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SOME RESULTS ON NONLINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH POSITIVE CONSTANT COEFFICIENT

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Abstract. In this paper, we study the main results of existence and uniqueness of mild solutions of fractional Volterra integrodifferential equation involving Caputo fractional derivative of special class $n-1 < \alpha \leq n, n > 1$. Furthermore, the various type of dependency of mild solutions of the proposed problem have been studied such as dependence on given initial data, continuously depends on the functions involved in the right side of the problem and on real parameters. The result of existence and uniqueness is obtained with help of well known Banach contraction principle and the integral inequality established by B. G. Pachpatte which provides explicit bound on the unknown function. A suitable example is given to demonstrate the obtained results.

1. Introduction

In the present paper, we study existence, uniqueness and other properties of mild solutions of the following nonlinear Caputo fractional integro-differential equations with constant coefficient $\lambda \in (0,1)$ of the type:

$$D_*{}^{\alpha} y(t) = \lambda y(t) + f\left(t, y(t), \int_0^t K(s, y(s))ds\right), \tag{1}$$

for $t \in I = [0,b],\ n-1 < \alpha \leq n,\ n > 1$; with initial conditions:

$$y^{(j)}(0) = c_j,\ (j = 0, 1, 2, \ldots, n-1) \tag{2}$$

where $f : I \times X \times X \to X$, $K : I \times X \to X$ are continuous functions and c_j $(j = 0, 1, 2, \ldots, n-1)$ are given points in X.

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For the most of differential or integro-differential equations of fractional order, the solution is presented in terms of equivalent integral equation with singular kernel and few inequalities are there to study other properties of special version of such equations. Further, in case of singular kernel, there are several research papers in the literature using the fact that $(t-s)^{\alpha-1} \leq b^{\alpha},\ s \leq t \in [0,b]$ with $0 < \alpha < 1$. This is incorrect, in fact for $\alpha = \frac{1}{2}$ and the interval $[0,1]$ with $t = \frac{1}{2},\ s = \frac{1}{3}$, one can observe that

$$(t-s)^{\alpha-1} = (\frac{1}{2} - \frac{1}{3})^{\frac{1}{2}-1} = (\frac{3-2}{6})^{-\frac{1}{2}} = (\frac{1}{6})^{-\frac{1}{2}} = \sqrt{6} \nleq b^{\alpha} = 1^{\frac{1}{2}} = 1.$$

By keeping these in mind, authors considered a class of special equations where singularities are removed and we are free to use general integral inequalities to discuss the various properties of solutions. This study may be the new motivation towards the class of more general type.

Recently, several researchers have been studied the results such as existence, uniqueness and other properties of solutions for the nonlinear fractional equations involving various types of fractional derivatives by different techniques, see [2, 3, 5, 7, 8, 9, 10, 12, 13, 15, 16, 17, 21, 22, 23, 24, 26] and the detailed literature for fractional calculus can be found in [1, 4, 6, 14, 18, 20, 25].

The paper is organized as follows. In section 2, we present the preliminaries and hypotheses. Section 3, deal with existence and uniqueness by contraction principle. Section 4 devoted to the existence of at most one mild solution and estimates on mild solutions via inequality. In Section 5, we discuss results on continuous dependence of mild solutions on initial data, functions involved therein and parameters. Finally, in section 6, we present a suitable example to demonstrate the obtained results.

2. Preliminaries

Before proceeding to the statement of our main results, we shall setforth some preliminaries and hypotheses that will be used in our subsequent discussion.

Let X be a Banach space with norm $\|\cdot\|$ and $I = [0,b]$ denotes an interval of the real line $\mathbb{R}$. Let $B = C(I,X)$ denote the Banach space of continuous functions from I into X, endowed with the norm

$$\|y\|_B = \sup_{t\in I}\{\|y(t)\| : y \in B\}.$$

Definition 2.1. [20] *The Riemann-Liouville fractional integral (left-sided) of a function $h \in C[a,b]$ of order $\alpha \in \mathbf{R}_+ = (0,\infty)$ is defined by*

$$I_a^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\, ds,$$

where Γ is the Euler gamma function.

Definition 2.2. [20] *Let $n-1 < \alpha < n,\ n \in \mathbb{N}$. Then the expression*

$$D_a^{\alpha} h(t) = \frac{d^n}{dx^n}\left[I_a^{n-\alpha} h(t)\right],\ x \in [a,b]$$

is called the (left sided) Riemann Liouville derivative of h of order α whenever the expression on the right-hand side is defined.

Definition 2.3. [14] *Let $h \in C^n[a,b]$ and $n-1 < \alpha \leq n,\ n \in \mathbb{N}$. Then the expression*

$$\big(D^{\alpha_i}_{*a}\big)h(t) = I^{n-\alpha}_a h^{(n)}(t),\ x \in [a,b]$$

is called the (left sided) Caputo derivative of h of order α.

