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# SOLVABILITY OF A FUNCTIONAL DIFFERENTIAL EQUATION WITH INTERNAL NONLOCAL INTEGRO-DIFFERENTIAL CONDITION

AHMED M. A. EL-SAYED, MOAMEN O. RADWAN, HANNA R. EBEAD

Abstract. Here, we study a nonlocal problem of a functional differential equation subject to an internal integro-differential condition. The existence and uniqueness of the solution will be proved. The Hyers-Ulam stability will be applied. The continuous dependence of the unique solution on some factors will be examined. Special cases and examples will be given.

## 1. Introduction

Functional differential equations are essential to understanding complex processes in a variety of scientific and practical domains, such as engineering, physics, biology, and economics [6, 22, 29]. Several authors have studied delay functional differential equations with parameters, which provide a crucial mathematical foundation for modeling real-world processes (see [14, 16, 20, 25, 28]). These equations become more complex when an integro-differential condition is applied, leading to what are commonly known as nonlocal problems which have been explored by various authors (see [1, 10, 15, 24, 26]).

Assuring the credibility of these models requires integrating the concepts of Hyers-Ulam stability and continuous dependence. This integration is vital for evaluating how the models respond to slight disturbances, shedding light on their robustness and reliability. Hyers-Ulam stability, when applied to the problem specifically, evaluates the model's robustness to disturbances, while continuous dependency examines the effect on its parameters (see [5, 12, 18, 19, 23, 27]).

Many techniques, such as operator theory and fixed point theorems, have been developed to analyze the solvability of such problems. One method is to consider the problem as a fixed-point problem and use the Schauder fixed point theorem

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to show that a solution exists. This method has been established by numerous amounts of publications and monographs (we refer to [3, 4, 7, 8, 21]).

Additional investigations on the existence of solutions are provided in [11], where authors explored a constrained problem of a fractional functional integro-differential equation. Their research included a detailed analysis of the solution's solvability, Hyers-Ulam stability, and continuous dependence on particular factors.

Motivated by the above mentioned outcomes, We are exploring the solvability of the nonlocal problem of a delay functional differential equation with parameter

$$\frac{dx(t)}{dt} = f\left(t, \lambda \frac{d}{dt} x\big(\phi(t)\big)\right), \ a.e.\ t \in (0, T], \tag{1}$$

subject to the internal integro-differential condition,

$$x(\tau) = x_0 + \int_0^{T-\tau} g\left(s, x(s), \frac{dx(s)}{ds}\right) ds, \tag{2}$$

where $\lambda > 0$ and $\tau \in (0, T)$.

In this paper, we focus on analyzing the existence and uniqueness of solutions of the problem (1)-(2) by converting it into a fixed point problem and employing the Schauder fixed point theorem to prove the solvability of it. Our research includes investigating the Hyers-Ulam stability of the problem. We also explore the continuous dependence of the unique solution on the initial data $x_0$, the functions $g$ and $f$, and parameter $\lambda$. Finally, we will illustrate the relevance of our conclusions through special cases and examples.

## 2. Existence of Solutions

In this section, we will justify the existence of at least one absolutely continuous solution $x \in AC[0, T]$ of (1)-(2). To achieve this purpose, we require the following key assumptions:

- $(i)$ $f : [0, T] \times R \to R$ fulfills the Carathéodory condition, being measurable in $t \in [0, T]$ for all $x \in R$ and continuous in $x \in R$ for almost all $t \in [0, T]$.
- $(ii)$ There exist an integrable function $a \in L^1[0, T]$ and a positive constant $b$ such that
$$|f(t, x)| \le |a(t)| + b|x|.$$
- $(iii)$ $\phi : [0, T] \to [0, T]$ is continuous and increasing function, satisfying $\phi(t) \le t$.
- $(iv)$ $b\lambda < 1$.
- $(v)$ $g : [0, T] \times R \times R \to R$ fulfills the Carathéodory condition, being measurable in $t \in [0, T]$ for all $x, y \in R$ and continuous in $x, y \in R$ for almost all $t \in [0, T]$.
- $(vi)$ There exist an integrable function $h \in L^1[0, T]$ and a positive constant $L$ such that
$$|g(t, x, y)| \le |h(t)| + L\big(|x| + |y|\big).$$
- $(vii)$ $LT < 1$.

The following lemma demonstrates the equivalence between the problem (1)-(2) and its corresponding problem (3)-(4).

