mathfrak{z},\mathfrak{w}))\leq\Pi(\mho(\mathfrak{z},\mathfrak{w}))$$

is a contradiction and $\mathfrak{z}=\mathfrak{w}$. Hence the f.p. is unique. This completes the proof.

Corollary 32*Let $(\mathbb{U},\wedge,\mho)$ be an O-complete Branciari type MS, and let $\bigtriangledown:\mathbb{U}\to\mathbb{U}$ be a self-map satisfying*

(i) $\forall\ \mathfrak{c},\rho\in\mathbb{U}$ with $\mathfrak{c}\wedge\rho$,

$$\mho(\bigtriangledown\mathfrak{c},\bigtriangledown\rho)>0\ [\Pi(\mho(\bigtriangledown\mathfrak{c},\bigtriangledown\rho)) \leq\Pi(\mho(\mathfrak{c},\rho))-\xi(\mho(\mathfrak{c},\rho))] \quad (27)$$

where Π and ξ are defined as in Theorem 31,
$(ii)\wedge$-continuous,
$(iii)\wedge$-preserving.

Then $\bigtriangledown$ has a ***ufp****.*

Similar results are obtained from Theorem 31 taking $\varsigma_1=\varsigma_3=0,\varsigma_2=1$ or $\varsigma_1=\varsigma_2=0,\varsigma_3=1$.

Corollary 33*Let $(\mathbb{U},\wedge,\mho)$ be an O-complete Branciari type MS, and let $\bigtriangledown:\mathbb{U}\to\mathbb{U}$ be a self-map satisfying*

(i) $\forall\ \mathfrak{c},\rho\in\mathbb{U}$ with $\mathfrak{c}\wedge\rho$,

$$\mho(\bigtriangledown\mathfrak{c},\bigtriangledown\rho)>0\ [\Pi(\mho(\bigtriangledown\mathfrak{c},\bigtriangledown\rho)) \leq\Pi(\mho(\mathfrak{c},\bigtriangledown\mathfrak{c}))-\xi(\mho(\mathfrak{c},\bigtriangledown\mathfrak{c})) \quad (28)$$

or

$$\Pi(\mho(\bigtriangledown\mathfrak{c},\bigtriangledown\rho))\leq\Pi(\mho(\rho,\bigtriangledown\rho))-\xi(\mho(\rho,\bigtriangledown\rho))] \quad (29)$$

where Π and ξ are defined as in Theorem 31,

(ii) $\wedge$-*continuous,*
(iii) $\wedge$-*preserving.*

Then $\bigtriangledown$ *has a* ***ufp***.

Corollary 34 *Let* $(\mathbb{U}, \wedge, \mho)$ *be an O-complete Branciari type MS, and let* $\bigtriangledown : \mathbb{U} \to \mathbb{U}$ *be a self-map satisfying*

(i) $\forall\ \mathfrak{c}, \rho \in \mathbb{U}$ *with* $\mathfrak{c} \wedge \rho$,

$$\begin{aligned} \mho(\bigtriangledown\mathfrak{c}, \bigtriangledown\rho) > 0\ [\Pi(\mho(\bigtriangledown\mathfrak{c}, \bigtriangledown\rho)) \\ \leq \Pi(\max\{\mho(\mathfrak{c}, \rho), \mho(\mathfrak{c}, \bigtriangledown\mathfrak{c}), \mho(\rho, \bigtriangledown\rho)\}) \\ - \xi(\max\{\mho(\mathfrak{c}, \rho), \mho(\mathfrak{c}, \bigtriangledown\mathfrak{c}), \mho(\rho, \bigtriangledown\rho)\})] \end{aligned} \quad (30)$$

where Π *and* ξ *are defined as in Theorem 31,*
(ii) $\wedge$-*continuous,*
(iii) $\wedge$-*preserving.*

Then $\bigtriangledown$ *has a* ***ufp***.

4 Existence of the local solution to a first-order periodic problem

Let $\mathbb{U} = \mathscr{C}(\mathscr{I})$ be the set of all continuous real functions on $\mathscr{I} = [0, \bigtriangledown]$ with $\bigtriangledown < 2.5$. Obviously, this space with the Branciari type **gms** given by

$$\mho(\mathfrak{c}, \rho) = \mathfrak{e}^{\max_{\varphi \in \mathscr{I}} |\mathfrak{c}(\varphi) - \rho(\varphi)|} - 1$$

$\forall\ \mathfrak{c}, \rho \in \mathbb{U}$ is an orthogonal complete Branciari type **gms** with $\Omega(\varphi) = \mathfrak{e}^{\varphi} - 1$. Additionally, $\mathbb{U}$ may have a partial order given by

$$\mathfrak{c} \leq \rho \text{ iff } \mathfrak{c}(\varphi) \leq \rho(\varphi)\ \ \forall\ \ \varphi \in \mathscr{I}.$$

