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ON Q-SHIFT DIFFERENCE-DIFFERENTIAL POLYNOMIALS OF ENTIRE FUNCTIONS THAT SHARE ONE VALUE

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Abstract. In this article we addresses the product of q-shift difference of transcendental entire functions. We primarily examine the zero distribution of the q-shift difference-differential polynomials of transcendental entire functions while simultaneously preparing the answers to the uniqueness problem in the case where the q-shift difference-differential polynomials of transcendental entire functions share a constant value. The findings are based on I. Lahiri's [9, 10] introduction of the concept of weighted sharing. The theory of Picard's exceptional value play an effective role for finding of our results. To discuss our results we create certain polynomial equation and analysed entire results of the article applying the theory of simple and multiple zeros of polynomial equation. We broadly elaborate our results with remark and corollary, and give an excellent example for proper justification of our results. Some open problems are generated from our results for future research. We extend and improve the results of R.S. Dyavanal and A.M. Hatticat [4], and generalized the result of P. Sahoo and G. Biswas [21] in effective manner.

1. Introduction, Definitions and Results

In this paper, meromorphic or entire function are defined on the complex plane and we used the standard notations, symbols, definition, theorems of the Nevanlinna's theory of meromorphic functions are explained in [8, 11, 26]. If f has no poles, then f is called entire function and if f be a nonconstant meromorphic function, then $T(r, f)$ is called Nevanlinna's characteristics function of f. We will use $S(r, f)$ as any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside a possible exceptional set E of the finite logarithmic measure $\lim_{r\to\infty} \int_{[1,r)\cap E} \frac{dt}{t} < \infty$. For our convenience we means that $\mathbb{S}(f)$ contains all constant functions and $\hat{\mathbb{S}} = \mathbb{S}(f) \cup \{\infty\}$.

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In addition, let f and g be two meromorphic functions defined in the complex plane and a be a value in the extended complex plane. Now we say that f and g share that value a CM(Counting Multiplicities) if the zeros of $f-a$ and $g-a$ coincide in location as well as in multiplicity and say that f and g share the value a IM if zeros of $f-a$ and $g-a$ coincide only in location but not in multiplicity. The counting function of zeros of $f-a$ where m-fold zero is counted m-times if $m \leq p$ and p times if $m > p$ is denoted by $N_p(r, a; f)$ where $p \in \mathbb{Z}^+$. In this paper we always use $\lambda = \Sigma_{j=1}^{d} s_j$ where $d, s_j (j = 1, 2, ..., d)$ are integers.

Many mathematicians already worked out many research papers on entire, meromorphic functions, their differential polynomials and sharing(see [5, 13, 14, 15, 19]). Recently mathematicians are attracted with the problem of difference equations and difference products in the complex plane . Already a numbers of papers have been published on the topics difference equation and difference product(see [3, 6, 7, 12, 20, 24]). Here we worked out another problem on difference product and it's derivative.

We introduce following standard definitions of Nevanlinna's value distribution theory which enlarge the article in sense of totality:

Definition 1.1. *Let $p(z) = \Sigma_{i=0}^{n} a_i z^i$ be a nonzero polynomial, where $a_i (i = 0, 1, 2, ..., n)$ are complex constants and $a_n \neq 0$ and n is an integer. We denote the numbers of single zeros of $p(z)$ by m_1 and the number of multiple zeros of $p(z)$ by m_2 and Γ_1, Γ_2 defined by $\Gamma_1 = m_1 + m_2$; $\Gamma_2 = m_1 + 2m_2$ respectively.*

Definition 1.2. [9, 10] *Let $a \in \mathbb{C} \cup \{\infty\}$ and l be a nonnegative integer or infinity, then we denote by $E_l(a; f)$ the set of all a-point of f where an a-point of multiplicity m is counted m times if $m \leq l$ and $l+1$ times if $m > l$. If $E_l(a; f) = E_l(a; g)$ we say that f, g share the value a with weight l.*

Zhang[26] established the following theorem in 2010:

Theorem A. [26] *Let $\alpha(z)$ be a small function of two transcendental entire functions of finite order f and g. Let $c \in \mathbb{C} \backslash \{0\}$, a constant and $n \geq 7$ be and integer then if $f(z)(f(z)-1)f(z+c)$ and $g(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.*

Qi, Yang and Liu [18], extend the result of theorem A as follows:

Theorem B. [18] *Let f and g be two transcendental entire functions of finite order, and c be a nonzero complex constant, and let $n \geq 6$ be an integer. If $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 CM, then either $fg = t_1$ or $f = t_2 g$ for some constant t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.*

In 2012, Chen and Chen [2] studies the uniqueness of difference polynomials $f^n(f^m-1)\prod_{j=1}^{d} f(z+c_j)^{s_j}$ and $g^n(g^m-1)\prod_{j=1}^{d} g(z+c_j)^{s_j}$ sharing small function, where $c_j \in \mathbb{C} \backslash \{0\}$ (j=1,2,...,d) are distinct constants, $n, m, d, s_j \in \mathbb{N}_+$ and obtained the following theorem:

Theorem C. [2] *Let f and g be two transcendental entire functions of finite order and $c_j \in \mathbb{C} \backslash \{0\}$ (j=1,2,...,d) be distinct constants, $n, m, d, s_j \in \mathbb{N}_+$ and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. If $n \geq m + 8\lambda$ and $f^n(f^m -$*

1) $\prod_{j=1}^{d} f(z+c_j)^{s_j}$ *and* $g^n(g^m-1)\prod_{j=1}^{d} g(z+c_j)^{s_j}$ *share* $\alpha(z)$ *CM, then* $f(z) \equiv tg(z)$*, where* $t^m = t^{n+\lambda} = 1$ *and* $\lambda = \Sigma_{j=1}^{d} s_j$.