**Lemma 2.1.** *The solution to the problem* (1)-(2)*, if it exists, is given by the integral equation*

$$x(t) = x_0 + \int_0^{T-\tau} g\big(s, x(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^t y(s)ds, \ t \in [0, T], \tag{3}$$

*where* $y(t)$ *satisfies the functional equation*

$$y(t) = f\big(t, \lambda\phi'(t)y(\phi(t))\big), \ t \in [0, T]. \tag{4}$$

*Proof.* Assuming $x$ is a solution of (1)-(2) and $\frac{dx(t)}{dt} = y \in L^1[0, T]$, then

$$x(t) = x(0) + \int_0^t y(s)ds, \tag{5}$$

at $t = \tau$, we have

$$x(\tau) = x(0) + \int_0^{\tau} y(s)ds,$$

substituting in (2), we get

$$x(0) = x_0 + \int_0^{T-\tau} g\big(s, x(s), y(s)\big)ds - \int_0^{\tau} y(s)ds,$$

using (5), we get (3)

$$x(t) = x_0 + \int_0^{T-\tau} g\big(s, x(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^t y(s)ds \in AC[0, T],$$

and

$$x(\phi(t)) = x_0 + \int_0^{T-\tau} g\big(s, x(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^{\phi(t)} y(s)ds,$$

then we obtain

$$\frac{d}{dt}x(\phi(t)) = \phi'(t)y(\phi(t)). \tag{6}$$

Substituting from (6) into (1), we obtain (4)

$$y(t) = f\big(t, \lambda\phi'(t)y(\phi(t))\big), \ t \in [0, T].$$

Additionally, we can return to equations to (1)-(2) by differentiating (3) and applying (4) and (6) in the following manner.

$$\begin{aligned}\frac{dx(t)}{dt} &= y(t), \ a.e. \ t \in (0, T]\\ &= f\big(t, \lambda\phi'(t)y(\phi(t))\big)\\ &= f\Big(t, \lambda\frac{d}{dt}x\big(\phi(t)\big)\Big),\end{aligned}$$

The condition (2) is met by substituting $t = \tau$ and $y = \frac{dx(t)}{dt}$ into (3). □

The following theorem establishes the existence of at least one integrable solution to the functional equation (4).

**Theorem 2.1.** *If the assumptions* $(i) - (iv)$ *are valid, then* (4) *has at least one integrable solution* $y \in L^1[0, T]$*.*

*Proof.* Define the set $Q_{r_1}$ and the operator $F_1$ correlated with (4) by

$$Q_{r_1} := \{y \in R : ||y||_{L_1} \leq r_1\} \subset L^1[0,T], \text{where } r_1 = \frac{||a||_{L^1}}{1-b\lambda},$$

$$F_1 y(t) = f\big(t, \lambda\phi'(t)y(\phi(t))\big),\ t \in [0,T].$$

It obvious that $Q_{r_1}$ is nonempty, closed, bounded and convex subset of $L^1[0,T]$.
Suppose $y \in Q_{r_1}$, then considering $t \in [0,T]$, we have

$$\begin{aligned} |F_1 y(t)| &= \big|f\big(t, \lambda\phi'(t)y(\phi(t))\big)\big| \\ &\leq |a(t)| + b\lambda\phi'(t)|y(\phi(t))|. \end{aligned}$$

Therefore

$$\begin{aligned} ||F_1 y||_{L^1} &:= \int_0^T |F_1 y(t)| dt \\ &\leq \int_0^T |a(t)| dt + b\lambda \int_0^T \phi'(t)|y(\phi(t))| dt. \end{aligned}$$

Let $\phi(t) = u$, this implies $\phi'(t)dt = du$, now

$$\begin{aligned} ||F_1 y||_{L^1} &\leq ||a||_{L^1} + b\lambda \int_{\phi(0)}^{\phi(T)} \phi'(t)|y(u)| \frac{du}{\phi'(t)} \\ &\leq ||a||_{L^1} + b\lambda \int_0^T |y(u)| du \\ &= ||a||_{L^1} + b\lambda ||y||_{L^1} \\ &\leq ||a||_{L^1} + b\lambda r_1 = r_1. \end{aligned}$$