Consider the following first-order periodic problems

$$\begin{cases} \mathfrak{c}'(\varphi) = \pounds(\varphi, \mathfrak{c}(\varphi)) \\ \mathfrak{c}(0) = \mathfrak{c}(\bigtriangledown). \end{cases} \quad (31)$$

where $\varphi \in \mathscr{I}$ and $\pounds =: \mathscr{I} \times \Re \to \Re$ is a given continuous function. A lower solution for (31) is a function $\beta \in \mathscr{C}'(\mathscr{I})$ s.t.

$$\begin{cases} \beta'(\varphi) = \pounds(\varphi, \beta(\varphi)) \\ \beta(0) = \beta(\bigtriangledown), \end{cases}$$

where $\varphi \in \mathscr{I}$.
Suppose $\exists\ \ell > 0$ s.t., $\forall\ \mathfrak{c}, \rho \in \mathbb{U}$ and $\varphi \in \mathscr{I}$, we have

$$|\pounds(\varphi, \mathfrak{c}(\varphi)) + \ell\mathfrak{c}(\varphi) - \pounds(\varphi, \rho(\varphi)) - \ell\rho(\varphi)| \leq \frac{\ell}{2}(|\mathfrak{c}(\varphi) - \rho(\varphi)|).$$

problem (31) can be rewritten as

$$\begin{cases} \mathfrak{c}'(\varphi) + \ell\mathfrak{c}(\varphi) = \pounds(\varphi, \mathfrak{c}(\varphi)) + \ell\mathfrak{c}(\varphi) = F(\varphi, \mathfrak{c}(\varphi)) \\ \mathfrak{c}(0) = \mathfrak{c}(\bigtriangledown). \end{cases}$$

where $\varphi \in \mathscr{I}$. It is well known that this problem is equivalent to the integral equation

$$\mathfrak{c}(\varphi) = \int_0^{\bigtriangledown} \mathscr{G}(\varphi, \flat) F(\flat, \mathfrak{c}(\flat)) \mho\flat.$$

Here $\mathbb{G}$ is the Green's function given as

$$\mathbb{G}(\varphi, \flat) = \begin{cases} \frac{\mathfrak{e}^{\ell(\bigtriangledown + \flat - \varphi)}}{\mathfrak{e}^{\ell\bigtriangledown} - 1}, 0 \leq \flat \leq \varphi \leq \bigtriangledown \\ \frac{\mathfrak{e}^{\ell(\flat - \varphi)}}{\mathfrak{e}^{\ell\bigtriangledown} - 1}, 0 \leq \varphi \leq \flat \leq \bigtriangledown. \end{cases}$$

Theorem 41 *Assume that the following axioms are satisfied:*

$(P1)$ $F : [0, \bigtriangledown] \times \Re \to \Re$ *is orthogonal continuous function*
$(P2)$ *Assume that* $\exists\ \ell > 0$ *s.t.,* $\forall\ \mathfrak{c}, \rho \in \mathbb{U}$ *and* $\varphi \in \mathscr{I}$, *we have*

$$|\pounds(\varphi, \mathfrak{c}(\varphi)) + \ell\mathfrak{c}(\varphi) - \pounds(\varphi, \rho(\varphi)) - \ell\rho(\varphi)| \leq \frac{\ell}{2}(|\mathfrak{c}(\varphi) - \rho(\varphi)|).$$

$(P3)$ *Now define an operator* $\mathbb{H} : \mathbb{U} \to \mathbb{U}$ *by*

$$\mathbb{H}\mathfrak{c}(\varphi) = \int_0^{\bigtriangledown} \mathbb{G}(\varphi, \flat) F(\flat, \mathfrak{c}(\flat)) \mho\flat.$$

Then, (31) *has a unique solution in* $\mathbb{U}$.

Proof. We define the orthogonal relation $\wedge$ on $\mathbb{U}$ by

$$\mathfrak{c} \wedge \rho \iff (\pounds\mathfrak{c} \wedge \pounds\rho) \text{ or } (\pounds\rho \wedge \pounds\mathfrak{c}).$$

We define $\mho : \mathbb{U} \times \mathbb{U} \to \Re_0^+$ by

$$\mho(\mathfrak{c}, \rho) = \mathfrak{e}^{\max_{\varphi \in \mathscr{I}} |\mathfrak{c}(\varphi) - \rho(\varphi)|} - 1$$

$\forall\ \mathfrak{c}, \rho \in \mathbb{U}$. Then, $(\mathbb{U}, \wedge, \mho)$ is an orthogonal complete MS, and hence, $(\mathbb{U}, \wedge, \mho)$ is an O- complete Branciari MS with $\Omega(\varphi) = \mathfrak{e}^{\varphi} - 1$. Observe that $\mathfrak{c} \in \mathbb{U}$ is a solution of (31) iff $\mathfrak{c} \in \mathbb{U}$ is a solution of the differential equation

$$\mathfrak{c}(\varphi) = \int_0^{\bigtriangledown} \mathscr{G}(\varphi, \flat) F(\flat, \mathfrak{c}(\flat)) \mho\flat.$$

