Concatenation of SISO log maximum a posteriori - polar codes at the receiver

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Abstract. Potential error-correction capabilities of polar codes rendered itself the ideal choice for 5G *New Radio's* (NR) control channels. Furthermore, the use of the CRC-Aided successive cancellation list (CA-SCL) algorithm of polar codes not only outperforms successive cancellation (SC) polar decoding algorithm but also has better error performance than Low-Density Parity Check (LDPC) and turbo codes for short block lengths. Even though its superior decoding error performance, CA-SCL offers a substantially higher complexity than the SC algorithm. This paper adopts a concatenated decoding approach to boost the SC algorithm's performance while keeping the computation complexity within a tolerable range. The received signal is oversampled at the receiver's front end with the oversampling ratio (OSR) faster than the Nyquist criterion to produce a repeated version of the transmitted symbols. Then, the oversampled soft data are processed by the concatenated decoding stage that encompasses the soft input soft output (SISO) Log-MAP decoding algorithm and conventional SC algorithm representing inner and outer decoders. Furthermore, this approach can potentially handle the issue of the sampling clock jitter. Finally, as per the simulation results, the proposed scheme's decoding performance using the SC algorithm outperforms that of previous polar Successive Cancellation List schemes (SCL), CA-SCL, and accomplishes coding gain up to 2.4 dB with OSR = 2.

Keywords— Polar Codes, SC, SCL, CA-SCL, OSR, SISO Log-MAP, Computational Complexity

1. Introduction

Polar codes, as a novel error-correcting code, have been chosen for control channels within the scope of the 5G NR benchmark, proving the essence of Arikan's invention in paving the way for commercial use [1]. The channel polarization has set the foundation for this capacity-achieving novel coding technique. It replaces the LTE tail-biting convolutional codes used for control channels, where the decoding performance is akin or superior to turbo and LDPC codes. Tal and Vardy's SCL approach was initially described by likelihoods, i.e., probabilities, during the decoding operations [2]. However, the intrinsic calculations were numerically unreliable. In this context, the soft information has been specified in the log-likelihood ratio (LLRs) domain [5] with a parameter called list size "L," describing the most likely surviving decoded paths. Once the decoding operation is finished, the decoder selects the most likely codeword path from a list of "L" available paths. Although there is a degrading-return impact, the decoding performance is still improved when "L" is raised. Deployment of the CA-SCL, as the decoding algorithm in 5G's eMBB for the control channels (uplink/downlink), is handled implicitly by CRC encoding [3]. It is typically known that the CA-SCL algorithm outperforms turbo or LDPC codes [4], regarded as a crucial selecting aspect in adopting 3GPP's polar codes.

Additionally, Log-Maximum A Posteriori (MAP) is employed with turbo codes, using a soft-input soft-output (SISO) algorithm to exchange the soft information between its constituent decoders through a decoding process that employs an iterative decoding nature until convergence is achieved. For simplicity, the log-MAP and max-log approximation algorithms are used in practice [10]. Furthermore, the process of oversampling the received signal in the time domain,i.e., sampling with fractional spaces or sampling faster than the Nyquist rate, has been studied for OFDM single-carrier systems [11]. More precisely, it has been revealed that time-domain oversampling could lead to multipath diversity gain, thereby remarkably enhancing performance. Recently, several studies have implemented the utilization of oversampling at the receiver's end combined with 1-bit quantization [12]. According to the results, the oversampling can attain a gain in signal-noise ratio (SNR) and a substantial enhancement in possible transmission throughput. Therefore, it has the ability to recover the loss of information during the coarse quantization process. Furthermore, different pulse shaping filters were investigated for the 1-bit quantized approaches with the aid of oversampling, where results have exhibited that information rates can be elevated by employing the root-raised-cosine (RRC) filters.

Moreover, thermal noise in the sampling clock's crystal can result in timing jitter; This might result in the signal being sampled at incorrect time intervals during the analog-to-digital conversion (ADCs) and DACs, which is analogous to yield a slight variable (+ or -) random delay that varies arbitrarily across samples; The limitation of performance, especially at *multi*-GS/s sampling speeds, may be caused by timing jitter [13], [14].