Luo and Lin [17], established the result for general polynomial and difference function:

Theorem D. [17] *Let f and g be two transcendental entire functions of finite order, c be a nonzero complex constant and $n > 2\Gamma_2 + 1$ be an integer. If $p(f)f(z+c)$ and $p(g)g(z+c)$ share 1 CM, then one of the following results hold:*
(i) $f = tg$, where $t^\eta = 1$ where $\eta = GCD(\lambda_0+1, \lambda_1+1, ..., \lambda_n+1)$ and

$$\lambda_i = \begin{cases} i, & if \quad a_i \neq 0, \\ n, & if \quad a_i = 0; \\ i = 0, 1, 2, ..., n. \end{cases}$$

(ii) f and g satisfy the algebraic equation $R(f,g) = 0$, where
$R(\gamma_1, \gamma_2) = p(\gamma_1)\gamma_1(z+c) - p(\gamma_2)\gamma_2(z+c)$;
(iii) $f = e^\alpha, g = e^\beta$, where $\alpha(z)$ and $\beta(z)$ are two polynomials and $\alpha + \beta = h$, h is a complex constant satisfying $a_n^2 e^{(n+1)h} = 1$.

Example 1.1. *Let $p(z) = (z-1)^8(z+1)^8 z^{15}$, $f(z) = \sin(z)$ and $g(z) = \cos(z)$ and $c = 2\pi$. Now it is clear $n > 2\Gamma_2 + 1$ and $p(f)f(z+c) = p(g)g(z+c)$. Then $p(f)f(z+c)$ and $p(g)g(z+c)$ share 1 CM. Then it satisfy condition of theorem D and f and g satisfy the algebraic equation $R(f,g) = 0$ where $R(\gamma_1,\gamma_2) = p(\gamma_1)\gamma_1(z+c) - p(\gamma_2)\gamma_2(z+c)$.*

Wang and Xu [22], developed two interesting results for product difference function:

Theorem E. [22] *Let f and g be transcendental entire functions of finite order such that $f$and g share 0 CM. Let $F(z) = p(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}$ and $G(z) = p(g)\prod_{j=1}^{d} g(z+c_j)^{s_j}$, where $c_j \in \mathbb{C}$ and $n, d, s_j \in \mathbb{N}_+$, (j=1,2,...,d). If $F(z)$ and $G(z)$ share 1 CM and $n > 2\Gamma_2 + \lambda$, then one of the following cases holds:*
(i) $f \equiv tg$ for a constant t such that $t^\chi = 1$ where $\chi = GCD(\lambda_0+\lambda, \lambda_1+\lambda, ..., \lambda_n+\lambda)$ and $\lambda_i (i = 0, 1, ..., n)$ are state as in theorem D;
(ii) $f = e^\gamma, g = \mu e^{-\gamma}$, where γ is a nonconstant polynomials, μ is a complex constant satisfying $a_n^2 \mu^{n+\lambda} \equiv 1$.

Theorem F. [22] *Under the assumption of theorem E, if $E_l(1; F(z)) = E_l(1; G(z))$ and $l, n, d(>0), s_j(>0)(j = 1, 2, ..., d)$ are integers satisfying one of the following conditions:*
(i) $l \geq 3; n > 2\Gamma_2 + \lambda$;
(ii) $l = 2; n > 2\Gamma_2 + \Gamma_1 + \lambda + d$;
(iii) $l = 1; n > 2\Gamma_2 + 2\Gamma_1 + \lambda + 2d$;
(iv) $l = 0; n > 2\Gamma_2 + 3\Gamma_1 + \lambda + 3d$;
then conclusions of theorem E holds.

In 2016, Sahoo and Biswas [21], considered q-shift difference function and established following uniqueness result on a difference differential polynomial:

Theorem G. [21] *Let f and g be two transcendental entire functions of zero order and let $q \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$. If $E_l(1; (p(f)f(qz+c))^{(k)}) = E_l(1; (p(g)g(qz+c))^{(k)})$ and l, m, n are integers satisfy one of the following conditions:*

(i) $l \geq 2; n > 2\Gamma_2 + 2km_2 + 1$;
(ii) $l = 1; n > \frac{1}{2}(\Gamma_1 + 4\Gamma_2 + 5km_2 + 3)$;
(iii) $l = 0; n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4$;
then one of the following results holds:
(i) $f = tg$ *for a constant* t *such that* $t^\eta = 1$;
(ii) f *and* g *satisfy the algebraic equation* $R(f, g) = 0$, *where*
$R(\gamma_1, \gamma_2) = p(\gamma_1)\gamma_1(qz + c) - p(\gamma_2)\gamma_2(qz + c)$;
(iii) $fg = \zeta$, *where* ζ *is a complex constant satisfying* $a_n^2\zeta^{n+1} \equiv 1$.

Question: Is it possible to derive uniqueness results for product of q-shift difference-differential polynomial functions?

We consider product of q-shift difference-differential polynomial functions and develop uniqueness result in concern of weighted sharing. We discuss our result elaborately in main section.

2. Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows :

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 2.1. [25] *Let* f *be a nonconstant meromorphic function, and* $p(f) = \sum_{i=0}^{n} a_i f^i$, *where* $a_0, a_1, a_2, ..., a_n$ *are complex constants and* $a_n \neq 0$. *Then* $T(r, p(f)) = nT(r, f) + S(r, f)$.

Note: Throughout the paper we use $p(f)$ as polynomial in f as defined in lemma 2.1.

Lemma 2.2. [27] *Let* f *be a nonconstant meromorphic function, and* $p, k \in \mathbb{Z}^+$. *Then*

$$N_p(r, \frac{1}{f^{(k)}}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f), \tag{2.1}$$

$$N_p(r, \frac{1}{f^{(k)}}) \leq k\overline{N}(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f). \tag{2.2}$$

Lemma 2.3. [10] *Let* f *and* g *be two non-constant meromorphic functions. If* $E_2(1; f) = E_2(1; g)$, *then one of the following relation holds:*
(i) $T(r) \leq N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g}) + N_2(r, f) + N_2(r, g) + S(r)$;
(ii) $f = g$;
(iii) $fg = 1$;
where $T(r) = max\{T(r, f), T(r, g)\}$ *and* $S(r) = o\{T(r)\}$.

Lemma 2.4. [1] *Let* F *and* G *be two non-constant meromorphic functions such that* $E_1(1; F) = E_1(1; G)$ *and* $H \not\equiv 0$, *then*
$T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + N_2(r, F) + N_2(r, G) + \frac{1}{2}\overline{N}(r, \frac{1}{F}) + \frac{1}{2}\overline{N}(r, F) + S(r, F) + S(r, G)$;
and we can deduce same result for $T(r, G)$.