This demonstrates that $F_1 : Q_{r_1} \to Q_{r_1}$ and the family $\{F_1 y(t)\}$ is uniformly bounded on $Q_{r_1}$.
Suppose $y \in \Omega \subset Q_{r_1}$, then

$$\begin{aligned} ||(F_1 y)_h - (F_1 y)||_{L^1} &= \int_0^T |(F_1 y(t))_h - (F_1 y(t))| dt \\ &= \int_0^T \left| \frac{1}{h} \int_t^{t+h} F_1 y(\theta) d\theta - F_1 y(t) \right| dt \\ &\leq \int_0^T \frac{1}{h} \int_t^{t+h} \big|F_1 y(\theta) - F_1 y(t)\big| d\theta dt \\ &= \int_0^T \frac{1}{h} \int_t^{t+h} \big|f\big(\theta, \lambda\phi'(\theta)y(\phi(\theta))\big) - f\big(t, \lambda\phi'(t)y(\phi(t))\big)\big| d\theta dt. \end{aligned}$$

It implies that $f \in L^1[0,T]$ based on assumptions $(i)$–$(ii)$.

$$\frac{1}{h} \int_t^{t+h} \big|f\big(\theta, \lambda\phi'(\theta)y(\phi(\theta))\big) - f\big(t, \lambda\phi'(t)y(\phi(t))\big)\big| d\theta \ \to\ 0 \ as\ h \to\ 0.$$

This concludes that $(F_1 y(t))_h \ \to\ (F_1 y(t))$ uniformly in $L^1[0,T]$.
According to the Kolmogorov compactness criterion [17], $F_1(\Omega)$ is relatively compact, which implies that $F_1$ is compact.

Suppose the sequence $\{y_n\} \subset Q_{r_1}$ such that $y_n \to y$, then

$$F_1 y_n(t) = f\big(t, \lambda\phi'(t) y_n(\phi(t))\big),$$

and

$$\begin{aligned}\lim_{n\to\infty} F_1 y_n(t) &= \lim_{n\to\infty} f\big(t, \lambda\phi'(t) y_n(\phi(t))\big)\\ &= f\big(t, \lambda\phi'(t) \lim_{n\to\infty} y_n(\phi(t))\big)\\ &= f\big(t, \lambda\phi'(t) y(\phi(t))\big)\\ &= F_1 y(t).\end{aligned}$$

Therefore, $F_1$ is continuous operator. Now all hypotheses of Schauder fixed point Theorem [7] are fulfilled, then $F_1$ has at least one fixed point $y \in Q_{r_1}$, implying that (4) has at least one solution $y \in L^1[0,T]$. □

The following theorem establishes the existence of at least one continuous solution to the integral equation (3), consequently leading to an absolutely continuous solution to the problem (1)-(2).

**Theorem 2.2.** *If the assumptions $(i)-(vii)$ are valid, then (3) has at least one continuous solution $x \in C[0,T]$. As a result, (1)-(2) has at least one solution $x \in AC[0,T]$.*

*Proof.* Define the set $Q_{r_2}$ and the operator $F_2$ correlated with (3) by

$$Q_{r_2} := \big\{x \in R : ||x||_C \le r_2\big\} \subset C[0,T], \text{where } r_2 = \frac{|x_0| + ||h||_{L^1} + (L+2)r_1}{1 - LT},$$

$$F_2 x(t) = x_0 + \int_0^{T-\tau} g\big(s, x(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^t y(s)ds,\ t \in [0,T].$$

It obvious that $Q_{r_2}$ is nonempty, closed, bounded and convex subset of $C[0,T]$. Suppose $y \in Q_{r_2}$, then considering $t \in [0,T]$, we have

$$\begin{aligned}|F_2 x(t)| &= \left| x_0 + \int_0^{T-\tau} g\big(s, x(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^t y(s)ds \right|\\ &\le |x_0| + \int_0^{T-\tau} \big|g\big(s, x(s), y(s)\big)\big|ds + \int_0^{\tau} |y(s)|ds + \int_0^t |y(s)|ds\\ &\le |x_0| + \int_0^T \big[|h(s)| + L|x(s) + y(s)|\big]ds + \int_0^T |y(s)|ds + \int_0^T |y(s)|ds\\ &\le |x_0| + \int_0^T |h(s)|ds + L\int_0^T |x(s)|ds + L\int_0^T |y(s)|ds + 2\int_0^T |y(s)|ds\\ &\le |x_0| + ||h||_{L^1} + L\int_0^T \sup_{s\in[0,T]} |x(s)|ds + L||y||_{L^1} + 2||y||_{L^1}\\ &= |x_0| + ||h||_{L^1} + LT||x||_C + (L+2)||y||_{L^1},\end{aligned}$$

