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Exponential stability of Petri net systems

By

Mehdi Foroozanfar* mehdi_f_s@yahoo.com

Ali Doustmohammadi* doustm@aut.ac.ir

S. K. Y. Nikravesh* nikravsh@aut.ac.ir

Abstract:

In this paper we first present the exponential stability of discrete event dynamic systems (DEDS) available in the literature. The result is then applied to a class of discrete event systems modeled by Petri net. In particular it is shown that a Petri net model of a discrete event system will be exponentially stable if the transition firing obeys certain Lyapunov type rules. In other words, the result obtained here shows that if the Petri net has a given marking at a certain time and the firing of the transitions are done according to certain rules and conditions then the marking of the system states will eventually go to zero in an exponential manner. As a result the finite capacity buffers in the system will not suffer any overflow. An example is given at the end in order to give more insight to the results obtained here.

Keywords:

Discrete Event Dynamic Systems; Petri Nets; Exponential Stability; Lyapunov Stability; Invariant Set;

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* Department of Electrical Engineering, Amirkabir University of Technology
Tehran, IRAN

1. Introduction:

Discrete Event Systems (DES) are dynamic systems which are discrete in time, discrete in space, and asynchronous (or event-driven). Application domains include manufacturing systems, database management systems, traffic systems, and communication protocols. A discrete event system can be modeled using different approaches such as queuing networks, automata, max-plus algebra, Petri net, etc. Similar to almost all control problems, stability is an important issue for DES, however one needs to define what he means by stability in this case.

Recently, much attention has been devoted to studying stability properties of DES [1], [2], [3], [4]. Different types of stability are defined in the literature such as stability in the sense of Lyapunov, asymptotic stability, exponential stability, Lagrange stability, practical stability, finite time stability, etc. In [4], Lyapunov stability of DES, modeled by Petri net, is defined. In this paper exponential stability of DES modeled by Petri net is described.

Section 2 gives some preliminaries and definitions that are needed for the proceeding sections. The basics of Petri nets are discussed in Section 3 and the sufficient conditions for exponential stability of Petri nets are presented in Section 4. Section 5 gives an example that uses the results of the theorem stated in Section 4 to show exponential stability of a simple discrete event system. The conclusions are stated in Section 6.

The following notations and definition will be used throughout this paper.
N ={1,2,3,...}; R^+=[0,∞);
$N^+_{n_0}$={$n_0, n_0+1, \ldots, n_0+k, \ldots$}; $n_0 \geq 0$;
A function f(n, x), f: $N^+_{n_0} \times R^n \rightarrow R^n$ is said to be non-decreasing in X if for any given two vectors X,Y∈R^n the following is true.
∀ x_i X , y_i∈Y , $x_i \geq y_i$, (i∈N), n∈ $N^+_{n_0}$, then f(n, x) ≥ f(n, y).

2. Preliminaries:

Consider a “discrete” system described by the following first order difference equation:

X(n+1)=f[n, x(n)], x(n_0) (1)

Where n$\in N_{n_0}^{+}$, x(n) $\in R^n$, and f: $N_{n_0}^{+} \times R^n \rightarrow R^n$ is continuous in x(n). We define a vector Lyapunov function such that it satisfies the Lyapunov properties given in [4]. As defined in that reference, V is a Lyapunov vector such that V(x) and V(n,x(n)) are both greater than zero (positive definite) and ΔV(n,x(n)) is defined as:

$\Delta V(n,x(n))=V(n+1,x(n+1))-V(n,x(n))$.

To proceed further, for a given discrete event system, the following notations are introduced.

- X : Is the set of states
- x: Is a state
- $X_m \subset X$: Is a closed invariant set.
- E: The set of finite event trajectories
- $E_v \subseteq$ E: A set of valid trajectories that is assumingly specified as part of the modeling process.
- $E_a \subseteq E_v$: A set of allowed trajectories.
- E_k: For k$\in$\{N$\cup\{0\}$\}, is used to denote the sequence of events($e_0, e_1, \ldots, e_{k-1}$)
- X(x_0, E_k, k): State that can be reached at time k from x_0 by $E_k E$.
- $\rho(M,M')=\sum_{i=1}^{m} |m_i - m'_i|$.
- S(X_m, r)=\{x: 0<ρ(x, X_m)< r\}.