Lemma 2.5. [1] *Let F and G be two non-constant meromorphic functions which are share 1 IM and $H \not\equiv 0$, then,*
$T(r,F) \leq N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+N_2(r,F)+N_2(r,G)+2\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})+2\overline{N}(r,F)+\overline{N}(r,G)+S(r,F)+S(r,G)$;
the same inequality holds for $T(r,G)$.

Lemma 2.6. [23] *Let f be a transcendental meromorphic function of order zero and q, c two nonzero complex constants. Then*
$T(r,f(qz+c)) = T(r,f(z)) + S(r,f)$;
$N(r,f(qz+c)) = N(r,f(z)) + S(r,f)$;
$N(r,\frac{1}{f(qz+c)}) = N(r,\frac{1}{f(z)}) + S(r,f)$;
$\overline{N}(r,f(qz+c)) = \overline{N}(r,f(z)) + S(r,f)$;
$\overline{N}(r,\frac{1}{f(qz+c)}) = \overline{N}(r,\frac{1}{f(z)}) + S(r,f)$.

Lemma 2.7. [16] *Let f(z) be a nonconstant zero-order meromorphic function and $q \in \mathbb{C}\setminus\{0\}$. Then, $m(r,\frac{f(qz+c)}{f(z)}) = S(r,f)$ on a set of logarithmic density 1.*

Remark 2.1. *For finite nonzero-order meromorphic functions, the conclusion of the lemma 2.7 may not be true. We can take the following example:*
Let $f(z) = e^{2z}$. Then $m(r,\frac{f(2z+1)}{f(z)}) = e^2 m(r,f) = e^2 T(r,f)$.

Lemma 2.8. *Let f be a transcendental meromorphic function of order zero and $q_j(\neq 0), c_j(j=1,2,...,d)$ are complex constants. Then*
$(n-\lambda+\tau)T(r,f)+S(r,f) \leq T(r,f^\tau p(f)\prod_{j=1}^{d} f(q_j z+c_j)^{s_j}) \leq (n+\lambda+\tau)T(r,f)+S(r,f)$
In addition, if f is a transcendental entire function of zero order, then
$T(r,f^\tau p(f)\prod_{j=1}^{d} f(q_j z+c_j)^{s_j}) = T(r,f^\tau p(f)\prod_{j=1}^{d} f^{s_j}(z)) + S(r,f) = (n+\lambda+\tau)T(r,f)+S(r,f)$.

Proof. Now $f(z)$ is a transcendental entire function and $F(z) = f^\tau p(f)\prod_{j=1}^{d} f(q_j z+c_j)^{s_j}$. Then after using lemma 2.7 we have

$$\begin{aligned}
T(r,F(z)) &= m(r,F(z)) \\
&= m(r,f^\tau p(f)\prod_{j=1}^{d} f(q_j z+c_j)^{s_j}) \\
&\leq m(r,f^\tau p(f)\prod_{j=1}^{d} f(z)^{s_j}) + m(r,\frac{\prod_{j=1}^{d} f(q_j z+c_j)^{s_j}}{\prod_{j=1}^{d} f(z)^{s_j}}) \\
&\leq (n+\tau)T(r,f) + \lambda T(r,f) + S_1(r,f).
\end{aligned}$$

Hence, $T(r,F(z)) \leq (n+\lambda+\tau)T(r,f) + S_1(r,f)$.
On the other hand from Lemma 2.7, we have

$$\begin{aligned}(n+\lambda+\tau)T(r,f) &= T(r, f^\tau p(f)\prod_{j=1}^{d} f(z)^{s_j}) + S(r,f)\\ &\leq m(r, f^\tau p(f)\prod_{j=1}^{d} f(z)^{s_j}) + S(r,f)\\ &\leq m(r,F(z)) + m(r, \frac{\prod_{j=1}^{d} f(z)^{s_j}}{\prod_{j=1}^{d} f(q_j z+c_j)^{s_j}})\\ &\leq T(r,F(z)) + S_1(r,f).\end{aligned}$$

Hence,
$T(r, f^\tau p(f)\prod_{j=1}^{d} f(q_j z+c_j)^{s_j}) = T(r, f^\tau p(f)\prod_{j=1}^{d} f(z)^{s_j}) + S_1(r,f) = (n+\lambda+\tau)T(r,f) + S_1(r,f)$.
If f is a meromorphic function of zero order from lemma 2.6 and lemma 2.1 we have,

$$\begin{aligned}T(r, f^\tau p(f)\prod_{j=1}^{d} f(q_j z+c_j)^{s_j}) &\leq T(r, f^\tau p(f)) + T(r, \prod_{j=1}^{d} f(q_j z+c_j)^{s_j})\\ &\leq (n+\tau)T(r,f) + \lambda T(r,f) + S_1(r,f)\\ &\leq (n+\lambda+\tau)T(r,f) + S_1(r,f),\end{aligned}$$

On the other hand from lemma 2.7 and lemma 2.1 we have

$$\begin{aligned}(n+\lambda+\tau)T(r,f) &\leq T(r, f^\tau p(f)\prod_{j=1}^{d} f(z)^{s_j}) + S(r,f)\\ &= m(r, f^\tau p(f)\prod_{j=1}^{d} f(z)^{s_j}) + N(r, f^\tau p(f)\prod_{j=1}^{d} f(z)^{s_j}) + S(r,f)\\ &\leq m(r, F(f)\frac{\prod_{j=1}^{d} f(z)^{s_j})}{\prod_{j=1}^{d} f(q_j z+c_j)^{s_j}})\\ &+ N(r, F(f)\frac{\prod_{j=1}^{d} f(z)^{s_j})}{\prod_{j=1}^{d} f(q_j z+c_j)^{s_j}}) + S(r,f)\\ &\leq T(r,F(f)) + 2\lambda T(r,f) + S(r,f),\end{aligned}$$

hence $(n-\lambda+\tau)T(r,f) \leq T(r,F(z)) + S(r,f)$.
Hence the result,
$(n-\lambda+\tau)T(r,f) + S_1(r,f) \leq T(r, f^\tau p(f)\prod_{j=1}^{d} f(q_j z+c_j)^{s_j}) \leq (n+\lambda+\tau)T(r,f) + S_1(r,f)$. □

Lemma 2.9. *Let f and g be two entire functions, $q_j, c_j (j = 1, 2, ..., d)$ complex constants and $q_j \neq 0$; n, k, d, s_j, τ are positive integers and let $F = (f^\tau p(f)\prod_{j=1}^{d} f(q_j z+c_j)^{s_j})^{(k)}$, $G = (g^\tau p(g)\prod_{j=1}^{d} g(q_j z+c_j)^{s_j})^{(k)}$. If there exists two nonzero constants b_1 and b_2 such that*
$\overline{N}(r, b_1; F) = \overline{N}(r, \frac{1}{G})$ and $\overline{N}(r, b_2; G) = \overline{N}(r, \frac{1}{F})$, then $n \leq 2\Gamma_1 + km_2 + \lambda + \tau$.