In the effort to meet the stringent prerequisites (Decoding complexity - Reliability) for the Ultra-Reliable and Low-Latency Communication (URLLC) applications [18], the concatenation decoding scheme at the receiver is presented to boost the bit-error-rate (BER) performance and benefit from the achieved coding gain in the aforementioned 5G NR applications. In this sense, the received signal is oversampled at a rate exceeding the Nyquist rate within the symbol period; this repeated M-folded data are then processed by the SISO-based serial concatenation decoding scheme of Log-MAP and SC algorithms, representing the inner and outer decoders, respectively. Finally, the waterfall simulation curves reveal that the proposed scheme enhances decoding performance, which can be exploited to diminish decoding latency.

The subsequent sections of this paper will be arranged as follows: Firstly, section II focuses on the decoding methodology for various polar decoding schemes, including successive cancellation (SC), SC-list (SCL), and CA-SCL. Following that, in section III, the proposed model is thoroughly examined. Lastly, section IV assesses the proposed algorithm and presents simulation results, while the conclusion is detailed in section VII.

2. Polar Decoding

Polar codes of length N, where N is a power of 2 (i.e., $N = 2^{\mathrm{n}}$), are utilized to transmit 'k' information bits via noiseless polarized channels. The '$N - k$' frozen bits are allocated to channels affected by noise in conformity with the polarization principle. The encoder structure is recursive, resulting in a recursive decoding structure facilitated by Arikan's innovative decoding algorithm, known as successive cancellation [1]. SC decoding algorithm performs sequential decoding of the bits using the previously decoded bits. For illustration, let's consider a ($N = 8$)-length polar codes in Fig.1, which can be rendered utilizing a factor graph or binary tree. The polar codes' kernel unit, concerning encoding and decoding operations, has an identical construction, with the smallest codeword of length 2. The kernel can be repeatedly extended to create any block length N. Moreover, the decoding kernel has two nodes: the check node (CN) and the variable node (VN), as seen in Fig.1. The necessary formulas for the CNs and VNs decoding operation are calculated using the δ and ρ, respectively.

$$\delta = 2\tanh^{-1}\left(\prod_{i=1}^{2}\tanh\left(\frac{L_i}{2}\right)\right) \tag{1}$$

$$\rho = L_1(-1)^{\hat{u}_1} + L_2 \tag{2}$$

Furthermore, the CNs update function " δ " can be modified to reduce the calculation complexity yielding

$$\delta = \min_{i\in\{1,2\}} |L_i| \prod_{i=1}^{2} \operatorname{sign}(L_i) \tag{3}$$

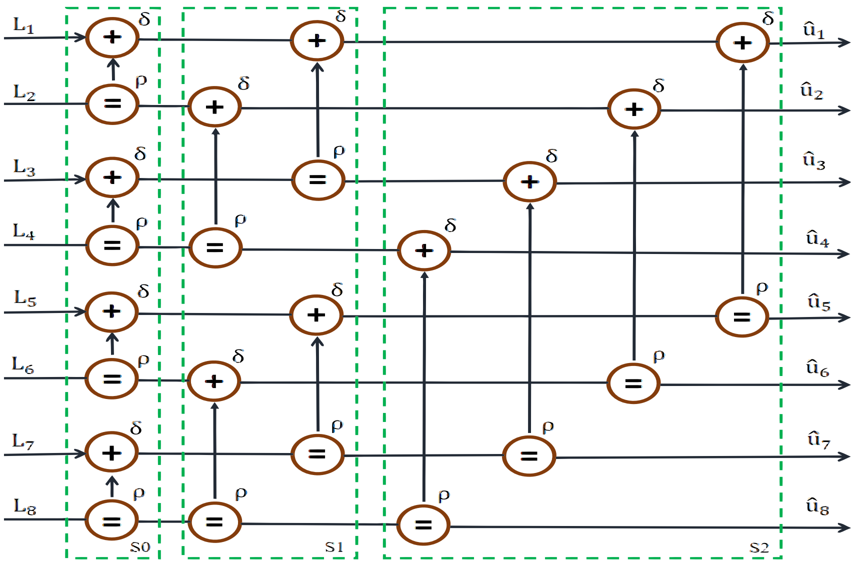


Figure 1: tanner graph structure of N = 8 for the decoding of SC algorithm.