In the further discussion, we will denote I^α_a, D^α_a and D^α_{*0} as I^α, D^α and D^α_*.

Lemma 2.1. [11] *If the function $f = (f_1, \cdots, f_n) \in C^1$, then the initial value problem*

$$D^{\alpha_i}_* y_i(t) = f_i(t, y_1, \cdots, y_n),\ y_i^{(k)}(0) = c^i_k,\ i = 1,2,\cdots,n,\ k = 1,2,\cdots,m_i$$

where $m_i < \alpha_i \leq m_i + 1$ and $D^{\alpha_i}_*$ denotes Caputo fractional derivative, is equivalent to Volterra integral equations:

$$y_i(t) = \sum_{k=0}^{m_i} c^i_k \frac{t^k}{k!} + I^{\alpha_i} f_i(t, y_1, \cdots, y_n),\ 1 \leq i \leq n.$$

Lemma 2.2. *If the function $H = \lambda y(t) + f\Big(t, y(t), \int_0^t K(s, y(s))\Big) \in C^1[0,b]$, then the corresponding Volterra integral equation of the initial value problem $(1)-(2)$ is*

$$\begin{aligned} y(t) = \sum_{j=0}^{n-1} \frac{c_j}{j!}\ t^j &+ \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \int_0^s K(\tau, y(\tau)) d\tau\right) ds, \end{aligned} \tag{3}$$

which represents the mild solution of the problem $(1)-(2)$.

Proof. The proof is similar to Lemma 2.1 as proved in [11]. Hence, we omit the details. □

We require the following Lemma known as Pachpatte's inequality in our further discussion.

Lemma 2.3. [19] *Let $u(t), q_1(t)$ and $q(t)$ be real valued nonnegative continuous functions defined on $\mathbb{R}_+$ and $w(t)$ be a positive and nondecreasing continuous function on $\mathbb{R}_+$, for which the inequality*

$$u(t) \leq w(t) + \int_0^t q_1(s) \left[u(s) + \int_0^s q_2(\tau) u(\tau) d\tau\right] ds,$$

holds for $t \in \mathbb{R}_+$. Then

$$u(t) \leq w(t) \left[1 + \int_0^t q_1(s) exp\left(\int_0^s [q_1(\tau) + q_2(\tau)] d\tau\right) ds\right],$$

for $t \in \mathbb{R}_+$.

We list the following hypotheses for our convenience.

(H_1) There exists a function $p \in C(I, \mathbb{R}_+)$ such that

$$||f(t,x,y) - f(t,\bar{x},\bar{y})|| \leq p(t)\big[\, ||x-\bar{x}|| + ||y-\bar{y}||\, \big],$$

for $t \in J$ and $x, y, \bar{x}, \bar{y} \in X$.

(H_2) There exists $q \in C(I, \mathbb{R}_+)$ such that

$$\|K(t,x) - K(t,\bar{x})\| \leq q(t)\|x - \bar{x}\|,$$

for $t \in J$ and $x, \bar{x} \in X$.

(H_3) Assume that $d_1 = \sup_{t\in I} \|f(t, 0, \int_0^t K(s,0)ds)\|$ and $M = \sup_{t\in I} \sum_{j=0}^{n-1} \frac{\|c_j\|}{j!}\, t^j$.

3. Existence and Uniqueness

The following theorem deals with existence and uniqueness of mild solution of the problem $(1) - (2)$.

Theorem 3.1. *Assume that hypotheses* $(H_1) - (H_3)$ *hold. If*

$$\beta = \left[\frac{\lambda + P(1 + bQ)}{\Gamma(\alpha+1)}\right] b^{\alpha} < 1,$$

where $P = \max_{t\in I}\{p(t)\},\ Q = \max_{t\in I}\{q(t)\}$, *then the IVP problem* $(1) - (2)$ *has a unique mild solution on* $y \in B$.

Proof. We use the Banach contraction principle to prove existence and uniqueness of solution to the problem $(1) - (2)$. Let $E_r = \{y \in B : \|y\|_B \leq r\}$, where $r \geq \left[1 - \left(\frac{\lambda+P(1+bQ)}{\Gamma(\alpha+1)}\right) b^{\alpha}\right]^{-1} \left[M + \frac{d_1 b^{\alpha}}{\Gamma(\alpha+1)}\right]$ be closed and bounded set. Define an operator on the Banach space B by

$$\begin{aligned}(Ty)(t) = &\sum_{j=0}^{n-1} \frac{c_j}{j!}\, t^j + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \int_0^s K(\tau, y(\tau))d\tau\right) ds. \qquad (4)\end{aligned}$$

Firstly, we show that the operator T maps E_r into itself.

