Information theoretic properties for mixed random variables and applications

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Abstract. The bimodal gaussian (BMG) distribution plays an important role representing the output of many stochastic information sources. In this paper we study the bimodal gaussian distribution as a mixture distribution resulting from mixing both discrete and continuous gaussian random variables (R.V.). This distribution is characterized with the mixture probability model with different moment. We argue the entropy of this R.V., consequently, we establish an information theoretic foundation for the recent signaling schemes such as subcarrier index modulation OFDM (SIM-OFDM) and spatial Multiple Input Multiple Output (MIMO) modulation.

1. Introduction

Mixing (addition or multiplication) both discrete and continuous R.V. results in the mixture distribution. This distribution has gained its importance for representing the stochastic information sources in many different applications such as digital communication, economy, and machine learning.

The resultant mixture distribution consists of replicas of the continuous distribution but with different probabilities (weights), means, and variance. When the continuous random variable is a gaussian distribution, the resultant mixture will be multimodal gaussian distribution. When the discrete R.V. has only two values, the multimodal gaussian distribution becomes BMG distribution, which is our interest in this paper as it used widely in the recent digital communication applications.

Information theoretic parameters such as entropy, mutual information, achievable rates and channel capacity are necessary to give both the accurate performance limits and the characterization of the information sources [1].

Subcarrier Index Modulation Orthogonal Frequency Division Multiplexing (SIM-OFDM) is one of the recent important techniques, in which, both the transmitter output and the output of an additive white Gaussian noise (AWGN) channel are modeled with the BMG R.V.

SIM-OFDM has gained its importance due to its privileges over the ordinary OFDM with respect to the energy efficiency and the spectral efficiency.

Unlike the ordinary OFDM, not all the subcarriers are activated, meanwhile, part of the subcarriers are deactivated according to a certain pattern (indices), this pattern is chosen according to part of the transmitted data, and the other part of data is transmitted on the active subcarriers with the ordinary modulation techniques.

In [2], the basic single mode SIM-OFDM is presented, in which the indices is determined by

On-Off-Key (OOK) using P_1 bits, while P_2 bits are modulated using Quadrature Amplitude Modulation (QAM) figure 1. The proposed scheme has achieved high spectral efficiency but suffer from error propagation due to the usage of OOK in determining the indices.
In order to mitigate this error, [3] has proposed an enhanced SIM-OFDM (ESIM-OFDM), in which the subcarriers are divided into subgroups, each subgroup has two subcarriers, only one of them are active and carry single bite of information achieving better BER but with low spectral efficiency.
A generalized technique (GSIM-OFDM) provided in [4], in which the subgroups has equal size of L subcarriers with K out of L active subcarriers. By changing the size of the subgroups and adjusting the ratio of active to inactive subcarriers, the proposed technique in [4] achieves higher spectral efficiency compared to the previous method [5].
Spacial modulation is used to refer to the index modulation in the space domain (MIMO techniques), in which, not all the available antennas are activated, instead, some of them are deactivated within a time slot and the indices of the deactivated antennas are used as a source of information [3, 6].
Index modulation can also be applied in time domain. In this case the available resources are the time slots.

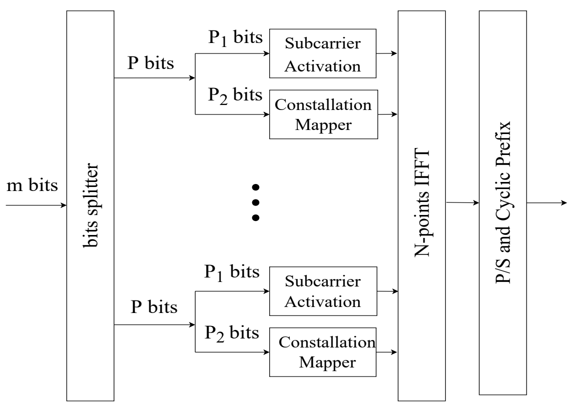

Figure 1. SIM-OFDM Transmitter structure

2. Probability Analysis for Mixture R.V.

In this section, we provide the required definitions for the set up of the paper framework.