Proof. Let $F_1 = f^\tau P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}$ and $G_1 = g^\tau P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}$. By the second main theorem of Nevanlinna, we have

$$T(r,F) \le \overline{N}(r,\frac{1}{F}) + \overline{N}(r,c_1;F) + S(r,F) \le \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + S(r,F). \quad (2.3)$$

Using (2.1),(2.2),(2.3) and lemmas 2.1, 2.6, 2.8, we get,

$$\begin{aligned}
(n+\lambda+\tau)T(r,f) &\le T(r,F) - \overline{N}(r,\frac{1}{F}) + N_{k+1}(r,\frac{1}{F_1}) + S(r,f)\\
&\le \overline{N}(r,\frac{1}{G}) + N_{k+1}(r,\frac{1}{F_1}) + S(r,f)\\
&\le N_{k+1}(r,\frac{1}{F_1}) + N_{k+1}(r,\frac{1}{G_1}) + S(r,g) + S(r,f)\\
&\le N_{k+1}(r,\frac{1}{f^\tau p(f)}) + N_{k+1}(r,\frac{1}{\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}}) + S(r,f)\\
&+ N_{k+1}(r,\frac{1}{g^\tau p(g)}) + N_{k+1}(r,\frac{1}{\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}}) + S(r,g)\\
&\le (m_1+m_2+km_2+\lambda+\tau)(T(r,f)+T(r,g)) + S(r,g) \quad (2.4)
\end{aligned}$$

Similarly,

$$\begin{aligned}
(n+\lambda+\tau)T(r,g) &\le (m_1+m_2+km_2+\lambda+\tau)(T(r,f)+T(r,g))\\
&+ S(r,f)+S(r,g). \quad (2.5)
\end{aligned}$$

In view of (2.4), (2.5) we have,

$$(n-2m_1-2m_2-2km_2-\lambda-\tau)(T(r,f)+T(r,g)) \le S(r,f)+S(r,g).$$

Which gives $n \le 2\Gamma_1 + 2km_2 + \lambda + \tau$. This proves the lemma. □

3. Main Results

Theorem 3.1. *Let f transcendental entire function of zero order and let $q_j \in \mathbb{C}\setminus\{0\}$, $c_j \in \mathbb{C}(j=1,2,...,d)$. If $n > \Gamma_1 + km_2$, then $(f^\tau p(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})^{(k)} - \alpha(z) = 0$ has infinitely many solutions where n, k,τ are integers and $\alpha(z) \in S(f)\backslash\{0\}$.*

Proof. We take $F_1 = f^\tau p(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}$. Then F_1 is transcendental entire function. In contrary, we assume that $F_1^{(k)} - \alpha(z)$ has finitely many solutions. Then

$$N(r,\alpha;F_1^{(k)}) = O\{\log r\} = S(r,f).$$

Now using the above result and (2.1) and (2.3) we have from Nevanlinna's theorem for three small functions

$$\begin{aligned}
T(r,F_1^{(k)}) &\le \overline{N}(r,\frac{1}{F_1^{(k)}}) + \overline{N}(r,\alpha;F_1^{(k)}) + S(r,f)\\
&\le T(r,F_1^{(k)}) - T(r,F_1) + N_{k+1}(r,\frac{1}{F_1}) + S(r,f).
\end{aligned}$$

Now using lemma 2.7 on the above results we have

$$\begin{aligned}(n+\lambda+\tau)T(r,f) &\leq N_{k+1}(r,\frac{1}{F_1})+S(r,f)\\ &\leq N_{k+1}(r,\frac{1}{p(f)})+N(r,\frac{1}{f^{\tau}})\\ &+ N(r,\frac{1}{\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}})+S(r,f)\\ &\leq (m_1+m_2+km_2+\lambda+\tau)T(r,f)+S(r,f),\end{aligned}$$

that is,

$$(n-m_1-m_2-km_2)T(r,f)\leq S(r,f),$$

which is a contradiction with our assumption that $n>\Gamma_1+km_2$.
This proves the theorem. □

Theorem 3.2. *Let f and g be two transcendental entire functions of zero order and let $q_j\in\mathbb{C}\setminus\{0\}$, $c_j\in\mathbb{C}(j=1,2,...,d)$. If $E_l(1;(f^{\tau}p(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j})^{(k)})=E_l(1;(g^{\tau}p(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j})^{(k)})$ and l,m,n,d,τ are integers satisfy one of the following conditions:*
(i) $l\geq 2; n>2\Gamma_2+2km_2+\lambda+\tau$;
(ii) $l=1; n>\frac{1}{2}(\Gamma_1+4\Gamma_2+5km_2+3\lambda+3\tau)$;
(iii) $l=0; n>3\Gamma_1+2\Gamma_2+5km_2+4\lambda+4\tau$;
then one of the following results holds:
(i) $f=tg$ for a constant t such that $t^{\chi}=1$;
(ii) f and g satisfy the algebraic equation $R(f,g)=0$, where
$R(\gamma_1,\gamma_2)=\gamma_1^{\tau}p(\gamma_1)\prod_{j=1}^{d}\gamma_1(q_jz+c_j)^{s_j}-\gamma_2^{\tau}p(\gamma_2)\prod_{j=1}^{d}\gamma_2(q_jz+c_j)^{s_j}$;
(iii) $f(z)=e^{\gamma(z)}$ and $g(z)=\kappa e^{-\gamma(z)}$, where κ is a complex constant satisfy $a_n^2\kappa^{n+\lambda+\tau}=1$.