3. Petri Net:

In this section a brief description of Petri net will be given. A Petri net is a 5-tuple written as PN=\{P,T,F,W,M_0\} where

- P = \{$p_1,p_2,\ldots,p_m$\} is a finite set of places
- T = \{$t_1, t_2, \ldots, t_n$\} is a finite set of transitions
- F$\subseteq$ (P×T)$\cup$(T×P) is a set of arcs
- W: F N, is a weight function; N = \{1, 2, 3, …\}
- M_0: P N is the initial marking

Note that P$\cap$T=$\emptyset$, P$\cup$T$\neq\emptyset$.

An ordinary Petri net without any initial marking is denoted by N and a Petri net with a specified initial marking is denoted by (N,M_0). Let $M_k(p_i)$ denote the marking (the number of tokens) at place $p_i \in$P at time k and let $M_k=[M_k(p_1), M_k(p_2), \ldots, M_k(p_m)]^T$ denote the marking (state) of PN at time k. A transition $t_j \in$T is said to be enabled at time k if $M_k(p_i) \geq$W(p_i, t_j) for all $p_i \in$P such that(p_i, t_j)$\in$F. A transition $t_j \in$T can fire if it is enabled at time k. The next marking for $p_i \in$P after a transition fires is given by:

$$M_{k+1}(p_i) = M_k(p_i) + W(t_j, p_i) - W(p_i, t_j). \tag{2}$$

Let a_{ij}^- =W(p_i, t_j), a_{ij}^+=W(t_j, p_i) and define an n×m integer matrix A as A=$[a_{ij}] = [a_{ij}^+ - a_{ij}^-]$. Furthermore, let $u_k \in \{0,1\}^n$ denote the firing vector. Then the matrix describing the dynamical behavior of a system represented by a Petri net will be given by:

$$M_{k+1} = M_k + A^T.u_k \tag{3}$$

Notice that if M' can be reached from some other marking M and if our sequence firing of d transition is with corresponding firing vector $u_0, u_1, \dots, u_{d-1}$; we obtain that

$$M' = M + A^T U, \quad U = \sum_{k=0}^{d-1} u_k \tag{4}$$

4. Exponential stability of DES & Petri Nets:

In this section, a theorem will be presented at first. The results obtained from this theorem are then used to propose a new theorem that gives the sufficient conditions for a PN to be exponentially stable.

Theorem 1: (see [4])

In order for the invariant set X_m to be exponentially stable with respect to E_a it is sufficient that in a neighborhood S(X_m;r) there exists a specified function V and three positive constants c_1, c_2, and c_3 such that

i) $c_2 > c_3$ &

ii) $c_1 \rho(x, X_m) \leq V(x) \leq c_2 \rho(x, X_m)$ &

iii) $V(X(x_o), E_{k+1}, k+1)) - V(X(x_o, E_k, k)) \leq -c_3 \rho(X(x_o, E_k, k), X_m)$ for all $x_0 \in$S(X_m;r) and for all E_k such that E_{k+1}=E_ke, (e $\in \varepsilon$) and E_{k+1}E $\in E_a(x_o)$ and all k≥0.

Proof :

Given $x_0 \in$S(X_m;r) let X_0(k)=X(x_o, E_k,k) for any E_k such that $E_k E \in E_a$ (x_o) and all $k \geq 0$. Also let $V'(k)=V(x_\circ(k))$ and assume $V'(k)$ satisfies $V'(k+1)-V'(k) \leq -(c3/c2)V'(k)$. Hence $V'(k+1) \leq (1-c_3/c_2)V'(k)$ or $V'(k) \leq \left(1-\frac{c3}{c2}\right)^k V'(0)$. As a result $c1.\rho(x_\circ(k), X_m) \leq \left(1-\frac{c3}{c2}\right)^k V'(0)$. Since $V'(0) \leq c_2\rho(x_\circ(0), X_m)$, then one can combine the previous equations to obtain

$$c_1\rho(x_\circ(k), X_m) \leq c_2\left(1-\frac{c3}{c2}\right)^k \rho(x_\circ(0), X_m) \tag{5}$$

Or

$$\rho(x_\circ(k), X_m) \leq c_2/c_1.\left(1-\frac{c3}{c2}\right)^k.\ \rho(x_\circ(0), X_m) \tag{6}$$

Therefore there must exist an α and δ such that

$$\rho(x_\circ(k), X_m) \leq \delta e^{-\alpha k}.\rho(x_\circ(0), X_m) \tag{7}$$

Hence the system is exponentially stable. If the properties of above theorem hold on all of X, then the invariant set X_m is exponentially stable in the large with respect to E_a.

