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Representing Exponents and Combinations as Summations of Smaller Combinations

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ABSTRACT

Exponential forms shall be run-time expensive, especially with the increase in the base and exponent numbers. Combinations operations shall also take excessive computing, if performed as a set of factorial multiplications based on its definition. This paper introduces with a full derivation for a useful representation of both the Exponential forms and the Combinations operations as summation of smaller combinations in the form of generic theorems. Such presentation shall be more efficient for recursive computing operations to perform expensive calculations efficiently with lower cost on parallel computing machines performing the different portions of presented summation series in parallel. The summation series components can be running concurrently on parallel computing targets. Moreover, it shall provide more insightful meanings for the exponential operations in the educational and visualization purposes. Finally, it shall provide a fair estimate for the maximum memory allocation (number of bits) needed for the obtained result of the exponential operation according to the concluded lemmas.

1. Introduction

Simplifying the multiplication operations in the exponential functions and the permutations and combinations calculations into set of summations shall be highly promising in solving complex algebraic non-linear equations [1] and/or optimizing the usage of parallel computing resources.

Moreover, the representation of the exponential operation as sets of summation series of smaller combinations and exponents with smaller bases shall provide more intuitive insights of the mathematical exponential operation either for the educational [2-5] or data visualization [6] purposes.

In this letter paper, new representation of both the exponential and permutations/combinations functions as summation of smaller combinations is derived and formulated as generic theorems. The next section details how the exponential function is represented as summations of combinations based on extending the Binomial Theorem [7], and how it is used to estimate the maximum memory allocation (number of bits) for the exponential operation result. The third section introduces a new presentation of permutations/combinations as summation of smaller combinations, uses it to derive a generic representation of exponential operation and its base increment as sets of summation series of smaller combinations, and then proves it by Induction [8]. Finally, derived formulas in terms of theorems and lemmas are collected and concluded, and further extended research is proposed in the final Conclusions and Future Work section.

2. Presenting Exponential Functions as Summations of Combinations

Based on the algebraic distributive property of multiplication of sets of added terms, the number of added terms resulted from multiplication can be easily estimated as the multiplication of the number of added terms in each set. This can be formulated as follows.

$$(a_1 + a_2 + a_3 + .. + a_n)(b_1 + b_2 + b_3 + .. + b_m)$$

$$= \left(\sum_{i=1}^{n}(a_i)\right)\left(\sum_{i=1}^{m}(b_i)\right)$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{m}(a_i b_j)\right)$$

$$\Rightarrow n \times m \; terms$$

Accordingly,

$$(a_1 + a_2 + a_3 + .. + a_n)^m \Rightarrow n^m \; terms$$

The number of terms can be deduced as well based on the basic Binomial Theorem as follows, where the combination operation for i of n can be presented as $({n_C}_i)$ or $\binom{n}{i}$.

$$(a+b)^n = \sum_{i=0}^{n}({n_C}_i)(a^i)(b^{n-i})$$

$$(a+b)^n \Longrightarrow 2^n \; terms$$

$$\sum_{i=0}^{n}({n_C}_i)(a^i)(b^{n-i}) \Longrightarrow \left(\sum_{i=0}^{n}({n_C}_i)\right) \; terms$$

$$2^n = \sum_{i=0}^{n}({n_C}_i)$$

Then, the Binomial Theorem can be extended for higher bases, and then generalized as follows.

$$(a+b+c)^n = \sum_{i=0}^{n}({n_C}_i)(a^i)((b+c)^{n-i})$$

$$= \sum_{i=0}^{n}({n_C}_i)(a^i)\left(\sum_{j=0}^{n-i}\binom{n-i}{j}(b^j)(c^{n-i-j})\right)$$

$$(a+b+c)^n \Longrightarrow 3^n \; terms$$

$$\sum_{i=0}^{n}({n_C}_i)(a^i)\left(\sum_{j=0}^{n-i}\binom{n-i}{j}(b^j)(c^{n-i-j})\right)$$