Proof. We have $f(z)$ is a transcendental entire functions and we take
$F_1=f^{\tau}p(f)\prod_{j=1}^{d} f(q_jz+c_j)^{s_j}$, $G_1=g^{\tau}p(g)\prod_{j=1}^{d} g(q_jz+c_j)^{s_j}$ and $F=F_1^{(k)}$ and $G=G_1^{(k)}$. Then F and G are transcendental meromorphic functions satisfy $E_l(1;F)=E_l(1;G)$. Now with help of lemma 2.8 and using (2.1) we have,

$$\begin{aligned}N_2(r,\frac{1}{F}) &\leq N_2(r,\frac{1}{F_1^{(k)}})+S(r,f)\\ &\leq T(r,F_1^{(k)})-T(r,F_1)+N_{k+2}(r,\frac{1}{F_1})+S(r,f)\\ &\leq T(r,F)-(n+\lambda+\tau)T(r,f)+N_{k+2}(r,\frac{1}{F_1})+S(r,f).\end{aligned}$$

Hence,

$$(n+\lambda+\tau)T(r,f)\leq T(r,F)-N_2(r,\frac{1}{F})+N_{k+2}(r,\frac{1}{F_1})+S(r,f). \tag{3.1}$$

We can show from (2.2),

$$\begin{aligned} N_2(r,\frac{1}{F}) &\leq N_2(r,\frac{1}{F_1^{(k)}})+S(r,f) \\ &\leq N_{k+2}(r,\frac{1}{F_1})+S(r,f). \end{aligned} \quad (3.2)$$

Now following three cases will be discuss separately:

Case I.

Let $l \geq 2$. If possible we assume that (i) of lemma 2.3 holds. We can deduce from (3.1) with help of (3.2),

$$\begin{aligned} (n+\lambda+\tau)T(r,f) &\leq N_2(r,\frac{1}{G})+N_2(r,F)+N_2(r,G)+N_{k+2}(r,\frac{1}{F_1}) \\ &+ S(r,f)+S(r,g) \\ &\leq N_{k+2}(r,\frac{1}{F_1})+N_{k+2}(r,\frac{1}{G_1})+S(r,f)+S(r,g) \\ &\leq N_{k+2}(r,\frac{1}{p(f)})+N_{k+2}(r,\frac{1}{p(g)})+N_{k+2}(r,\frac{1}{f^\tau}) \\ &+ N_{k+2}(r,\frac{1}{g^\tau})+N_{k+2}(r,\frac{1}{\prod_{j=1}^d f(q_jz+c_j)^{s_j}}) \\ &+ N_{k+2}(r,\frac{1}{\prod_{j=1}^d g(q_jz+c_j)^{s_j}})+S(r,f)+S(r,g) \\ &\leq (m_1+2m_2+km_2+\lambda+\tau)(T(r,f)+T(r,g)) \\ &+ S(r,f)+S(r,g). \end{aligned} \quad (3.3)$$

Same we can show for $T(r,g)$ i.e

$$(n+\lambda+\tau)T(r,g) \leq (m_1+2m_2+km_2+\lambda+\tau)(T(r,f)+T(r,g))+S(r,f)+S(r,g). \quad (3.4)$$

We can obtain from (3.3) and (3.4)

$$(n-2m_1-4m_2-2km_2-\lambda-\tau)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

which contradict the fact $n > 2\Gamma_2+2km_2+\lambda+\tau$. Then by lemma 2.3 we claim that either $FG=1$ or $F=G$.
Let $FG=1$. Then,

$$(f^\tau p(f)\prod_{j=1}^d f(q_jz+c_j)^{s_j})^{(k)}(g^\tau p(g)\prod_{j=1}^d g(q_jz+c_j)^{s_j})^{(k)}=1. \quad (3.5)$$

If possible, let $p(z)=0$ has m roots $\alpha_1,\alpha_2,\alpha_3,...,\alpha_m$ with multiplicity $n_1,n_2,n_3,...,n_m$. Then we have $n_1+n_2+n_3+...+n_m=n$. Now from (3.5) we have,

$$\begin{aligned} &[f^\tau a_n(f-\alpha_1)^{n_1}(f-\alpha_2)^{n_2}...(f-\alpha_m)^{n_m}\prod_{j=1}^d f(q_jz+c_j)^{s_j}]^{(k)} \times \\ &[g^\tau a_n(g-\alpha_1)^{n_1}(g-\alpha_2)^{n_2}...(g-\alpha_m)^{n_m}\prod_{j=1}^d g(q_jz+c_j)^{s_j}]^{(k)}=1. \end{aligned} \quad (3.6)$$

Since f and g are nonconstant entire functions from (3.6), we see that $\alpha_1 = \alpha_2 = ... = \alpha_m = 0$. Also we can say that $\alpha_1, \alpha_2, ..., \alpha_m$ are picard's exceptional values. By picard's theorem of entire function, we have at least three picard's exceptional values of f and if $m \geq 2$ and $\alpha_i \neq 0 (i = 1, 2, ..., m)$, then we obtain a contradiction. Next we assume that $p(z) = 0$ has only one root. Then $p(f) = a_n(f-a)^n$ and $p(g) = a_n(g-a)^n$, where a is any complex constant. Now from (3.5) we can write

$$[f^\tau a_n(f-a)^n \prod_{j=1}^{d} f(q_j z + c_j)^{s_j}]^{(k)} [g^\tau a_n(g-a)^n \prod_{j=1}^{d} g(q_j z + c_j)^{s_j}]^{(k)} = 1. \quad (3.7)$$

By picard's theorem and as f, g are transcendental entire functions, then we can say that $f - a = 0$ and $g - a = 0$ do not have zeros. Then, we obtain that $f(z) = e^{\gamma(z)} + a$ and $g(z) = e^{\beta(z)} + a$, $\gamma(z), \beta(z)$ being nonconstant polynomials. From (3.7), we also see that $\prod_{j=1}^{d} f(q_j z + c_j)^{s_j} \neq 0$ and $\prod_{j=1}^{d} g(q_j z + c_j)^{s_j} \neq 0$ and therefore $a = 0$. Thus $f(z) = e^{\gamma(z)}, g(z) = e^{\beta(z)}, p(z) = a_n z^n$ and

$$[a_n e^{n\gamma(z) + \tau\gamma(z) + \Sigma_{j=1}^{d} s_j \gamma(q_j z + c_j)}]^{(k)} [a_n e^{n\beta(z) + \tau\gamma(z) + \Sigma_{j=1}^{d} s_j \beta(q_j z + c_j)}]^{(k)} = 1. \quad (3.8)$$