Theorem 2:
A discrete event system modeled by Petri net is exponentially stable if there exists a specified vector φ with three positive constants c1, c2, and c3 such that the following conditions are satisfied.

i) $c_2 > c_3$ &
ii) $c_1.r \leq M^T.\emptyset \leq c_2.M^T.\emptyset$ &
iii) $[(c_3/c_2).M_k^T + u_k^T.A].\ \varphi \leq 0$

Proof:
We choose a specified function V such that the above conditions are satisfied. That is

$$V(M)=\inf\{\sum_{i=1}^{m} \emptyset_i\,|M(p_i)-M''(p_i)| : M'' \in X_b\}=M^T.\emptyset \tag{8}$$

where

- V(M): Lyapunov function
- $\emptyset$: an "m vector" to be chosen

We also select three positive constants c_1,c_2, and c_3 such that $c_2>c_3$ and $c_1\rho(M,X_m)\leq \inf\{\rho(M,X_m)\}\leq c_2\rho(M,X_m)$ è $c_1\leq \inf\{\rho(M,X_m)\}/\sup\{\rho(M,X_m)\}$
&
$c_2\geq(\inf\{\rho(M,X_m)\}/\inf\{\rho(M,X_m)\})=1$
Now we choose M such that:
$S(X_m,r)=\{M:0<\rho(M,X_m)\leq r\}$ è $c_1.r\leq M^T.\emptyset \leq c_2.M^T.\emptyset$
notice that V must only satisfy the appropriate properties on $\{M: 0<\rho(M,X_m)\leq r\}$
$V(M(k+1))-V(M(k))\leq -c_3\rho(M(k),X_m)$
Let $V(M(k))=V'(k)$;
so we have

$$V'(k+1)-V'(k)\leq -c_3\rho(M(k),X_m)$$

and

$$V'(k)\leq c_2\rho(M(k),X_m)$$

Therefore

$$V'(k+1)-V'(k)\leq -(c_3/c_2).V'(k) \qquad (*)$$

then

$$V'(K+1)\leq(1-c_3/c_2)V'(k)$$
$$V'(K)\leq(1-c_3/c_2)V'(k-1)$$

From the above one can write

$$V'(1)\leq(1-c_3/c_2).V'(0)$$

So for V'(k) have

$$V'(k)\leq\left(1-\frac{c3}{c2}\right)^k V'(0)$$

From condition (i) and the last result one can write:

$$c_1\rho(Mk,X_m)=c_1\rho(k,X_m)\leq\left(1-\frac{c3}{c2}\right)^k V'(0)$$
$$V'(0)\leq c_2\rho(0,X_m)$$

Hence $\rho(x_\circ(k),X_m)\leq c_2/c_1\left(1-\frac{c3}{c2}\right)^k\rho(x_\circ(0),X_m)$. So there exists an α and δ such that $\rho(x_\circ(k),X_m)\leq\delta e^{-\alpha k}\rho(x_\circ(0),X_m)$. Therefore the system is exponentially stable if we choose $V'(k)=M_k^T\varphi$, and let φ be a positive vector. This implies

$$V'(K+1)\leq(1-c_3/c_2).V'(k) \qquad (9)$$
$$(M_{k+1}^T\varphi)\leq(1-c_3/c_2).M_k^T.\varphi \qquad (10)$$

From matrix equation we can write $(M_k^T+u_k^T.A)\varphi\leq(1-c_3/c_2).M_k^T\varphi$ or $[M_k^T+u_k^T.A+((c3/c2)-1)M_k^T]\varphi\leq 0$ which leads to the following:

$$[(c_3/c_2) \cdot M_k^T + u_k^T \cdot A]\varphi \leq 0 \qquad (11)$$

So we have shown that we can find positive φ such that when (**) is satisfied our system will be exponentially stable without any limitation on our firing vector (u_k). However, if we select $\varphi = [r, r, \ldots, r]^T$, r > 0, we must select a particular firing vector (u_k) to satisfy equation (**) and as a result we are forced to eliminate some markings.