$$\Longrightarrow \left(\sum_{i=0}^{n}({n_C}_i)\left(\sum_{j=0}^{n-i}\binom{n-i}{j}\right)\right) \; terms$$

$$3^n = \sum_{i=0}^{n}\binom{n}{i}\left(\sum_{j=0}^{n-i}\binom{n-i}{j}\right)$$

$$(a+b+c+d)^n = \sum_{i=0}^{n}({n_C}_i)(a^i)((b+c+d)^{n-i})$$

$$= \sum_{i=0}^{n}({n_C}_i)(a^i)\left(\sum_{j=0}^{n-i}\binom{n-i}{j}(b^j)\left(\sum_{k=0}^{n-i-j}\binom{n-i-j}{k}(c^k)(d^{n-i-j-k})\right)\right)$$

$$(a+b+c+d)^n \Longrightarrow 4^n \; terms$$

$$\sum_{i=0}^{n}({n_C}_i)(a^i)\left(\sum_{j=0}^{n-i}\binom{n-i}{j}(b^j)\left(\sum_{k=0}^{n-i-j}\binom{n-i-j}{k}(c^k)(d^{n-i-j-k})\right)\right)$$

$$\Rightarrow \left(\sum_{i=0}^{n}({n_C}_i)\left(\sum_{j=0}^{n-i}\binom{n-i}{j}\left(\sum_{k=0}^{n-i-j}\binom{n-i-j}{k}\right)\right)\right) \; terms$$

$$4^n = \sum_{i=0}^{n}\binom{n}{i}\left(\sum_{j=0}^{n-i}\binom{n-i}{j}\left(\sum_{k=0}^{n-i-j}\binom{n-i-j}{k}\right)\right)$$

Based on the formula obtained for 2^n, 3^n and 4^n, a generic form for m^n can be deduced and stated as a theorem as follows.

$Theorem$ 1:

$$m^n = \sum_{i_1=0}^{n}\binom{n}{i_1}\left(\sum_{i_2=0}^{n-i_1}\binom{n-i_1}{i_2}\left(\sum_{i_3=0}^{n-i_1-i_2}\binom{n-i_1-i_2}{i_3}\left(\dots\left(\sum_{i_{n-1}=0}^{n-\sum_{j=1}^{m-2} i_j}\binom{n-\sum_{j=1}^{m-2} i_j}{i_{n-1}}\right)\right)\dots\right)\right)\right)$$

Based on the form stated in $Theorem$ 1, an upper limit can be estimated for the exponential as the multiplication of $(m-1)$ terms of combinations sums.

$$m^n \leq \sum_{a_1=0}^{n}\left[\binom{n}{a_1}\left(\prod_{i=2}^{m-1}\left[\sum_{a_i=0}^{n-\sum_{j=1}^{i-1}a_j}\binom{n-\sum_{j=1}^{i-1}a_j}{a_i}\right]\right)\right]$$

$$\leq\left(\sum_{a_1=0}^{n}\left[\binom{n}{a_1}\right]\right)\left(\prod_{i=2}^{m-1}\left[\sum_{a_i=0}^{n-\sum_{j=1}^{i-1}a_j}\binom{n-\sum_{j=1}^{i-1}a_j}{a_i}\right]\right)$$

$$m^n \leq (2^n)\left(\prod_{i=2}^{m-1}[2^n]\right)$$

$$m^n \leq (2^n)\,(2^n)^{(m-2)}$$

$$m^n \leq \,(2^n)^{(m-1)}$$

$$m^n \leq \,2^{n(m-1)}$$

Accordingly, $Lemma\ 1$ can be deduced to put an upper limit for the number of bits (maximum bit size) for memory allocation of exponential calculation (m^n).

> $Lemma\ 1$:
> *The maximum bit size (number of bits) for memory allocation of exponential operation* (m^n) *is* $\left(n \times (m-1)\right)$.