By using hypotheses, we have

$$\begin{aligned}&\|(Ty)(t)\| \\ &\leq \sum_{j=0}^{n-1} \frac{\|c_j\|}{j!}\, t^j + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s)\|\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f\left(s, y(s), \int_0^s K(\tau, y(\tau))d\tau\right)\|\, ds \\ &\leq M + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} r\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f\left(s, y(s), \int_0^s K(\tau, y(\tau))d\tau\right) - f\left(s, 0, \int_0^s K(\tau, 0)d\tau\right)\|\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f\left(s, 0, \int_0^s K(\tau, 0)d\tau\right)\|\, ds \\ &\leq M + \frac{\lambda r}{\Gamma(\alpha)} \frac{t^{\alpha}}{\alpha} + \frac{d_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \left[\|y(s)\| + \int_0^s q(\tau) \|y(s)\| d\tau \right] ds \\
&\leq M + \frac{\lambda r b^\alpha}{\Gamma(\alpha+1)} + \frac{d_1 b^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \left[r + \int_0^s q(\tau) r d\tau \right] ds \\
&\leq M + \left(\frac{\lambda r + d_1}{\Gamma(\alpha+1)} \right) b^\alpha + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} P \left[r + \int_0^s Q r d\tau \right] ds \\
&\leq M + \left(\frac{\lambda r + d_1}{\Gamma(\alpha+1)} \right) b^\alpha + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} P \left[r + Q r b \right] ds \\
&\leq M + \left(\frac{\lambda r + d_1}{\Gamma(\alpha+1)} \right) b^\alpha + \frac{P r \left[1 + Q b\right]}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha} \\
&\leq M + \left(\frac{\lambda r + d_1}{\Gamma(\alpha+1)} \right) b^\alpha + \frac{P r \left[1 + Q b\right]}{\Gamma(\alpha+1)} b^\alpha \\
&\leq M + \left(\frac{d_1 b^\alpha}{\Gamma(\alpha+1)} \right) + \left[\frac{\lambda + P\left[1 + Q b\right]}{\Gamma(\alpha+1)} \right] r b^\alpha \\
&\leq \left(1 - \left[\frac{\lambda + P\left[1 + Q b\right]}{\Gamma(\alpha+1)} \right] b^\alpha \right) r + \left[\frac{\lambda + P\left[1 + Q b\right]}{\Gamma(\alpha+1)} \right] r b^\alpha \\
&= r - \left[\frac{\lambda + P\left[1 + Q b\right]}{\Gamma(\alpha+1)} \right] r b^\alpha + \left[\frac{\lambda + P\left[1 + Q b\right]}{\Gamma(\alpha+1)} \right] r b^\alpha \\
&= r
\end{aligned}
$$

Thus,

$$
\|(Ty)\|_B \leq r. \tag{5}
$$

The equation (5) shows that the operator T maps E_r into itself.

Now, for every $x, y \in E_r$ and for $t \in I$, we obtain

$$
\begin{aligned}
&\|(Tx)(t) - (Ty)(t)\| \\
&\quad \leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s) - y(s)\| \, ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| f\left(s, x(s), \int_0^s K(\tau, x(\tau)) d\tau\right) \\
&\qquad - f\left(s, y(s), \int_0^s K(\tau, y(\tau)) d\tau\right) \| \, ds \\
&\quad \leq \frac{\lambda}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha} \|x - y\|_B \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \left[\|x(s) - y(s)\| + \int_0^s q(\tau) \|x(\tau) - y(\tau)\| d\tau \right] ds \\
&\quad \leq \frac{\lambda}{\Gamma(\alpha+1)} b^\alpha \|x - y\|_B \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} P \left[\|x - y\|_B + \int_0^s Q \|x - y\|_B d\tau \right] ds \\
&\quad \leq \left(\frac{\lambda}{\Gamma(\alpha+1)} b^\alpha \right) \|x - y\|_B + \frac{P(1 + Q b)}{\Gamma(\alpha+1)} b^\alpha \|x - y\|_B
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{\lambda + P(1+Qb)}{\Gamma(\alpha+1)} b^{\alpha}\right) \|x-y\|_B \\
&\leq \beta \|x-y\|_B.
\end{aligned}
$$

Hence, we have

$$\|(Tx)-(Ty)\|_B \leq \beta \|x-y\|_B,$$

where $0 < \beta < 1$. This proves that the operator T is a contraction on the complete metric space B. Therefore, by Banach fixed point theorem, the operator T has a unique fixed point in the space B and this is the required unique mild solution of the IVP $(1)-(2)$ on I. □

4. Estimates on Mild Solutions

The following Theorem shows uniqueness of mild solutions to the IVP $(1)-(2)$ without the existence part.