2.1. Preliminaries

For the measure space $(\Omega, \mathcal{F}, \mathbb{P})$, with probability measure $\mathbb{P}$. Let $(\mathbb{R}, B_{\mathbb{R}})$ represents the measurable space on $\mathbb{R}$ with the Boral σ-algebra. The subset $\{d_1, d_2, ..\}$ of $\mathbb{R}$ is countable when D is an absolutely discrete R.V. with Probability Mass Function (PMF) $p_i = \mathbb{P}(D = d_i)$, where $\sum_i p_i = 1$, and an induced probability measure μ_D on $(\mathbb{R}, B_{\mathbb{R}})$.

For a continuous R.V. C, the probability measure μ_C, induced on $(\mathbb{R}, B_{\mathbb{R}})$ is absolutely continuous with respect to Lebesgue measure. These probability measure is characterized with the probability density function (pdf) $f(C)$, where $\int_{\mathbb{R}} f(C)dc = 1$.

2.2. Setup

The mixture R.V. X formed by mixing (addition/multiplication) operations an arbitrary discrete and continuous R.Vs. will have pdf on the form of:

$$f(x) = \sum_i p_i f_i(C) \tag{1}$$

When the continuous R.V. has a gaussian distribution $f(C) \sim \mathcal{N}(\mu_c, \sigma_c^2)$, the mixture R.V. is multimodal gaussian with pdf:

$$f(x) = \sum_i p_i \mathcal{N}(\mu_i, \sigma_i^2) \tag{2}$$

The number of modes depend on the number of the values of the discrete R.V., for a discrete R.V. $D \in \{a, b\}$ with PMF of $p(D = a) = p_1$, and $p(D = b) = p_2$, the resulting mixture $X = CD$ is a Bimodal gaussian R.V. with the pdf of:

$$f(x) = p_1 \mathcal{N}(\mu_1, \sigma_1^2) + p_2 \mathcal{N}(\mu_2, \sigma_2^2) \tag{3}$$

Where,

$$\begin{aligned}
&\mu_1 = a\mu_c, \quad \mu_2 = b\mu_c \\
&\sigma_1^2 = a^2\sigma_c^2, \quad \sigma_2^2 = b^2\sigma_c^2 \\
&\mu_x = p_1\mu_1 + p_2\mu_2 \\
&\sigma_x^2 = p_1[\sigma_1^2 + (\mu_1 - \mu_x)^2] + p_2[\sigma_2^2 + (\mu_2 - \mu_x)^2]
\end{aligned} \tag{4}$$

Remark 1 *Gaussian distribution converges to delta Dirac as the variance tends to zero.*

According to the distribution theory [7], any distribution T_f can be generated from a generic function $f : \mathcal{S} \rightarrowtail \mathcal{S}$ by:

$$T_f = \int_{-\infty}^{\infty} f(t)\phi(t)dt \tag{5}$$

where $\phi(t)$ is the delta dirac function.
Therefore, a gaussian distribution is:

$$T_{N(\mu,\sigma^2)} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(t-\mu)^2}{2\sigma^2}} \phi(t)dt \tag{6}$$

By using the substitution $s = \frac{t-\mu}{\sigma}$, Eq.6 will be:

$$T_{N(\mu,\sigma^2)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{2}} \phi(\mu + \sigma s) ds \tag{7}$$

Therefore,

$$\begin{aligned} \lim_{\sigma^2 \to 0} T_{N(\mu,\sigma^2)} &= \phi(\mu) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{2}} ds \\ &= \phi(\mu) \end{aligned} \tag{8}$$

Remark 2 *As a result of remark 1. The Bimodal Gaussian Distribution (BMG) will converge to either discrete distribution as the variance tends to zero, or single normal distribution as the variance tends to* ∞

In figure2 the two extreme cases of BMG distribution $f_t(t) = 0.5N(\mu, \sigma^2) + 0.5N(-\mu, \sigma^2)$. when σ^2 tends to zero, the random variable T will converge to a discrete random variable with PMF of $p(T = \mu) = 0.5$ and $p(T = -\mu) = 0.5$. However, when σ^2 tends to ∞, the distribution of T will converge to a single normal .