The terms obtained for 3^n and 4^n can be reformulated so that another theorem for the exponential function in a recursive form of lower base can be obtained as follows.

$$2^n = \sum_{i=0}^{n}\left({n_C}_i\right) = \sum_{i=0}^{n}\binom{n}{i}$$

$$3^n = \sum_{i=0}^{n}\binom{n}{i}\left(\sum_{j=0}^{n-i}\binom{n-i}{j}\right) = \sum_{i=0}^{n}\binom{n}{i}\,2^{(n-i)}$$

$$4^n = \sum_{i=0}^{n}\binom{n}{i}\left(\sum_{j=0}^{n-i}\binom{n-i}{j}\left(\sum_{k=0}^{n-i-j}\binom{n-i-j}{k}\right)\right)$$

$$= \sum_{i=0}^{n}\binom{n}{i}\,3^{(n-i)}$$

> $Theorem\ 2$:
> $$m^n = \sum_{i=0}^{n}\binom{n}{i}\,(m-1)^{(n-i)}$$

It shall be noted that the upper limit for the exponential (m^n) deduced from $Theorem\ 2$ is reasonable as it leads to the proper condition on the integer base (m) to be $(m \geq 2)$ corresponding to the assumption that $(m-1)^n > 1$ as follows. However, regardless of this assumption the base (m) can be any rational value.

$$m^n = \sum_{i=0}^{n}\binom{n}{i}\,(m-1)^{(n-i)} \leq \left((m-1)^n\right)\left(\sum_{i=0}^{n}\binom{n}{i}\right)$$

$$m^n \leq (m-1)^n\ 2^n$$

$$m^n \leq (2m-2)^n$$

$$m \leq 2m-2$$

$$m \geq 2$$

$Theorem\ 2$ can be also easily validated using the Binomial Theorem [7] for the representation of (m^n) as $\left(\left(1+(m-1)\right)^n\right)$ that ensures the base (m) shall be any rational value.

3. Presenting Permutations/Combinations as Summation of Smaller Permutations/Combinations

As an extension to the previous section, the Combinations and the Permutations can be presented as summation series of smaller Combinations/Permutations as follows, where the combination operation for i of n can be presented as $\left(\boldsymbol{n}_{\boldsymbol{C}_{\boldsymbol{k}}}\right)$ or $\binom{n}{k}$.

> $Theorem\ 3$:
> $$n_{C_k} = \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} + \cdots + \binom{k-1}{k-1}$$
> $$n_{C_k} = \sum_{i=k-1}^{n-1}\binom{i}{k-1}$$

Such presented relation can be extended to Permutations as follows, where the permutation operation for k of n can be presented as $\left(\boldsymbol{n}_{\boldsymbol{P}_{\boldsymbol{k}}}\right)$.

$$n_{C_k} = \frac{n_{P_k}}{k!} = \sum_{i=k-1}^{n-1}\binom{i}{k-1} = \left(\frac{1}{(k-1)!}\right)\sum_{i=k-1}^{n-1} i_{P_{k-1}}$$

$$\frac{n_{P_k}}{k!} = \left(\frac{1}{(k-1)!}\right)\sum_{i=k-1}^{n-1} i_{P_{k-1}}$$

$Theorem\ 4$:

$$n_{P_k} = k \sum_{i=k-1}^{n-1} i_{P_{k-1}}$$

Combining $Theorem\ 2$ and $Theorem\ 3$ together shall provide a generic summation series form for the exponential operation that can be presented in $Lemma\ 2$ and $Lemma\ 3$ as follows.

$Lemma\ 2$:

Generic representation of exponential operation (m^n) *as sets of summation series of smaller combinations and exponent terms with lower base*

$$m^n = (m-1)^n + \sum_{i=1}^{n}\left[\left(\sum_{j=i-1}^{n-1}\binom{j}{i-1}\right)(m-1)^{(n-i)}\right]$$

$Lemma\ 3$:

Generic representation of exponential operation base increment $(m^n-(m-1)^n)$ *as sets of summation series of smaller combinations and exponent terms with the lower base*

$$m^n - (m-1)^n = \sum_{i=1}^{n}\left[\left(\sum_{j=i-1}^{n-1}\binom{j}{i-1}\right)(m-1)^{(n-i)}\right]$$