5. An Example:

In this section an example is given to show that if the sufficient condition stated in the previous theorem is satisfied then the system is exponentially stable. In particular we let the system have some initial marking M(0) (number of parts in the buffers) with matrix A given below and show that the tokens from the buffers are removed with an exponential rate. In this example vector φ is considered to be fixed and the firing vector (U) is selected to achieve the stability.

For the example under consideration let the following be defined:

$$\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{bmatrix}, \ M(0) = \begin{bmatrix} 10 \\ 15 \\ 20 \end{bmatrix}$$

Furthermore, let the positive constants c_1 and c_2 be selected as c_1=0.5 and $c_2 > 1$. From the second condition of Theorem 2 we have:

$$\{(c_3/c_2)\, M(k) + u(k)\, A\}\ \varphi \leq 0$$

So,

$$\{(c_3/c_2)\, [10\ \ 15\ \ 20]^T + [u_1\ \ u_2\ \ u_3] \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{bmatrix}\} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \leq 0$$

Now let (c_3/c_2)=q for some positive real q. We then obtain the following:

$$45q + [u_1\ \ u_2\ \ u_3] \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = 45q - 2u_1 + u_2 - 3u_3 \leq 0$$

We now select $U(0) = [1\ \ 1\ \ 1]^T$ and select q such that above non-equality is satisfied. As an example we let $q = (c_3{=}1)/(c_2{=}15)$. Therefore the above equation becomes:

$$45(1/15) + [1\ \ 1\ \ 1] \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = -1 < 0$$

At step 3 ,the marking at step 1 can now be obtained from the following:

$$M(1)=M(0)+A^T u(0) = \begin{bmatrix}10\\15\\20\end{bmatrix}+\begin{bmatrix}-1 & -1 & 0\\ 0 & 1 & 0\\ -1 & -1 & -1\end{bmatrix}^T\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}8\\14\\19\end{bmatrix}$$

At Step 3 we have:

$$(1/15)[8+14+19]+u(1).A.\varphi \leq 0$$
$$(41/15)+ -2u_1 + u_2 - 3u_3 \leq 0$$

Then we choose

$$u(1)=\begin{bmatrix}1\\0\\1\end{bmatrix} \rightarrow (41/15)-5 < 0$$

step 4:

$$M(2)=M(1)+A^T.u(1) = \begin{bmatrix}8\\14\\19\end{bmatrix}+\begin{bmatrix}-1 & -1 & 0\\ 0 & 1 & 0\\ -1 & -1 & -1\end{bmatrix}^T.\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}6\\12\\18\end{bmatrix}$$

Step 5:

$$(1/15)\ .\ [6+12+18]+u\ (2).A.\varphi \leq 0$$
$$(36/15)+ -2u_1 + u_2 - 3u_3 \leq 0$$

Then we choose

$$u(2)=\begin{bmatrix}1\\1\\1\end{bmatrix} \rightarrow (36/15)-4 < 0$$

step 6:

$$M(3) =M(2)+A^T.u(2) = \begin{bmatrix}6\\12\\18\end{bmatrix}+\begin{bmatrix}-1 & -1 & 0\\ 0 & 1 & 0\\ -1 & -1 & -1\end{bmatrix}^T.\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}4\\11\\17\end{bmatrix}$$

If we continue this process we will eventually empty all buffers as can be seen from first few steps of the example. In particular, the markings are given by:

$$\begin{array}{ccccccc} M_0 & & M_1 & & M_2 & & M_3 \\ \begin{bmatrix}10\\15\\20\end{bmatrix} & \text{è} & \begin{bmatrix}8\\14\\19\end{bmatrix} & \text{è} & \begin{bmatrix}6\\12\\18\end{bmatrix} & \text{è} & \begin{bmatrix}4\\11\\17\end{bmatrix}\end{array} \qquad (44)$$

Notice how the markings are decaying in an exponential manner.

<u>6. Conclusions:</u>

An approach to investigate exponential stability of discrete event systems modeled by Petri net is presented in this paper. In particular, it is shown that if certain sufficient conditions are satisfied, then the buffer markings will decay exponentially. The significance of this result is that it guarantees the buffer will not overflow in a physical system such as a production system. It is important to note that the conditions given here are the sufficient conditions only. Similar to Lyapunov stability, finding a vector φ that satisfies the conditions given in Theorem 2 may be a difficult task and not being able to find one does not imply that the system is unstable. Finding the necessary conditions for exponential stability of discrete event systems modeled by Petri net is an interesting topic for future research.

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