Theorem 4.2. *Suppose that the hypotheses $(H_1)-(H_2)$ hold. Then the IVP problem $(1)-(2)$ has at most one mild solution on I.*

Proof. Let $x(t)$ and $y(t)$ be two mild solutions of the problem $(1)-(2)$ and $u(t) = \|x(t)-y(t)\|,\ t \in I$. Now by using hypotheses, we have

$$
\begin{aligned}
u(t) &= \|x(t)-y(t)\| \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s)-y(s)\|\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \left[\|x(s)-y(s)\| + \int_0^s q(\tau)\|x(\tau)-y(\tau)\| d\tau \right] ds \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \left[u(s) + \int_0^s q(\tau) u(\tau) d\tau \right] ds \\
&= \int_0^t \left(\frac{\lambda + p(s)}{\Gamma(\alpha)} \right) (t-s)^{\alpha-1} \left[u(s) + \int_0^s q(\tau) u(\tau) d\tau \right] ds.
\end{aligned}
$$

Thus,

$$u(t) < \epsilon + \int_0^t \left(\frac{\lambda + p(s)}{\Gamma(\alpha)} \right) (b-s)^{\alpha-1} \left[u(s) + \int_0^s q(\tau) u(\tau) d\tau \right] ds, \tag{6}$$

where $\epsilon > 0$.

By applying Pachpatte's inequality to the inequality (6) with

$$w(t) = \epsilon,\ q_1(s) = \left(\frac{\lambda + p(s)}{\Gamma(\alpha)} \right) (b-s)^{\alpha-1},\ q_2(\tau) = q(\tau),$$

we obtain,

$$
\begin{aligned}
u(t) \leq \epsilon \Big[1 + \int_0^t & \left(\frac{\lambda + p(s)}{\Gamma(\alpha)} \right) (b-s)^{\alpha-1} \\
& \times \exp\left\{ \int_0^s \left[\left(\frac{\lambda + p(\tau)}{\Gamma(\alpha)} \right) (b-\tau)^{\alpha-1} + q(\tau) \right] d\tau \right\} ds \Big]
\end{aligned}
$$

$$\leq \epsilon\Big[1 + b^{\alpha}\Big(\frac{\lambda + P}{\Gamma(\alpha+1)}\Big)\exp\Big(\frac{\lambda + P}{\Gamma(\alpha+1)}b^{\alpha} + Qb\Big)\Big]$$

Thus,

$$u(t)\Big[1 + b^{\alpha}\Big(\frac{\lambda + P}{\Gamma(\alpha+1)}\Big)\exp\Big(\frac{\lambda + P}{\Gamma(\alpha+1)}b^{\alpha} + Qb\Big)\Big]^{-1} \leq \epsilon, \tag{7}$$

for every $\epsilon > 0$. This shows that every non-negative value is less than every positive real number. Therefore it is possible only if

$$u(t)\Big[1 + b^{\alpha}\Big(\frac{\lambda + P}{\Gamma(\alpha+1)}\Big)\exp\Big(\frac{\lambda + P}{\Gamma(\alpha+1)}b^{\alpha} + Qb\Big)\Big]^{-1} = 0 \Rightarrow u(t) = 0,$$

since

$$\Big[1 + b^{\alpha}\Big(\frac{\lambda + P}{\Gamma(\alpha+1)}\Big)\exp\Big(\frac{\lambda + P}{\Gamma(\alpha+1)}b^{\alpha} + Qb\Big)\Big] \neq 0.$$

Therefore $x(t) = y(t)$, which proves that there exists at most one mild solution. □

First, we prove the following result concerning the estimates on the mild solution of the problem $(1) - (2)$ and shows that the boundedness of mild solution.

Theorem 4.3. *Suppose that the hypotheses* $(H_1) - (H_3)$ *hold. If* $y(t),\ t \in I$ *is a mild solution of the problem* $(1) - (2)$, *then*

$$\|y\|_B \leq \left(M + \frac{d_1 b^{\alpha}}{\Gamma(\alpha+1)}\right)\Big[1 + \Big(\frac{\lambda + P}{\Gamma(\alpha+1)}\Big)b^{\alpha}\exp\Big(\frac{\lambda + P}{\Gamma(\alpha+1)}b^{\alpha} + Qb\Big)\Big].$$

Proof. By using the fact that the mild solution $y(t)$ of the problem $(1)-(2)$ satisfies the equivalent equation (1) and the hypotheses, we have

$$\begin{aligned}
\|y(t)\| &\leq \sum_{j=0}^{n-1}\frac{\|c_j\|}{j!}\ t^j + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y(s)\|\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big\|f\Big(s, y(s), \int_0^s K(\tau, y(\tau))d\tau\Big)\Big\|\,ds \\
&\leq M + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y(s)\|\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big\|f\Big(s, y(s), \int_0^s K(\tau, y(\tau))d\tau\Big) \\
&\quad - f\Big(s, 0, \int_0^s K(\tau, 0)d\tau\Big)\Big\|\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big\|f\Big(s, 0, \int_0^s K(\tau, 0)d\tau\Big)\Big\|\,ds \\
&\leq M + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}d_1 ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y(s)\|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\left[\|y(s)\| + \int_0^s q(\tau)\|y(\tau)\|d\tau\right]ds \\
&\leq M + \frac{d_1 t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y(s)\|ds
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\left[\|y(s)\|+\int_0^s q(\tau)\|y(\tau)\|d\tau\right]ds$$