Despite the SC decoding scheme's attempt to attain Shannon's capacity, it only improves performance at high block lengths, while the performance is inferior at small and moderate block lengths. In [5], a novel technique called SCL was declared to tackle this problem, where there are (L) surviving paths (possible codewords) from the relevant list decoder. In addition, A CRC check can be implemented for faster convergence to the correct codeword. Finally, the algorithm complexity, $O\left(L \cdot N \cdot \log_2 N\right)$, is constrained by the list size (L); therefore, complexity and SCL decoding performance ought to be traded off.

3. Proposed Decoder

The proposed approach is developed to enhance the SC algorithm's BER performance for short and intermediate block lengths without significantly increasing decoding complexity. As shown in Fig. 2, the initial stage entails oversampling the received signal. After that, the inner SISO Log-MAP rule is implemented and followed by the maximization process, which results in more reliable soft information. Finally, the outer SC decoder is triggered by more reliable soft data. Eventually, the decoder will proceed until it reaches the decoded word.

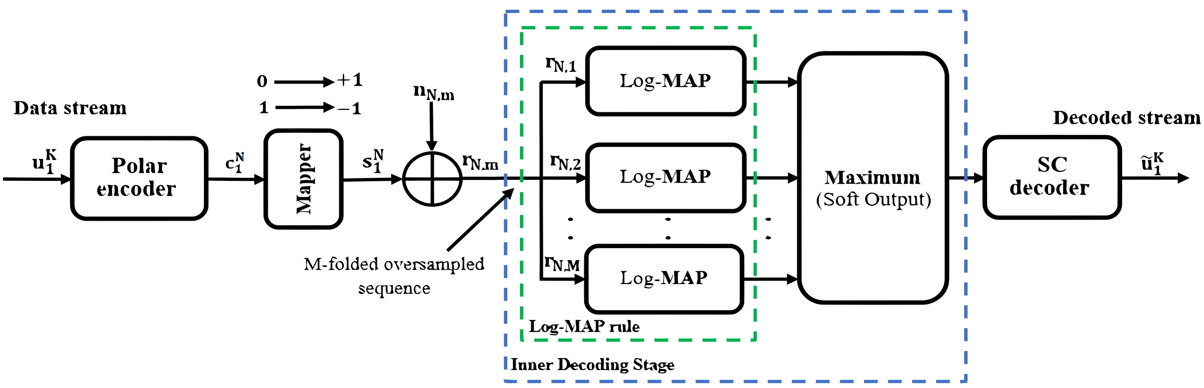


Figure 2: Discrete system model with polar encoder and BPSK mapper over AWGN channel with the oversampling at the receiver followed by two decoding stages (soft log-MAP + SC).

3.1. System Model

The overall system model is explained using a block diagram in Fig.2. The AWGN communication channel is defined with the conditional probability $P(\mathrm{r} \mid \mathrm{s})$. The polar encoder mapping C: $\{0,1\}^k \rightarrow C^n$, then the BPSK modulation maps the binary data into symbols $\mathrm{S} : \{-1,+1\}^n \rightarrow S^n$ [16]. Finally, the two-stage decoding process tries to map $\hat{u} : C^N \rightarrow \{0,1\}^k$. At the receiving side, the vector is temporally oversampled faster than the Nyquist criterion within the 3- dB of the symbol period at the time instances $(\mathrm{N} + \frac{\mathrm{m}}{M} - 1) \cdot T_s$, where $\mathrm{m} = 1, 2, \ldots, \mathrm{M}$ with M is an oversampling factor that is an integer. While T_s denotes the symbol duration [11], [12]. The oversampled received vector expressed by

$$r_{N,i} = s_N + n_{N,i} \quad \forall \quad i \quad \in \{1, 2, \ldots, M\} \tag{4}$$