If $k = 0$, then from (3.8) we have,

$$a_n^2 e^{(n+\tau)(\gamma(z)+\beta(z)) + \Sigma_{j=1}^{d} s_j (\gamma(q_j z + c_j) + \beta(q_j z + c_j))} = 1.$$

Since $\gamma(z)$ and $\beta(z)$ are two nonconstant polynomials, we get $\gamma(z) + \beta(z) = \rho$ where ρ is a constant. From this we can easily see that $f(z) = e^{\gamma(z)}$ and $g(z) = \kappa e^{-\gamma(z)}$ where γ is a nonconstant polynomial and κ is a complex constant satisfying $a_n^2 \kappa^{n+\lambda+\tau} \equiv 1$ and $\kappa = e^\rho$.

If $k \geq 1$, then we get

$$[a_n e^{(n+\tau)\gamma(z) + \Sigma_{j=1}^{d} s_j \gamma(q_j z + c_j)}]^{(k)} = a_n e^{(n+\tau)\gamma(z) + \Sigma_{j=1}^{d} s_j \gamma(q_j z + c_j)} p(\gamma' \gamma'_{c_j}, ..., \gamma^{(k)} \gamma_{c_j}^{(k)}),$$

where $\gamma_{c_j} = \gamma(q_j z + c_j)(j = 1, 2, ..., d)$. Obviously, $p(\gamma' \gamma'_{c_j}, ..., \gamma^{(k)} \gamma_{c_j}^{(k)})$ has infinitely many zeros, and which contradict with (3.8).

Now let $F = G$. Then
$(f^\tau p(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j})^{(k)} = (g^\tau p(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j})^{(k)}$,
Integrating one time we have

$$(f^\tau p(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j})^{(k-1)} = (g^\tau p(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j})^{(k-1)} + \mu_{k-1},$$

where μ_{k-1} is a constant. If $\mu_{k-1} \neq 0$ using lemma 2.9 we say that $n \leq 2\Gamma_1 + 2km_2 + \lambda + \tau$, which contradict with the fact that $n > 2\Gamma_2 + 2km_2 + \lambda + \tau$ and $\Gamma_2 \geq \Gamma_1$. Hence $\mu_{k-1} = 0$. Now repeating the process upto k-times, we can established

$$f^\tau p(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j} = g^\tau p(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j}. \quad (3.9)$$

Let $h = \frac{f}{g}$. If h is a not a constant, then from last equation, we can say that f and g satisfy the algebraic equation, $R(f,g) = 0$ where,

$$R(\gamma_1, \gamma_2) = \gamma_1^{\tau} p(\gamma_1) \prod_{j=1}^{d} \gamma_1(q_j z + c_j)^{s_j} - \gamma_2^{\tau} p(\gamma_2) \prod_{j=1}^{d} \gamma_2(q_j z + c_j)^{s_j}.$$

If h be a constant, then substituting $f = gh$ into (3.9) we have,

$$(hg)^{\tau} p(hg) \prod_{j=1}^{d} (hg)(q_j z + c_j)^{s_j} = g^{\tau} p(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j},$$

,

$$g^{\tau} \prod_{j=1}^{d} g(q_j z+c_j)^{s_j} [a_n g^n (h^{n+\lambda+\tau} - 1) + a_{n-1} g^{n-1} (h^{n+\lambda+\tau-1} - 1) + \cdots + a_0 (h^{\lambda+\tau} - 1)] = 0, \tag{3.10}$$

where $a_n (\neq 0), a_{n-1}, \cdots, a_0$ are constants. Since g is transcendental entire function, we have $g^{\tau} \prod_{j=1}^{d} g(q_j z + c_j)^{s_j} \not\equiv 0$. Then from (3.10) we have,

$$[a_n g^n (h^{n+\lambda+\tau} - 1) + a_{n-1} g^{n-1} (h^{n+\lambda+\tau-1} - 1) + \cdots + a_0 (h^{\lambda+\tau} - 1)] = 0. \tag{3.11}$$

If $a_n \neq 0$ and $a_{n-1} = a_{n-2} = \cdots = a_0 = 0$, then from (3.11) we have $h^{n+\lambda+\tau} = 1$. If $a_n \neq 0$ and there exists $a_i \neq 0 (i \in \{0, 1, 2, \cdots, n-1\})$. Suppose that $h^{n+\lambda+\tau} \neq 1$. From (3.11), we have $T(r,g) = S(r,g)$ which is a contradiction with that g is a transcendental function. Then $h^{n+\lambda+\tau} = 1$. Similar to this discussion, we can see that $h^{\lambda+\tau+j} = 1$ when $a_j \neq 0$ for same $j = 0, 1, 2, \cdots, n$. Thus, we have $f = tg$ for a constant t such that $t^{\chi} = 1, \chi = GCD(\lambda_0 + \lambda + \tau, \lambda_1 + \lambda + \tau, ..., \lambda_n + \lambda + \tau)$.

Case II.