which implies

$$\begin{aligned}\|y(t)\| \leq M + &\frac{d_1 t^\alpha}{\Gamma(\alpha+1)}\\ &+\int_0^t\left(\frac{\lambda+p(s)}{\Gamma(\alpha)}\right)(b-s)^{\alpha-1}\left[\|y(s)\|+\int_0^s q(\tau)\|y(\tau)\|d\tau\right]ds \end{aligned}\tag{8}$$

Hence, by an application of Lemma 2.3 to (8) with

$$u(t)=\|y(t)\|,\ w(t)=M+\frac{d_1t^\alpha}{\Gamma(\alpha+1)}, q_1(s)=\frac{\lambda+p(s)}{\Gamma(\alpha)}(b-s)^{\alpha-1}, q_2(\tau)=q(\tau),$$

we obtain

$$\begin{aligned}\|y(t)\| &\leq \left(M+\frac{d_1t^\alpha}{\Gamma(\alpha+1)}\right)\left[1+\int_0^t\left(\frac{\lambda+p(s)}{\Gamma(\alpha)}\right)(b-s)^{\alpha-1}\right.\\ &\qquad\left.\times\exp\left(\int_0^s\left(\frac{\lambda+p(\tau)}{\Gamma(\alpha)}\right)(b-\tau)^{\alpha-1}+q(\tau)\right)ds\right]\\ &\leq \left(M+\frac{d_1t^\alpha}{\Gamma(\alpha+1)}\right)\left[1+b^\alpha\Big(\frac{\lambda+P}{\Gamma(\alpha+1)}\Big)\exp\Big(\frac{\lambda+P}{\Gamma(\alpha+1)}b^\alpha+Qb\Big)\right]\\ &\leq \left(M+\frac{d_1b^\alpha}{\Gamma(\alpha+1)}\right)\left[1+b^\alpha\Big(\frac{\lambda+P}{\Gamma(\alpha+1)}\Big)\exp\Big(\frac{\lambda+P}{\Gamma(\alpha+1)}b^\alpha+Qb\Big)\right], t\in I.\end{aligned}\tag{9}$$

Hence, we have

$$\|y\|_B \leq \left(M+\frac{d_1b^\alpha}{\Gamma(\alpha+1)}\right)\left[1+b^\alpha\Big(\frac{\lambda+P}{\Gamma(\alpha+1)}\Big)\exp\Big(\frac{\lambda+P}{\Gamma(\alpha+1)}b^\alpha+Qb\Big)\right],\tag{10}$$

which implies the boundedness of all mild solutions of the problem $(1)-(2)$. □

5. Continuous Dependence

In this section, we shall deal with continuous dependence of the problem $(1)-(2)$ on the initial data, functions induced therein and also on parameters.

5.1. **Dependence on initial data.** We first discuss dependence of mild solution on given initial data.

Theorem 5.4. *Suppose that the hypotheses* $(H_1)-(H_2)$ *hold. If* $y(t)$ *and* $z(t)$ *are mild solutions of* (1) *with initial data*

$$y^{(j)}(0)=c_j,\ (j=0,1,2,\ldots,n-1)\tag{11}$$

and

$$z^{(j)}(0)=d_j,\ (j=0,1,2,\ldots,n-1)\tag{12}$$

respectively, then

$$\|y-z\|_B\leq\sum_{j=0}^{n-1}\frac{\|c_j-d_j\|}{j!}b^j\Big[1+b^\alpha\Big(\frac{\lambda+P}{\Gamma(\alpha+1)}\Big)\exp\Big(\frac{\lambda+P}{\Gamma(\alpha+1)}b^\alpha+Qb\Big)\Big].\tag{13}$$

Proof. By using the fact that $y(t)$ and $z(t)$ are mild solutions of (1). Then by the hypotheses, we have

$$\begin{aligned}
&\|y(t)-z(t)\| \\
&\le \sum_{j=0}^{n-1} \frac{\|c_j-d_j\|}{j!} t^j + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|y(s)-z(s)\|\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \left[\|y(s)-z(s)\| + \int_0^s q(\tau)\|y(\tau)-z(\tau)\|d\tau\right] ds \\
&\le \sum_{j=0}^{n-1} \frac{\|c_j-d_j\|}{j!} t^j + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \left[u(s) + \int_0^s q(\tau)u(\tau)d\tau\right] ds \\
&\le \sum_{j=0}^{n-1} \frac{\|c_j-d_j\|}{j!} t^j + \int_0^t \left(\frac{\lambda+p(s)}{\Gamma(\alpha)}\right)(b-s)^{\alpha-1} \left[u(s) + \int_0^s q(\tau)u(\tau)d\tau\right] ds. \qquad (14)
\end{aligned}$$