Then, $\mathbb{H}$ is an $\perp$-continuous. Now, we show that $\mathbb{H}$ is $\perp$-preserving, in (P2), $\forall\ \mathfrak{c}, \rho \in \mathbb{U}$ with $\mho(\mathbb{H}\mathfrak{c}, \mathbb{H}\rho) > 0$ and $\forall\ \varphi \in [0, 1]$. Then, $\mathbb{H}$ is an $\perp$-reserving. Let $\mathfrak{c}, \rho \in \mathbb{U}$. Next, we claim that $\mathbb{H}$ is an orthogonal (Π, ξ)-weak contractions. Then, we have

$$\begin{aligned}
\Omega(\mho(\mathbb{H}\mathfrak{c},\mathbb{H}\rho)) &= \mathfrak{e}^{\mathfrak{e}^{\max_{\varphi\in\mathscr{I}}|\flat(\mathfrak{c}(\varphi))-\flat(\rho(\varphi))|}-1}-1\\
&= \mathfrak{e}^{\mathfrak{e}^{\max_{\varphi\in\mathscr{I}}|\int_0^{\triangledown}\mathbb{G}(\varphi,\flat)\mathcal{F}(\flat,\mathfrak{c}(\flat))\mho\flat-\int_0^{\triangledown}\mathbb{G}(\varphi,\flat)\mathcal{F}(\flat,\rho(\flat))\mho\flat|}-1}-1\\
&\leq \mathfrak{e}^{\mathfrak{e}^{\max_{\varphi\in\mathscr{I}}\int_0^{\triangledown}|\mathbb{G}(\varphi,\flat)||\mathcal{F}(\flat,\mathfrak{c}(\flat))-\mathcal{F}(\flat,\rho(\flat))|\mho\flat}-1}-1\\
&\leq \mathfrak{e}^{\mathfrak{e}^{\max_{\varphi\in\mathscr{I}}\int_0^{\triangledown}|\mathbb{G}(\varphi,\flat)|\frac{\ell}{2}|\mathfrak{c}(\varphi)-\rho(\varphi)|\mho\flat}-1}-1\\
&\leq \mathfrak{e}^{\mathfrak{e}^{\frac{\ell}{2}\max_{\varphi\in\mathscr{I}}|\mathfrak{c}(\varphi)-\rho(\varphi)|\left(\int_0^{\triangledown}\frac{\mathfrak{e}^{\ell(\triangledown+\flat-\varphi)}}{\mathfrak{e}^{\ell\triangledown}-1}\mho\flat+\int_{\varphi}^{\triangledown}\frac{\mathfrak{e}^{\ell(\flat-\varphi)}}{\mathfrak{e}^{\ell\triangledown}-1}\mho\flat\right)}-1}-1\\
&= \mathfrak{e}^{\mathfrak{e}^{\frac{\ell}{2}\max_{\varphi\in\mathscr{I}}|\mathfrak{c}(\varphi)-\rho(\varphi)|\frac{1}{\ell(\mathfrak{e}^{\ell\triangledown}-1)}\left(\mathfrak{e}^{\ell\triangledown}-\mathfrak{e}^{\ell(\triangledown-\varphi)}+\mathfrak{e}^{\ell(\triangledown-\varphi)}-1\right)}-1}-1\\
&= \mathfrak{e}^{\frac{1}{2}\mho(\mathfrak{c},\rho)}-1\\
&= \Pi(\mathscr{M}(\mathfrak{c},\rho)).
\end{aligned}$$

where from $\Omega(\mho(\mathbb{H}\mathfrak{c},\mathbb{H}\rho)) \leq \Pi(\mathscr{M}(\mathfrak{c},\rho))$, where

$$\Pi(\mathscr{M}(\mathfrak{c},\rho)) = \Pi(\varsigma_1\mho(\mathfrak{c},\rho)+\varsigma_2\mho(\mathfrak{c},\triangledown\mathfrak{c})$$
$$+\varsigma_3\mho(\rho,\triangledown\rho)) - \xi(\varsigma_1\mho(\mathfrak{c},\rho)+\varsigma_2\mho(\mathfrak{c},\triangledown\mathfrak{c})+\varsigma_3\mho(\rho,\triangledown\rho)).$$

Hence, the conditions of Theorem 41 are fulfilled with $\Pi(\varphi) = \mathfrak{e}^{\frac{1}{2}\varphi}-1$. Therefore, $\exists$ a f.p. $\mathfrak{c} \in \mathscr{C}(\mathscr{I})$ s.t.$\mathbb{H}\mathfrak{c} = \mathfrak{c}$. Hence, $\mathfrak{c}$ is a solution of problem (31).

5 Conclusion

In the context of orthogonal Branciari metric spaces, the goal of this publication is to derive certain fixed point results for generalized orthogonal (Π,ξ)-weak contraction mappings. Axillary functions are also provided to support our findings. Additionally, some of the main theorem's corollaries are offered as results that can be drawn from them. Finally, the theoretical result is used to solve a differential equation in order to confirm and support the conclusions presented.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflict of interest

The authors declare that they have no conflicts of interest.

Author's contributions

All authors contributed equally in the writing and editing of this article. All authors read and approved the final version of the manuscript.

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