Let $l = 1$ and $H \not\equiv 0$. Using lemma 2.4 and (3.2) we can established from (3.1)

$$\begin{aligned}
(n+\lambda+\tau)T(r,f) &\leq T(r,F) - N_2(r,\frac{1}{F}) + N_{k+2}(r,\frac{1}{F_1})\\
&\leq N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) + \frac{1}{2}\overline{N}(r,\frac{1}{F})\\
&+ \frac{1}{2}\overline{N}(r,F) + N_{k+2}(r,\frac{1}{F_1}) + S(r,f) + S(r,g)\\
&\leq N_2(r,\frac{1}{G}) + \frac{1}{2}\overline{N}(r,\frac{1}{F}) + N_{k+2}(r,\frac{1}{F_1}) + S(r,f) + S(r,g)\\
&\leq N_{k+2}(r,\frac{1}{F_1}) + N_{k+2}(r,\frac{1}{G_1}) + \frac{1}{2}N_{k+1}(r,\frac{1}{F_1})\\
&+ S(r,f) + S(r,g)\\
&\leq N_{k+2}(r,\frac{1}{p(f)}) + N_{k+2}(r,\frac{1}{p(g)}) + \frac{1}{2}N_{k+1}(r,\frac{1}{p(f)})\\
&+ N_{k+2}(r,\frac{1}{f^\tau}) + N_{k+2}(r,\frac{1}{g^\tau}) + \frac{1}{2}N_{k+1}(r,\frac{1}{f^\tau})\\
&+ N_{k+2}(r,\frac{1}{\prod_{j=1}^{d} f(q_j z + c_j)^{s_j}}) + N_{k+2}(r,\frac{1}{\prod_{j=1}^{d} g(q_j z + c_j)^{s_j}})\\
&+ \frac{1}{2}N_{k+1}(r,\frac{1}{\prod_{j=1}^{d} f(q_j z + c_j)^{s_j}}) + S(r,f) + S(r,g)\\
&\leq \frac{1}{2}[3m_1 + (3k+5)m_2 + 3\lambda + 3\tau]T(r,f) + S(r,f)\\
&+ [m_1 + (k+2)m_2 + \lambda + \tau]T(r,g) + S(r,g).
\end{aligned}$$

Similarly we can show that

$$\begin{aligned}
(n+\lambda+\tau)T(r,g) &\leq \frac{1}{2}[3m_1 + (3k+5)m_2 + 3\lambda + 3\tau]T(r,g) + S(r,f)\\
&+ [m_1 + (k+2)m_2 + \lambda + \tau]T(r,f) + S(r,g),
\end{aligned}$$

we have from two inequalities,

$$(n - \frac{1}{2}(5m_1 + (5k+9)m_2 + 3\lambda + 3\tau))(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g),$$

which contradict the fact $n > \frac{1}{2}(\Gamma_1 + 4\Gamma_2 + 5km_2 + 3\lambda + 3\tau)$.

Now, let $H \equiv 0$, i.e $(\frac{F''}{F'} - \frac{2F'}{F-1}) - (\frac{G''}{G'} - \frac{2G'}{G-1}) = 0$.
After two times integration we have,

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.12}$$

where A, B are constants and $A \neq 0$. From (3.12) it is clear that F, G share the value 1 CM and then they share (1,2). Hence we have $n > 2\Gamma_2 + 2km_2 + \lambda + \tau$. Now we study the following cases:

Subcase I.

Let $B \neq 0$ and $A = B$. Then from (3.12) we get,

$$\frac{1}{F-1} = \frac{BG}{G-1}, \tag{3.13}$$

If $B = -1$, then from (3.13) $FG = 1$, from which, we get $f(z) = e^{\gamma(z)}$, and $g(z) = \kappa e^{-\gamma(z)}$, where κ is a constant satisfying $a_n^2 \kappa^{n+\lambda+\tau} = 1$ as in Case I. Now if $B \neq -1$, then from (3.10), we have, $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and then, $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, \frac{1}{F})$. Now from the second main theorem of Nevanlinna, we get using (2.1) and (2.2) that

$$\begin{aligned} T(r,G) &= \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{1+B}; G) + \overline{N}(r,G) + S(r,G) \\ &\leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r,G) + S(r,G) \\ &\leq N_{k+1}(r, \frac{1}{F_1}) + T(r,G) + N_{k+1}(r, \frac{1}{G_1}) - (n+\lambda+\tau)T(r,g) + S(r,g). \end{aligned}$$

This gives,

$$\begin{aligned} (n+\lambda+\tau)T(r,g) &\leq N_{k+1}(r, \frac{1}{F_1}) + N_{k+1}(r, \frac{1}{G_1}) + S(r,g) \\ &\leq N_{k+1}(r, \frac{1}{p(f)}) + N_{k+1}(r, \frac{1}{p(g)}) + N_{k+1}(r, \frac{1}{f^\tau}) \\ &+ N_{k+1}(r, \frac{1}{g^\tau}) + N_{k+1}(r, \frac{1}{\prod_{j=1}^{d} f(q_j z + c_j)^{s_j}}) \\ &+ N_{k+1}(r, \frac{1}{\prod_{j=1}^{d} g(q_j z + c_j)^{s_j}}) + S(r,f) + +S(r,g) \\ &\leq (m_1 + (k+1)m_2 + \lambda + \tau)(T(r,f) + T(r,g)) + S(r,g). \end{aligned}$$

We can show same result for $T(r,f)$ i.e

$$(n+\lambda+\tau)T(r,f) \leq (m_1 + (k+1)m_2 + \lambda)(T(r,f) + T(r,g)) + S(r,f),$$

Thus combining both we obtain

$$(n - 2m_1 - 2(k+1)m_2 - \lambda - \tau)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g),$$

a contradiction as $n > 2\Gamma_2 + 2km_2 + \lambda + \tau$.

Subcase II.

Let $A \neq 0$ and $B = 0$. Now from (3.12) we have $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, we have $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, \frac{1}{G})$ and $\overline{N}(r, 1-A; G) = \overline{N}(r, \frac{1}{F})$. Then by lemma 2.9, we have $n \leq 2\Gamma_1 + 2km_2 + \lambda + \tau$, which is a contradiction. Thus $A = 1$ and $F = G$, then the result follows from the proof of Case I.

Subcase III.

Let $B \neq 0$ and $A \neq B$. Then from (3.12), we obtain $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and therefore $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r, \frac{1}{F})$. Proceeding similarly as in Subcase I, we can get a contradiction.

Case III.