3.2. Concatenated Decoding Scheme

The block diagram in Fig.2 illustrates the elements and construction of the concatenated decoding process. The outer SISO Log-MAP decoder minimizes the error probability by making a separate decision for each sample of the M-folded samples. It selects the corresponding soft information that maximizes the Log-MAP soft decoding rule. Specifically, for each received sample $r_{i,m}$ where $\mathrm{i} = 1, \ldots, N$, and $m = 1, \ldots, M$ and to minimize the bit error probability, the decoder chooses an estimation $\hat{b} \in \mathbb{F}_2$, and $r_{\max}$ values of the appropriate mth received samples associated with the i^{th} information bit that maximizes the log a posteriori probability $P_{s_i|r_{i,1} \ldots r_M}(s \mid r_{i,m})$, yielding

$$\left\langle \hat{b}, r_{\max} \right\rangle = \underset{\substack{s \in \{+1,-1\} \\ r \in r_{i,m}}}{\arg\max} \; \log_2 P_{s|\mathbf{r}_1,\ldots \mathbf{r}_M}(s \mid r_{i,m}) \qquad \forall \quad i = 1, \ldots, N, m = 1, \ldots, M \tag{5}$$

$$\omega_{\text{opt.}} = \log_2 P\left(S = \hat{b} \mid r_i = r_{\max}\right) \cdot (-1)^{\hat{b}} \tag{6}$$

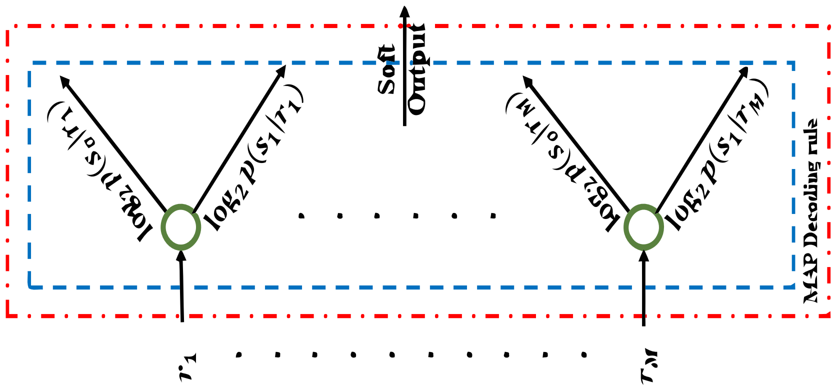


Figure 3: SISO Log-MAP proposed decoding process for the M-folded oversampled sequence.

The Log-MAP soft decoding approach can be utilized as a preprocessing stage at the front end of the successive cancellation decoder to handle the oversampled sequence. It comprises two steps, where the maximization is performed over the transmitted symbols and the oversampled received vector, as explained in equations (5) and (6). Unfortunately, implementing this algorithm requires large computation and storage; therefore, the aforementioned decoding rule can be simplified by using the likelihood ratio. Bayes' rule [18] explains how the likelihood and a posteriori probabilities relate to one another such that

$$P(s \mid r)=\frac{P(r \mid s) \cdot P(s)}{P(r)} \tag{7}$$

$$\begin{aligned} P_{s \mid r}\left(s=+1 \mid r_m\right) &= \frac{P_{r \mid s}\left(r_m \mid +1\right) \cdot P_s(+1)}{P_{r \mid s}\left(r_m \mid -1\right) \cdot P_s(-1)+P_{r \mid s}\left(r_m \mid +1\right) \cdot P_s(+1)} \\ &= \frac{1}{e^{+LLR(r)}+1} \end{aligned} \tag{8}$$

$$P_{s \mid r}\left(s=-1 \mid r_n\right)=\frac{1}{e^{-LLR(r)}+1} \tag{9}$$

Furthermore, the log-likelihood ratio and the log ratio of a posteriori probabilities will be identical if the input prior information for the transmitted symbols is equiprobable $P\left(s_i=0\right)=P\left(s_i=1\right)=\frac{1}{2}$.