Now, on application of Lemma 2.3 to (14), we obtain

$$\begin{aligned}
\|y(t)-z(t)\| \le \sum_{j=0}^{n-1} \frac{\|c_j-d_j\|}{j!} t^j \Big[1 + \int_0^t \left(\frac{\lambda+p(s)}{\Gamma(\alpha)}\right)(b-s)^{\alpha-1} \\
\times \exp\left(\int_0^s \left(\frac{\lambda+p(\tau)}{\Gamma(\alpha)}\right)(b-\tau)^{\alpha-1} + q(\tau)\right) ds\Big], \qquad (15)
\end{aligned}$$

implies that

$$\|y-z\|_B \le \sum_{j=0}^{n-1} \frac{\|c_j-d_j\|}{j!} b^j \Big[1 + b^\alpha\Big(\frac{\lambda+P}{\Gamma(\alpha+1)}\Big) \exp\Big(\frac{\lambda+P}{\Gamma(\alpha+1)} b^\alpha + Qb\Big)\Big]. \qquad (16)$$

The term $\sum_{j=0}^{n-1} \frac{\|c_j-d_j\|}{j!} b^j$ present from the right hand side of (16) depends continuously on initial conditions $c_j,\ d_j,\ (j = 0, 1, 2, \ldots, n-1)$. Therefore, the mild solutions of equation (1) are continuously depends on given initial data. □

5.2. **Dependence on functions.** Consider the problem $(1)-(2)$ and the corresponding problem

$$D_*{}^{\alpha} z(t) = \lambda z(t) + \bar{f}\left(t, z(t), \int_0^t \bar{K}(s, z(s))ds\right), \qquad (17)$$

for $t \in I := [0, b],\ b > 0,\ n-1 < \alpha \le n,\ n > 1, \lambda \in (0,1)$ with condition (2), where $\bar{f}$ and $\bar{K}$ are defined as f and K.

The following Theorem deals with the continuous dependence of mild solutions of the problem $(1)-(2)$ on the functions involved therein.

Theorem 5.5. *Suppose that the hypotheses* $(H_1)-(H_2)$ *hold. Furthermore, suppose that*

$$\|f\left(t, z(t), \int_0^t K(s, z(s))ds\right) - \bar{f}\left(t, z(t), \int_0^t \bar{K}(s, z(s))ds\right)\| < \epsilon,$$

where $\epsilon > 0$, is an arbitrary small constant and $z(t)$ is a mild solution of the problem (17) *with* (2). *Then the mild solution $y(t),\ t \in I$ of the problem* $(1) - (2)$ *depends continuously on the functions involved in the right side of the equation* (1).

Proof. Let $y(t)$ and $z(t)$ be mild solutions of the problem $(1) - (2)$ and (17) with (2) respectively. Then, by hypotheses, we have

$$
\begin{aligned}
&\|y(t) - z(t)\| \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s) - z(s)\|\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f\left(s, y(s), \int_0^s K(\tau, y(\tau)) d\tau\right) \\
&\qquad - \bar{f}\left(s, z(s), \int_0^s \bar{K}(\tau, z(\tau)) d\tau\right)\| ds \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s) - z(s)\|\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f\left(s, y(s), \int_0^s K(\tau, y(\tau)) d\tau\right) \\
&\qquad - f\left(s, z(s), \int_0^s K(\tau, z(\tau)) d\tau\right)\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f\left(s, z(s), \int_0^s K(\tau, z(\tau)) d\tau\right) \\
&\qquad - \bar{f}\left(s, z(s), \int_0^s \bar{K}(\tau, z(\tau)) d\tau\right)\| ds \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s) - z(s)\|\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \left[\|y(s) - z(s)\| + \int_0^s q(\tau) \|y(\tau) - z(\tau)\| d\tau\right] ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \epsilon ds \\
&\leq \frac{\epsilon}{\Gamma(\alpha+1)} t^\alpha + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s) - z(s)\|\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \left[\|y(s) - z(s)\| + \int_0^s q(\tau) \|y(\tau) - z(\tau)\| d\tau\right] ds \\
&\leq \frac{\epsilon}{\Gamma(\alpha+1)} t^\alpha \\
&\quad + \int_0^t \left(\frac{\lambda + p(s)}{\Gamma(\alpha)}\right) (b-s)^{\alpha-1} \left[\|y(s) - z(s)\| + \int_0^s q(\tau) \|y(\tau) - z(\tau)\| d\tau\right] ds
\end{aligned} \tag{18}
$$