and

$$||F_2 x||_C \le |x_0| + ||h||_{L^1} + LTr_2 + (L+2)r_1 = r_2.$$

Which proves that the family $\{F_2x(t)\}$ is uniformly bounded on $Q_{r_2}$.
Suppose $x \in Q_{r_2}$ and $t_1, t_2 \in [0,T]$, such that $t_2 > t_1$ and $|t_2 - t_1| \le \delta$, therefore

$$\begin{aligned} |F_2x(t_2) - F_2x(t_1)| &= \Big| x_0 + \int_0^{T-\tau} g\big(s, x(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^{t_2} y(s)ds \\ &\quad - x_0 - \int_0^{T-\tau} g\big(s, x(s), y(s)\big)ds + \int_0^{\tau} y(s)ds - \int_0^{t_1} y(s)ds \Big| \\ &\le \int_{t_1}^{t_2} |y(s)|ds \le \epsilon. \end{aligned}$$

This shows that $F_2 : Q_{r_2} \to Q_{r_2}$ and the family $\{F_2x(t)\}$ is equi-continuous on $Q_{r_2}$. According to the Arzela-Ascoli Theorem [2], $\{F_2x(t)\}$ is relatively compact and $F_2$ is compact operator.
suppose the sequence $\{x_n\} \subset Q_{r_1}$ such that $x_n \to x$, then

$$F_2x_n(t) = x_0 + \int_0^{T-\tau} g\big(s, x_n(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^{t} y(s)ds,$$

and

$$\lim_{n\to\infty} F_2x_n(t) = x_0 + \lim_{n\to\infty} \int_0^{T-\tau} g\big(s, x_n(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^{t} y(s)ds.$$

Utilizing assumptions $(v), (vi)$ along with application of the the Lebesgue dominated convergence Theorem [9], we obtain

$$\begin{aligned} \lim_{n\to\infty} F_2x_n(t) &= x_0 + \int_0^{T-\tau} \lim_{n\to\infty} g\big(s, x_n(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^{t} y(s)ds \\ &= x_0 + \int_0^{T-\tau} g\big(s, \lim_{n\to\infty} x_n(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^{t} y(s)ds \\ &= x_0 + \int_0^{T-\tau} g\big(s, x(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^{t} y(s)ds \\ &= F_2x(t). \end{aligned}$$

Therefore, $F_2$ is continuous operator, and by applying the Schauder fixed point Theorem [7], $F_2$ has at least one fixed point $y \in Q_{r_2}$, implying that (3) has at least one solution $x \in C[0,T]$.
As a result, (1)-(2) has at least one solution $x \in AC[0,T]$. □

## 3. Uniqueness of solution

Here, we will ensure the uniqueness of the solution of (1)-(2) by implementing the following assumptions:

$(i)'$ $f : [0,T] \times R \to R$ is measurable in $t \in [0,T]$ and satisfy the Lipschitz condition in $x \in R$ such that
$$|f(t,x) - f(t,y)| \le b|x-y| \quad \text{where } b > 0.$$
$(ii)'$ $f(t,0) \in L^1[0,T]$.
$(iii)'$ $g : [0,T] \times R \times R \to R$ is measurable in $t \in [0,T]$ and satisfy the Lipschitz condition in $x, y \in R$ such that
$$|g(t,x,y) - g(t,w,z)| \le L\big(|x-w| + |y-z|\big) \quad \text{where } L > 0.$$
$(iv)'$ $g(t,0,0) \in L^1[0,T]$.

The following theorem demonstrates the uniqueness of the solution of the functional equation (4) .

**Theorem 3.3.** *If the assumptions* $(iii)$-$(iv)$ *(of Theorem 2.1) and* $(i)'$-$(ii)'$ *are valid, then the solution* $y \in L^1[0,T]$ *of* (4) *is unique.*

*Proof.* assumptions $(i)$-$(ii)$ of Theorem 2.1 can be inferred from $(i)'$ and $(ii)'$. Setting $y = 0$ in $(i)'$, we obtain

$$|f(t,x)| \leq b|x| + |f(t,0)|,$$

where $a(t) = f(t,0) \in L^1[0,T]$.
Therefore, we conclude that all the assumptions of Theorem 2.1 are met, and (4) has at least one solution $y \in L^1[0,T]$. Now, if $y_1, y_2$ are two solutions of (4), then

$$\begin{aligned} |y_2(t) - y_1(t)| &= \big|f\big(t, \lambda\phi'(t)y_2(\phi(t))\big) - f\big(t, \lambda\phi'(t)y_1(\phi(t))\big)\big| \\ &\leq b\lambda\phi'(t)\big|y_2(\phi(t)) - y_1(\phi(t))\big|. \end{aligned}$$

Hence

$$||y_2 - y_1||_{L^1} \leq b\lambda||y_2 - y_1||_{L^1}.$$

As $b\lambda < 1$, this implies $y_1 = y_2$, leading to the uniqueness of the solution of (4). □

Here, we prove the uniqueness of an absolutely continuous solution to the problem (1)-(2).