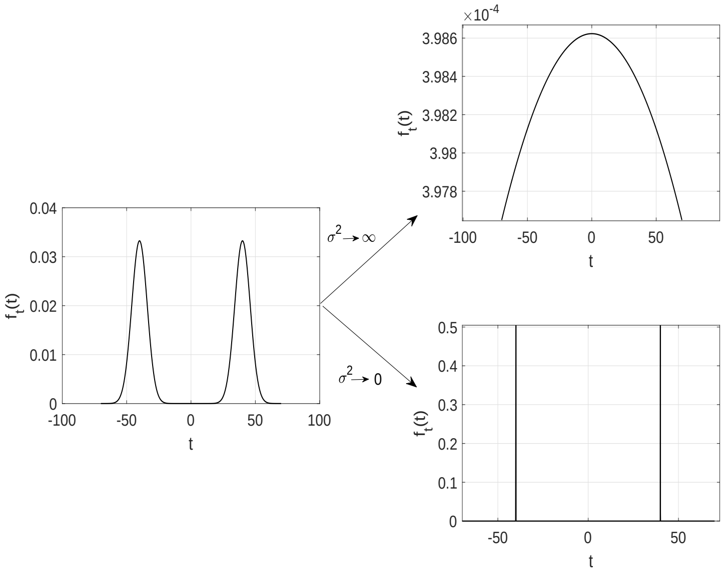

Figure 2. The convergence of BMG distribution to either a discrete or continuous single normal according to the variance change

2.3. Applications

The output X of a SIM-OFDM transmitter can be represented as a BMG R.V. results from the multiplication of a discrete and a continuous R.Vs.. The discrete random variable $D \in \{0, 1\}$ represents the state of the subcarriers, where "0" represents inactive subcarrier, and "1" represents active subcarrier, with probability mass function PMF of $p(D = 0) = 1 - p$, and

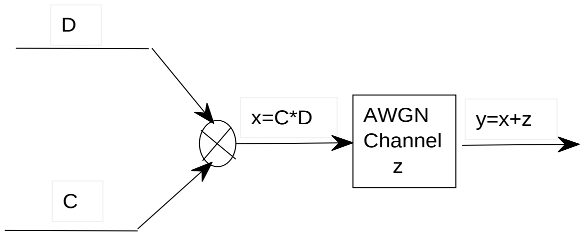

Figure 3. SIM-OFDM probability model under AWGN channel

$p(D = 1) = p$ respectively.
The second part of data to be transmitted using the active subcarrier is represented with a continuous random variable C with probability density function $f_C(C) \sim N(0, \sigma_c^2)$.

The output of the SIM-OFDM transmitter will have a pdf of:

$$f(x) = (1 - p)\delta(0) + pN(0, \sigma_c^2) \tag{9}$$

$f(x)$ is a BMG distribution with one mode of a zero variance remark 1. The output of a channel with gaussian noise distribution of $Z \sim N(0, \sigma_n^2)$ is $Y = X + Z$ and will have a pdf of:

$$f(y) = (1 - p)N(0, \sigma_n^2) + pN(0, \sigma_c^2 + \sigma_n^2) \tag{10}$$

Therefore, both the transmitter output and the receiver input are bimodal gaussian with different means and variances.