$$LLR\left(r_m\right)=\log_2(APP)$$

Hence,

$$\log_2 \frac{P_{r \mid s}\left(r_m \mid \mathrm{s}=+1\right)}{P_{r \mid s}\left(r_m \mid \mathrm{s}=-1\right)}=\log_2 \frac{P_{s \mid r}\left(\mathrm{~s}=+1 \mid r_m\right)}{P_{s \mid r}\left(\mathrm{~s}=-1 \mid r_m\right)} \tag{10}$$

The log-likelihood ratio is divided into two main elements, which merit attention, where the absolute value $|LLR\left(\mathrm{r}_{i,m}\right)|$, and the sign indicate the decoded bit's reliability and hard decision, respectively. In this particular context, the Bitwise rule for soft decoding a set of oversampled sequences can be simplified into

$$LLR_{Out}^i=\max \left\{\left|LLR\left(\mathrm{r}_{i,m}\right)\right|\right\} * \operatorname{sign}\left\{\max \left(LLR\left(\mathrm{r}_{i,m}\right)\right)\right\} \quad \forall \quad m=1, \ldots, M \tag{11}$$

Upon completing this decoding stage, it can be inferred that the inner soft decoding rule relies on identifying the most reliable soft information from the over-sampled LLR sequence while preserving the sign, which is imperative for successful decoding The outer SC decoder will be initialized with the most reliable soft information from the outer decoding stage. Subsequently, at every node of the SC decoder, the LLRs are processed by "δ" and "ρ" functions to find a rough estimation of the transmitted symbol, and the hard decision can be given by

$$\hat{u}_i=\begin{cases}0, & \gamma_{\delta/\rho}^{(i)}\left(r_1^N, a_1^{i-1}\right) \geq 0 \\ 1, & \text{Otherwise.}\end{cases} \tag{12}$$

Then the sequential decoding property will continue decoding the soft input data in a bit-by-bit manner until the decoding process is fulfilled. The SC algorithm's output, which represents the decoded word, is marked as $\hat{u}=\left[\hat{u}_1, \hat{u}_2, \hat{u}_3, \ldots \ldots \ldots, \hat{u}_k\right]^T$.
Finally, The proposed method strives to decrease jitter noise by mitigating sampling uncertainty, i.e., introducing a variable delay from sample to sample during the analog-to-digital conversion. The proposed approach is based on maximization over multiple samples $\left[L_1(1) L_1(2) \ldots L_1(M), L_2(1) L_2(2) \ldots L_2(M), \ldots L_N(1) L_N(2) \ldots L_N(M)\right]^T$ within a Nyquist interval instead of a single sample as conventional, which makes it unsusceptible to the sampling clock jitter effect.

4. System Complexity

Computational complexity is a vital factor in many applications. Due to its significance, the proposed approach's complexity will be examined in this section. Initially, the oversampling stage complexity is $(N \cdot M)$, where M denotes the OSR. After that, the inner decoding stage complexity is given by $(2N \cdot M)$, and the outer decoding stage is $\left(N \cdot \log_2 N\right)$ [15]. Hence, the proposed approach's overall computational complexity is given by

$$\begin{aligned} \text{Overall Complexity} &= (N \cdot M)+(2N \cdot M)+(N)+\left(N \cdot \log_2 N\right) \\ &= \left(N \cdot M+2N \cdot M+N+N \cdot \log_2 N\right) \\ &= \left(N \cdot\left[3M+\log_2 N+1\right]\right) \end{aligned} \tag{13}$$

As a consequence, hosting the likelihood ratio approximation can lead to a decrease in computational complexity "$\Delta_{Complexity}$," which yields

$$\begin{aligned} \Delta_{Complexity} &= (N \cdot M) + (N) + (N \cdot \log_2 N) \\ &= (N \cdot [M + \log_2 N + 1]) \end{aligned} \tag{14}$$

Ultimately, table 1 enumerates the high computational intricacies of diverse polar decoding algorithms and the proposed approximate scheme. Furthermore, the approximate proposed scheme ($\Delta_{Complexity}$) attains convergence with a standard SC algorithm when controlling the oversampling ratio (M).