Therefore, on application of Lemma 2.3 to (18), we get

$$\begin{aligned}\|y(t)-z(t)\| &\leq \frac{\epsilon t^{\alpha}}{\Gamma(\alpha+1)}\Big[1+\int_0^t\left(\frac{\lambda+p(s)}{\Gamma(\alpha)}(b-s)^{\alpha-1}\right)\\&\qquad\times\exp\left[\int_0^s\left\{\left(\frac{\lambda+p(s)}{\Gamma(\alpha)}\right)(b-s)^{\alpha-1}+q(\tau)\right\}d\tau\right]ds\Big]\\&\leq \frac{\epsilon b^{\alpha}}{\Gamma(\alpha+1)}\Big[1+b^{\alpha}\Big(\frac{\lambda+P}{\Gamma(\alpha+1)}\Big)\exp\Big(\frac{\lambda+P}{\Gamma(\alpha+1)}b^{\alpha}+Qb\Big)\Big], t\in I.\end{aligned}$$

This gives

$$\|y-z\|_B \leq \frac{\epsilon b^{\alpha}}{\Gamma(\alpha+1)}\Big[1+b^{\alpha}\Big(\frac{\lambda+P}{\Gamma(\alpha+1)}\Big)\exp\Big(\frac{\lambda+P}{\Gamma(\alpha+1)}b^{\alpha}+Qb\Big)\Big]. \tag{19}$$

From (19), it follows that the mild solutions of problem $(1)-(2)$ and (17) with (2) are close to each other if f is near to $\bar{f}$. Thus, the mild solutions of the problem $(1)-(2)$ depends continuously on the functions involved in the right side of the problem (1). □

5.3. **Dependence on Parameters.** We next consider the following problem

$$D_*{}^{\alpha}y(t)=\lambda y(t)+F\left(t,y(t),\int_0^t K(s,y(s))ds,\mu_1\right), \tag{20}$$

for $t\in I:=[0,b],\ b>0,\ n-1<\alpha\leq n,\ n>1, \lambda\in(0,1)$ with condition (2) and

$$D_*{}^{\alpha}y(t)=\lambda y(t)+F\left(t,y(t),\int_0^t K(s,y(s))ds,\mu_2\right), \tag{21}$$

for $t\in I:=[0,b],\ b>0,\ n-1<\alpha\leq n,\ n>1, \lambda\in(0,1)$ with condition (2), where $F\in C(I\times X\times X\times\mathbb{R},X),\ K\in C(I,X)$ and constants μ_1 and μ_2 are real parameters.

The following Theorem shows that the dependency of mild solutions of the problem (20) with condition (2) and (21) with condition (2) on parameters.

Theorem 5.6. *Assume that* (H_2) *holds and the function* F *satisfying the conditions*

$$\begin{aligned}&\|F\left(t,y(t),z(t),\mu_1\right)-F\left(t,\bar{y}(t),\bar{z}(t),\mu_1\right)\|\\&\leq p_1(t)[\|y(t)-\bar{y}(t)\|+\|z(t)-\bar{z}(t)\|],\end{aligned} \tag{22}$$

and

$$\|F\left(t,y(t),z(t),\mu_1\right)-F\left(t,y(t),z(t),\mu_2\right)\|\leq p_2(t)|\mu_1-\mu_2|, \tag{23}$$

where $p_1,p_2\in C(I,\mathbb{R}_+)$. *Let* $y(t)$ *and* $z(t)$ *be the mild solutions of the problem* (20) *with condition* (2) *and* (21) *with condition* (2). *Then*

$$\|y-z\|_B\leq \frac{|\mu_1-\mu_2|\bar{P}_2}{\Gamma(\alpha+1)}b^{\alpha}\Big[1+b^{\alpha}\Big(\frac{\lambda+\bar{P}_1}{\Gamma(\alpha+1)}\Big)\exp\Big(\frac{\lambda+\bar{P}_1}{\Gamma(\alpha+1)}b^{\alpha}+Qb\Big)\Big].$$