Illustrative numeric example to reveal the proposed representation in $Lemma\ 2$ and $Lemma\ 3$ can be detailed as follows.

$$3^n - 2^n = \sum_{i=1}^{n}\left[\left(\sum_{j=i-1}^{n-1}\binom{j}{i-1}\right)2^{(n-i)}\right]$$

$$3^3 - 2^3 = 27 - 8 = 19 = \sum_{i=1}^{3}\left[\left(\sum_{j=i-1}^{2}\binom{j}{i-1}\right)2^{(3-i)}\right]$$

$$\sum_{i=1}^{3}\left[\left(\sum_{j=i-1}^{2}\binom{j}{i-1}\right)2^{(3-i)}\right] = \left(\binom{0}{0}+\binom{1}{0}+\binom{2}{0}\right)2^2 + \left(\binom{1}{1}+\binom{2}{1}\right)2^1 + \left(\binom{2}{2}\right)2^0$$

$$\sum_{i=1}^{3}\left[\left(\sum_{j=i-1}^{2}\binom{j}{i-1}\right)2^{(3-i)}\right] = (1+1+1)4 + (1+2)2 + (1)1$$

$$= 3\times 4 + 3\times 2 + 1\times 1 = 12 + 6 + 1 = 19$$

The formulated presentation in $Theorem\ 3$ and $Theorem\ 4$ shall be proved by Induction as follows.

1) $For\ n = 1 \implies$

$$1_{C_1} = \sum_{i=0}^{0}\binom{i}{0} = \binom{1}{1} = \binom{0}{0} = 1$$

✓

2) $For\ \ n = m \implies$

$$m_{C_k} = \binom{m}{k} = \sum_{i=k-1}^{m-1}\binom{i}{k-1}$$

3) $For\ \ n = m+1 \implies$

$$(m+1)_{C_k} = \binom{m+1}{k} =$$

$$= \sum_{i=k-1}^{m}\binom{i}{k-1}\ \ ??$$

$$R.H.S = \sum_{i=k-1}^{m}\binom{i}{k-1} = \binom{m}{k-1} + \sum_{i=k-1}^{m-1}\binom{i}{k-1}$$

According to 2)

$$R.H.S = \sum_{i=k-1}^{m}\binom{i}{k-1} = \binom{m}{k-1} + \binom{m}{k}$$

$$L.H.S = \binom{m+1}{k}$$

$$\binom{m+1}{k}\ == \binom{m}{k-1} + \binom{m}{k}\ \ ??$$

$$\frac{(m+1)!}{k!\times(m+1-k)!} == \frac{m!}{(k-1)!\times(m+1-k)!} + \frac{m!}{k!\times(m-k)!}\ \ ??$$

By multiplying both sides by

$$\left(\frac{k!\times(m+1-k)!}{m!}\right)$$

$$m+1\ == \ k + (m+1-k)\ \ ??$$

$$m+1 = m+1$$

✓

4. Conclusions and Future Work

The following representations of the Exponential and Combinations functions as summation of smaller combinations have been proved and formalized in Theorem 1, Theorem 2, Theorem 3, and Theorem 4. Further conclusions from the proposed representations are obtained by the analysis formalized in Lemma 1, Lemma 2, and Lemma 3.

Theorem 1:

$$m^n = \sum_{i_1=0}^{n} \binom{n}{i_1} \left(\sum_{i_2=0}^{n-i_1} \binom{n-i_1}{i_2} \left(\sum_{i_3=0}^{n-i_1-i_2} \binom{n-i_1-i_2}{i_3} \left(\ldots \left(\sum_{i_{n-1}=0}^{n-\sum_{j=1}^{m-2} i_j} \binom{n-\sum_{j=1}^{m-2} i_j}{i_{n-1}} \right) \right) \ldots \right) \right) \right)$$

Theorem 2:

$$m^n = \sum_{i=0}^{n} \binom{n}{i} (m-1)^{(n-i)}$$

Theorem 3:

$$n_{C_k} = \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} + \cdots + \binom{k-1}{k-1}$$

$$n_{C_k} = \sum_{i=k-1}^{n-1} \binom{i}{k-1}$$

Theorem 4:

$$n_{P_k} = k \sum_{i=k-1}^{n-1} i_{P_{k-1}}$$

Lemma 1:

The maximum bit size (number of bits) for memory allocation of exponential operation (m^n) *is* $\big(n \times (m-1)\big)$.