Algorithm	Computation Complexity
successive cancellation (SC)	$N \cdot \log_2 N$
CA-SCL	$N \cdot L \cdot \log_2 N$
proposed	$[N \cdot (3M + \log_2 N + 1)]$
proposed (Δ)	$[N \cdot (M + \log_2 N + 1)]$

Table 1: Complexity Breakdown Analysis

5. Results

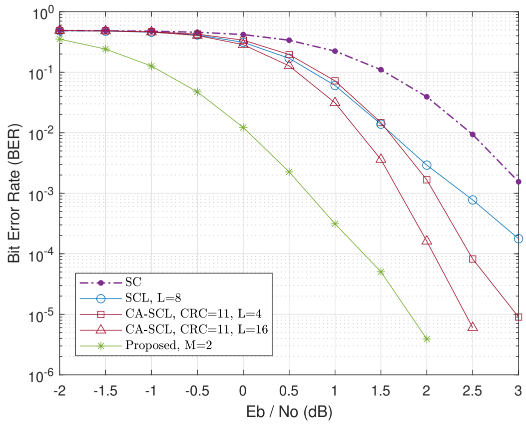
This section aims to evaluate the efficiency and BER performance of the proposed concatenated decoding approach and compare it with the prevailing polar decoding techniques in the literature, in particular, the standard SC, SCL, and CA-SCL decoding algorithms. Additionally, table 2 introduces an overview of the simulation parameters. Where the message length (k = 256) is established to examine the performance for short and moderate block lengths that are attractive for 5G NR's URLLC applications [18]. Moreover, miscellaneous code rates are studied at rates of R = $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$. Different modulation schemes are examined for further evaluation of the performance; the CRC length implemented with the CA-SCL algorithm is 11, while the SCL decoder uses list sizes of $L = 4,\ L = 16$. Finally, the utilized oversampling factor is (OSR = 2).

Table 2: Simulation Guidelines.

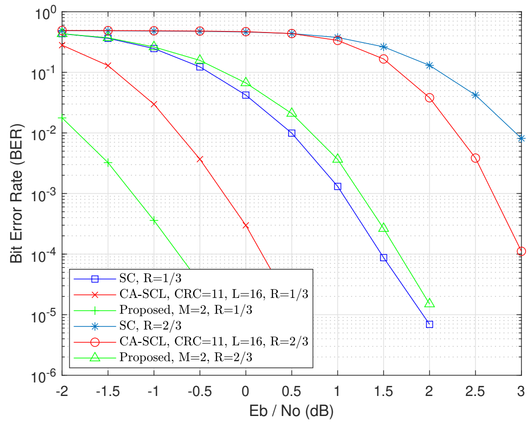
Parameter	Value
Simulation tool	MATLAB
Channel model	AWGN
Modulation scheme	BPSK, 4-QAM, 16-QAM
Code rate	$\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$
# of List sizes	4, 8, 16
Block lengths	256, 512
CRC length	11
Oversampling ratio (M)	2

The results in Fig.4a compare the proposed approach's performance against the well-known algorithms, where the dashed lines represent the classical SC algorithm that offers better complexity performance at the cost of poor BER performance in the case of short and medium blocks. Moreover, the proposed approach utilizing an oversampling factor of $M = 2$ demonstrates a superior coding gain. Specifically, it achieves a coding gain of 2.4 dB at a BER of 10^{-3}, representing a considerable enhancement in comparison to the typical SC decoder. While for the SCL SCL decoder with L = 8, the proposed approach shows a substantial gain of 1.7 dB. Furthermore, compared to the CA-SCL decoder with (CRC = 11) and a list size of L = 4, it yields a remarkable increase of 1.2 dB in coding gain. Regardless of the list size being further increased to L = 16, the proposed approach remains capable of achieving a notable gain of 0.6 dB at a BER of 10^{-5}.