Proof. From the hypotheses, it follows that

$$\|y(t)-z(t)\|$$

$$
\begin{aligned}
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s)-z(s)\|\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Bigg\| F\Bigg(s, y(s), \int_0^s K(\tau, y(\tau)) d\tau, \mu_1\Bigg) \\
&\qquad - F\Bigg(s, z(s), \int_0^s K(\tau, z(\tau)) d\tau, \mu_2\Bigg) \Bigg\| ds \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s)-z(s)\|\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Bigg\| F\Bigg(s, y(s), \int_0^s K(\tau, y(\tau)) d\tau, \mu_1\Bigg) \\
&\qquad - F\Bigg(s, z(s), \int_0^s K(\tau, z(\tau)) d\tau, \mu_1\Bigg) \Bigg\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Bigg\| F\Bigg(s, z(s), \int_0^s K(\tau, z(\tau)) d\tau, \mu_1\Bigg) \\
&\qquad - F\Bigg(s, z(s), \int_0^s K(\tau, z(\tau)) d\tau, \mu_2\Bigg) \Bigg\| ds \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s)-z(s)\|\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_1(s) \left[\|y(s)-z(s)\| + \int_0^s q(\tau) \|y(\tau)-z(\tau)\| d\tau \right] ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_2(s) |\mu_1 - \mu_2| ds \\
&\leq \frac{|\mu_1-\mu_2|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_2(s) ds \\
&\quad + \int_0^t \left(\frac{\lambda + p_1(s)}{\Gamma(\alpha)} \right) (b-s)^{\alpha-1} \left[\|y(s)-z(s)\| + \int_0^s q(\tau) \|y(\tau)-z(\tau)\| d\tau \right] ds \\
&\leq \left[\frac{|\mu_1-\mu_2| \bar{P}_2}{\Gamma(\alpha+1)} t^\alpha \right] \\
&\quad + \int_0^t \left(\frac{\lambda + \bar{P}_1}{\Gamma(\alpha)} \right) (b-s)^{\alpha-1} \left[\|y(s)-z(s)\| + \int_0^s Q \|y(\tau)-z(\tau)\| d\tau \right] ds, \qquad (24)
\end{aligned}
$$

where $\bar{P}_1 = \sup_{t\in I}\{p_1(t)\}$, $\bar{P}_2 = \sup_{t\in I}\{p_2(t)\}$.
Now, an application of Lemma 2.3 to (24), we obtain

$$
\begin{aligned}
\|y(t)-z(t)\| \leq \frac{|\mu_1-\mu_2| \bar{P}_2}{\Gamma(\alpha+1)} t^\alpha \Big[1 + \int_0^t \left(\frac{\lambda+\bar{P}_1}{\Gamma(\alpha)} \right) (b-s)^{\alpha-1} \\
\times \exp\left[\int_0^s \left\{ \left(\frac{\lambda+\bar{P}_1}{\Gamma(\alpha)} \right) (b-\tau)^{\alpha-1} + Q \right\} d\tau \right] ds \Big], t \in I
\end{aligned}
$$

which gives

$$\|y-z\|_B \leq \frac{|\mu_1-\mu_2|\bar{P}_2}{\Gamma(\alpha+1)} b^\alpha \Big[1+b^\alpha\Big(\frac{\lambda+\bar{P}_1}{\Gamma(\alpha+1)}\Big)\exp\Big(\frac{\lambda+\bar{P}_1}{\Gamma(\alpha+1)}b^\alpha+Qb\Big)\Big]. \qquad (25)$$

This shows the dependence of mild solutions of the problems (20) with condition (2) and (21) with condition (2)on parameters μ_1 and μ_2. □

6. Example

Here, we illustrate our results through the following example by taking the fractional order $\alpha,\ 1<\alpha\leq 2$.
Consider the following fractional integro-differential equation

$$D_*{}^{3/2}y(t) = \frac{1}{10}y(t) + \frac{e^{-t}}{(8+e^t)}\left[\frac{|y(t)|}{1+|y(t)|}\right] + \frac{1}{9}\int_0^t \frac{e^{-s}}{(2+s)^2}y(s)ds, \qquad (26)$$

for $t\in I:=[0,1],\ 1<\alpha\leq 2,\ \lambda\in(0,1)$ with conditions:

$$y(0)=c_1,\ y'(0)=c_2. \qquad (27)$$

Problem $(26)-(27)$ is of the form $(1)-(2)$ with $\alpha=\frac{3}{2},\ \lambda=\frac{1}{10}$,

$$f\left(t,y(t),\int_0^t K(s,y(s))ds\right) = \frac{e^{-t}}{(9+e^t)}\left[\frac{|y(t)|}{1+|y(t)|}\right] + \frac{1}{9}\int_0^t \frac{e^{-s}}{(2+s)^2}y(s)ds$$

Clearly, for each $y,z\in X=\mathbb{R}$ and $t\in[0,1]$,

$$||f(t,y,z)-f(t,\bar{y},\bar{z})|| \leq \frac{1}{9}\Big[\ ||y-\bar{y}||+||z-\bar{z}||\Big].$$

Also, we have

$$\|K(t,y)-K(t,\bar{y})\| \leq \frac{1}{9}\|y-\bar{y}\|.$$

Hence all hypotheses $(H_1)-(H_3)$ are satisfied with $\lambda=\frac{1}{10},\ P=\frac{1}{9},\ Q=\frac{1}{9}$.
Thus, we have

$$\beta = \left[\frac{\lambda+P(1+bQ)}{\Gamma(\alpha+1)}\right]b^\alpha = \left[\frac{\frac{1}{10}+\frac{1}{9}\left(1+\frac{1}{9}\right)}{\Gamma\left(\frac{5}{2}\right)}\right] = 0.0845 < 1.$$

It follows from Theorem 3.1 that the problem $(26)-(27)$ has a unique mild solution in $C([0,1],\mathbb{R})$.

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