3. Notes on The Information Theory for The BMG Distribution

Information theory is used to give limits and bounds for a stochastic information sources such as the channel capacity and the achievable rates of different modulation schemes. Entropy is used to calculate this limits. In [1], entropy is defined either for discrete random variable (discrete entropy $H(D)$) or continuous random variable (differential entropy $h(c)$) without giving a definition for the mixture entropy.

$$H(D) = -\sum_{i=-\infty}^{\infty} p_i \log(p_i) \tag{11}$$

$$h(C) = -\int_{-\infty}^{\infty} f(C) \log(f(C)) dc \tag{12}$$

The mutual information between the channel input and output $I(X; Y)$ is required to calculate the achievable rate of SIM-OFDM

$$\begin{aligned} I(X;Y) &= \mathcal{H}(X) - \mathcal{H}(X/Y) \\ &= h(Y) - h(Y/X) \\ &= h(Y) - h(z) \\ &= h(y) - 0.5\log(2\pi e \sigma_n^2) \end{aligned} \tag{13}$$

where $\mathcal{H}$ is the mixture entropy. The output of a noisy gaussian channel y is a BMG distribution as defined in Eq.10, which has two modes with different probabilities, means, and variance. The differential entropy of BMG distribution can't be found in a closed analytical form [8, 9, 10], instead, many researches have proposed either bounds or approximations [9]for the differential entropy of BMG distribution.
In [11] the bounds are calculated with the assumption that the two modes are with the same variance, probability, and the means are symmetric (symmetric bimodal gaussian distribution). For more general distribution, [10] calculated the bounds for distribution with modes differ in both the means and probabilities but with the same variance. In [8] another bounds are provided, but with the constrain that the variance of each mode is $\geqslant$0.5.
No bounds are available for BMG distribution with modes having different variances. Therefore, simulations are used to exploit the properties of the entropy of the mixture R.V..

In figure 4, a comparison between e differential entropy of a BMG distribution and normal distribution with the same second moment. The simulation shows that the entropy of the mixture random variable is higher than the entropy of the single normal which gives the interpretation of the better spectral efficiency of the SIM-OFDM over the ordinary OFDM. Moreover, as a result of remark 2, the simulated mixture entropy has two limiting cases according to the variance:

(i) When the variance tends to zero, the mixture entropy will tends to the discrete entropy.
(ii) When the variance tends to infinity, the mixture entropy tends to the entropy of the single normal random variable.

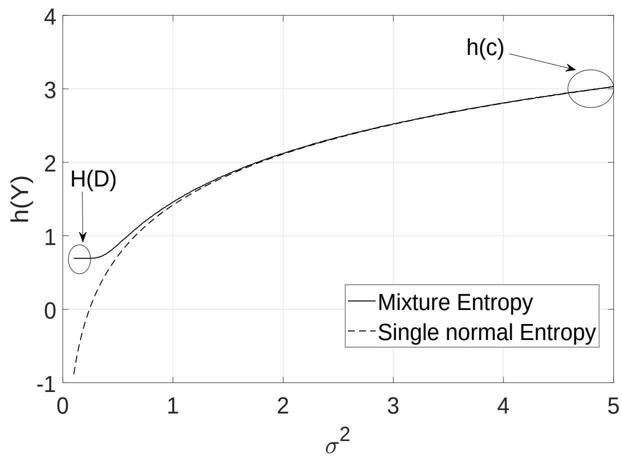

Figure 4. Comparison between the differential entropy of BMG and normal distribution with the same second moment

Remark 3 *The entropy of a random variable with BMG distribution will converge to the entropy of a discrete random variable when the variance tends to zero, and will converge to the differential entropy of a normal distribution as the variance tends to infinity.*

4. Conclusion and Future Work

SIM-OFDM achieve better spectral efficiency and energy efficiency over the ordinary OFDM, which makes SIM-OFDM an attractive field of research for 5G communications and beyond. A survey on the SIM-OFDM is proposed, comparing different techniques. Moreover, the properties

of SIM-OFDM from an information theoretic point of view is presented.
SIM-OFDM is a promising topic of research. In order to complete the information theoretic characterization, bounds on the entropy of the channel output is required to be calculated, by which, we can optimize between spectral efficiency and energy efficiency. Moreover, the optimum number of active/inactive subcarrier can be calculated for a variable length indices.

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