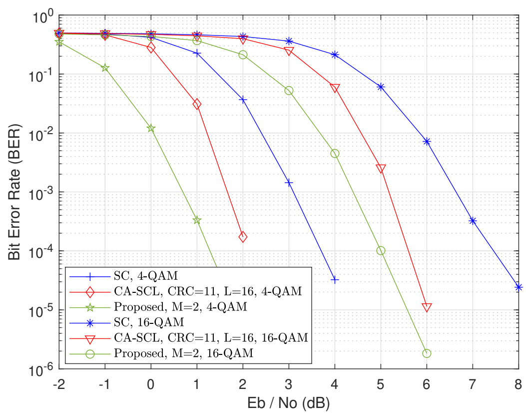
In Fig.4b and Fig.4c, the proposed scheme's performance is further evaluated in many aspects, such as several code rates and modulation schemes. As illustrated in Fig.4b, 1/3 and 2/3 code rates were



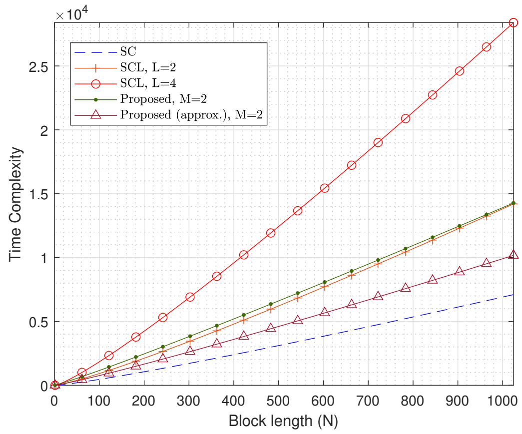
(a) Analysis of different decoding algorithms' performance versus the proposed scheme at M = 2



(b) Comparison of SC and CA-SCL and the proposed scheme at code rates of R=$\frac{1}{3}$, $\frac{2}{3}$, and M = 2



(c) Comparison of SC/CA-SCL versus the proposed scheme at modulation schemes (4-QAM, 16-QAM) and M =2



(d) Comparison of different complexities of polar decoding algorithms and the proposed system with OSR = 2

Figure 4: Analysis of the proposed scheme performance in many aspects (decoding algorithms - modulation schemes - code rates - complexity)

employed to verify the impact of changing the code rate on the proposed scheme, whereby the presence of coding gain is still observable despite varying the coding rate. Finally, in Fig.4c, both 4-QAM and 16-QAM modulation schemes were considered, with Gray coding utilized to assign the information bits to the constellation points. Results demonstrated a substantial coding gain for the proposed scheme compared to the conventional polar decoding algorithms. Furthermore, fig. 4d compares the complexity to assess the effectiveness of the proposed technique and the consequences of changing the list size and the oversampling ratio.

6. Conclusions

In this work, a new SISO concatenation decoding algorithm was put forward to enhance the performance of the polar SC decoding algorithm without significantly affecting the computational complexity. Specifically, temporal oversampling was carried out at the receiver prior to introducing the concatenated decoding process of SISO Log-MAP and conventional SC decoding algorithms, resulting in improving the LLR reliability applied to the SC decoder. Furthermore, the proposed approach can mitigate the impact of sampling clock jitter since we are searching for the most reliable LLRs within the symbol period.

Table 3: Summary performance comparison between the proposed scheme and the CA-SCL algorithm.

Designation	Polar codes	Proposed Decoder
Decoding algorithm	CA-SCL	Concatenation (log-MAP + SC)
BER Performance @ 10^{-3}	1.4 dB Coding gain	2.4 dB Coding gain
Complexity	Scales with list size (L) $[N.L.\log_2 N]$	Converges to SC $[N \cdot (M + \log_2 N + 1)]$

Finally, the proposed approach yields noticeable benefits, as demonstrated by the simulation results, which can substantially boost the performance of BER while maintaining low computational cost, gradually approach the standard SC decoding algorithm complexity, and offers decoding performance superior to SCL and CA-SCL decoding algorithms. Additionally, the proposed approach accomplishes a coding gain up to 2.4 dB with $OSR = 2$.

In general, the remarkable coding gain accomplished by the proposed approach can be exploited in many low-cost and low-latency wireless communications systems, such as 5G NR and *mm*-wave satellite communications.

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