Let $l=0$ and $H\not\equiv 0$, we can established from (3.1) after using lemma 2.5 and (3.2)

$$\begin{aligned}(n+\lambda+\tau)T(r,f) &\leq N_2(r,\frac{1}{G})+N_2(r,F)+N_2(r,G)+2\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})\\ &+ 2\overline{N}(r,F)+\overline{N}(r,G)+N_{k+2}(r,\frac{1}{F_1})+S(r,f)+S(r,g)\\ &\leq N_{k+2}(r,\frac{1}{F_1})+N_{k+2}(r,\frac{1}{G_1})+2N_{k+1}(r,\frac{1}{F_1})+N_{k+1}(r,\frac{1}{G_1})\\ &+ S(r,f)+S(r,g)\\ &\leq N_{k+2}(r,\frac{1}{p(f)})+N_{k+2}(r,\frac{1}{p(g)})+2N_{k+1}(r,\frac{1}{p(f)})\\ &+ N_{k+1}(r,\frac{1}{p(g)})+N_{k+2}(r,\frac{1}{f^\tau})+N_{k+2}(r,\frac{1}{g^\tau})+2N_{k+1}(r,\frac{1}{f^\tau})\\ &+ N_{k+1}(r,\frac{1}{g^\tau})+N_{k+2}(r,\frac{1}{\prod_{j=1}^d f(q_jz+c_j)^{s_j}})\\ &+ N_{k+2}(r,\frac{1}{\prod_{j=1}^d g(q_jz+c_j)^{s_j}})+2N_{k+1}(r,\frac{1}{\prod_{j=1}^d f(q_jz+c_j)^{s_j}})\\ &+ N_{k+1}(r,\frac{1}{\prod_{j=1}^d g(q_jz+c_j)^{s_j}})+S(r,f)+S(r,g)\\ &\leq [3m_1+(3k+4)m_2+3\lambda+3\tau]T(r,f)\\ &+ [2m_1+(2k+3)m_2+2\lambda+2\tau]T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

Similarly it follows that

$$\begin{aligned}(n+\lambda+\tau)T(r,g) &\leq [3m_1+(3k+4)m_2+3\lambda+3\tau]T(r,g)\\ &+ [2m_1+(2k+3)m_2+2\lambda+2\tau]T(r,f)+S(r,f)+S(r,g).\end{aligned}$$

From the above two inequalities we have
$(n-5m_1-(5k+7)m_2-4\lambda-4\tau](T(r,f)+T(r,g))\leq S(r,f)+S(r,g)$,
which contradict with our assumption that $n>3\Gamma_1+2\Gamma_2+5km_2+4\lambda+4\tau$. Therefore $H=0$ and then proceeding in similar manner as Case II, we get the results. This complete the proof of the theorem. □

Theorem 3.3. *Let f and g be two transcendental entire functions of zero order and let $q\in\mathbb{C}\setminus\{0\}$, $c\in\mathbb{C}$. If $E_l(1;(f^\tau p(f)f(qz+c))^{(k)})=E_l(1;(g^\tau p(g)g(qz+c))^{(k)})$ and l,m,n,τ are integers satisfy one of the following conditions:*
(i) $l\geq 2; n>2\Gamma_2+2km_2+\tau+1$;
(ii) $l=1; n>\frac{1}{2}(\Gamma_1+4\Gamma_2+5km_2+3\tau+3)$;
(iii) $l=0; n>3\Gamma_1+2\Gamma_2+5km_2+4\tau+4$;
Then one of the following results holds:
(i) $f=tg$ for a constant t such that $t^\eta=1$;
(ii) f and g satisfy the algebraic equation $R(f,g)=0$, where
$R(\gamma_1,\gamma_2)=\gamma_1^\tau p(\gamma_1)\gamma_1(qz+c)-\gamma_2^\tau p(\gamma_2)\gamma_2(qz+c)$;
(iii) $f(z)=e^{\gamma(z)}$ and $g(z)=\kappa e^{-\gamma(z)}$, where κ is a complex constant satisfy $a_n^2\kappa^{n+\tau+1}=1$.

Proof. The theorem 3.3 is special case of theorem 3.2, where $\prod_{j=1}^{d} f(q_j z + c_j)^{s_j}$ is taken as $f(qz+c)$ i.e λ is equivalent to 1. $\square$

Remark 3.1. *The theorem 3.1 and theorem 3.2 may not hold for non-zero finite order transcendental entire function.*

Corollary 3.1. *We can obtain another interesting result by considering difference-differential polynomial function like as* $f^m(p(f)\prod_{j=1}^{d} f(q_j z + c_j)^{s_j})^{(k)}$.

Example 3.1. *Let* $p(z) = (z-1)^8(z+1)^8 z^{15}$, $f(z) = \sin(z)$ *and* $g(z) = \cos(z)$. *we assume* $d = 1$, $S_j = 1$, $k = 0$, $q_j = 1$,$m = 0$ *and* $c_j = 2\pi$ *for problem of theorem 3.2. Now it is clear* $n > 2\Gamma_2 + 2km_2 + \tau + 1$ *and* $(f^m p(f)\prod_{j=1}^{d} f(q_j z + c_j)^{s_j})^{(k)} = (g^m p(g)\prod_{j=1}^{d} g(q_j z + c_j)^{s_j})^{(k)}$. *Then* $(f^m p(f)\prod_{j=1}^{d} f(q_j z + c_j)^{s_j})^{(k)} = (g^m p(g)\prod_{j=1}^{d} g(q_j z + c_j)^{s_j})^{(k)}$ *share 1 CM. Hence it satisfy condition of theorem 3.2 and* f *and* g *satisfy the algebraic equation* $R(f,g) = 0$ *where* $R(\gamma_1,\gamma_2) = \gamma_1^m p(\gamma_1)\prod_{j=1}^{d}\gamma_1(q_j z + c_j)^{s_j} - \gamma_2^m p(\gamma_2)\prod_{j=1}^{d}\gamma_2(q_j z + c_j)^{s_j}$.

4. Open Problems

We can pose following problems from our results:
1. Can n be further reduced in theorem 3.2?
2. Is the theorem 3.2 would be valid for any non-constant meromorphic function?
3. Is it possible to discuss the problem of theorem 3.2 under the concept of weakly weighted sharing?
4. What is the extra condition which need to introduced to develop an uniqueness result for the problem of theorem 3.2 for a transcendental entire function of finite non-zero order?

5. Declarations

1. Conflict of Interest: The author of the manuscript declare that there is no conflict of interest for publication of the article.

2. Data Availability: The author of the manuscript declare that there is no statistical or survey data. The article is prepared on the investigation results of the articles which are include as reference of the manuscript. The paper is purely theoretical.

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