Lemma 2:

Generic representation of exponential operation (m^n) *as sets of summation series of smaller combinations and exponent terms with lower base*

$$m^n = (m-1)^n + \sum_{i=1}^{n} \left[\left(\sum_{j=i-1}^{n-1} \binom{j}{i-1} \right) (m-1)^{(n-i)} \right]$$

Lemma 3:

Generic representation of exponential operation base increment $(m^n - (m-1)^n)$ *as sets of summation series of smaller combinations and exponent terms with the lower base*

$$m^n - (m-1)^n = \sum_{i=1}^{n} \left[\left(\sum_{j=i-1}^{n-1} \binom{j}{i-1} \right) (m-1)^{(n-i)} \right]$$

In order to summarize, such proposed and proved presentations of sets of summation series shall be beneficial in:

1. Solving algebraic equations and implementing such operations on computer programs based on building block iterative functions running independently on parallel computing resources.
2. Estimating the maximum memory allocation (number of bits) for the exponential operation result.
3. Providing more intuitive insights of the mathematical exponential operation either for the educational or data visualization purposes, as both the main input exponent and the exponent operation base increment are decomposed to set of smaller exponent terms.

The optimized allocation of the summation sets independently on the parallel computing resources and its application on solving complex exponential operations need to be elaborated deeply in further extended work in order to reveal the first mentioned benefit of the proposed exponential presentation. Such further research shall investigate in the gained benefit on the run time optimization and the corresponding limits to be defined in the degree of complexity of the input operation, and the minimum requirements for the available computing resources. Finally, the future work shall be extended to cover with figures real use cases of complex exponential operations decomposition in educational and visualization tools applications.

5. References

1. **Śmietański., Marek. (2023).** Some Inexact Version of the Iterative Exponential Method for Solving Nonsmooth Equations. Algorithms. **16. 27. 10.3390/a16010027.**
2. **Utami., Nurul., Sa'dijah., Cholis., and Chandra., Tjang. (2023).** Students' Mathematical Representation in Solving Exponential Function. AKSIOMA: Jurnal Program Studi Pendidikan Matematika. **12. 450. 10.24127/ajpm.v12i1.6166.**
3. **Díaz-Berrios., Tomás., and Martinez-Planell., Rafael. (2022). High** school student understanding of exponential and logarithmic functions. The Journal of Mathematical Behavior. **66. 100953. 10.1016/j.jmathb.2022.100953.**
4. **Weber., Keith. (2002).** Students' understanding of exponential and logarithmic functions. Second international conference on the teaching of mathematics.
5. **Campo., Karen., Font., Vicenç., García-García., Javier., and Sánchez., Alicia. (2021).** Mathematical Connections Activated in High School Students' Practice Solving Tasks on the Exponential and Logarithmic Functions. Eurasia Journal of Mathematics, Science and Technology Education. **17. 1-14. 10.29333/ejmste/11126.**
6. **Tuda., S., and Rexhepi., S. (2023).** Exploring Exponential Functions Using Geogebra. Brillo Journal, **3(1), 43-58.**
7. **Flusser., Peter., and Iii., Guillermo. (2000).** Derivation and Visualization of the Binomial Theorem. International Journal of Computers for Mathematical Learning. **5. 3-24. 10.1023/A: 1009873212702.**
8. **Handbook of Mathematical Induction:** Theory and Applications (Discrete Mathematics and Its Applications) 1st Edition **(September 14, 2010), ISBN-10 : 9781420093643.**