**Theorem 3.4.** *If the assumptions* $(iii) - (iv)$ *and* $(vii)$ *(of Theorem 2.2) and* $(i)'$-$(iv)'$ *are valid, then the solution* $x \in AC[0,T]$ *of* (1)-(2) *is unique.*

*Proof.* assumptions $(v)$-$(vi)$ of Theorem 2.2 can be inferred from $(iii)'$ and $(iv)'$. Setting $w = z = 0$ in $(iii)'$, we obtain

$$|g(t,x,y)| \leq |g(t,0,0)| + L(|x| + |y|),$$

where $h(t) = g(t,0,0) \in L^1[0,T]$.
Therefore, we conclude that all the assumptions of Theorem 2.2 are met, and (3) has at least one solution $x \in C[0,T]$. Now, if $x_1, x_2$ are two solutions of (4), then

$$\begin{aligned} |x_2(t) - x_1(t)| &= \Big|x_0 + \int_0^{T-\tau} g\big(s, x_2(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^t y(s)ds \\ &\quad - x_0 - \int_0^{T-\tau} g\big(s, x_1(s), y(s)\big)ds + \int_0^{\tau} y(s)ds - \int_0^t y(s)ds\Big| \\ &\leq \int_0^T \big|g\big(s, x_2(s), y(s)\big) - g\big(s, x_1(s), y(s)\big)\big|ds \\ &\leq L\int_0^T |x_2(s) - x_1(s)|ds. \end{aligned}$$

Hence

$$||x_2 - x_1||_C \leq LT||x_2 - x_1||_C.$$

As $LT < 1$, this implies $x_1 = x_2$ and the solution $x \in C[0,T]$ of (3) is unique.
As a result, the solution $x \in AC[0,T]$ of (1)-(2) is unique. □

## 4. Hyers-Ulam stability

Here, we study the first concept of stability which is Hyers-Ulam stabiltiy of our problem (1)-(2), assessing its resilience under slight disturbances.

**Definition 4.1.** [11, 12, 13] *Let the solution to* (1)-(2) *be exists uniquely. The problem* (1)-(2) *is Hyers-Ulam stable, if* $\forall\ \epsilon > 0,\ \exists\ \delta(\epsilon) > 0$ *such that for any* $\delta$*-approximate solution* $x_s \in AC[0,T]$ *of* (1)-(2) *satisfying*

$$\left|\frac{dx_s(t)}{dt} - f\left(t, \lambda \frac{d}{dt} x_s\big(\phi(t)\big)\right)\right| \leq \delta,$$

*then*

$$||x - x_s||_C \leq \epsilon.$$

**Theorem 4.5.** *If the assumptions of Theorem 3.4 are met, then the problem* (1)-(2) *is Hyers-Ulam stable.*

*Proof.* suppose $\left|\frac{dx_s(t)}{dt} - f\left(t, \lambda \frac{d}{dt} x_s\big(\phi(t)\big)\right)\right| \leq \delta$, this implies

$$-\delta \leq \frac{dx_s(t)}{dt} - f\left(t, \lambda\phi'(t)\frac{dx_s\big(\phi(t)\big)}{d\big(\phi(t)\big)}\right) \leq \delta,$$
$$-\delta \leq y_s(t) - f\big(t, \lambda\phi'(t) y_s(\phi(t))\big) \leq \delta.$$

Now

$$\begin{aligned}
|y(t) - y_s(t)| &= \big|f\big(t, \lambda\phi'(t)y(\phi(t))\big) - y_s(t)\big| \\
&= \big|f\big(t, \lambda\phi'(t)y(\phi(t))\big) - y_s(t) - f\big(t, \lambda\phi'(t)y_s(\phi(t))\big) + f(t, \lambda\phi'(t)y_s(\phi(t)))\big| \\
&\leq \big|f\big(t, \lambda\phi'(t)y(\phi(t))\big) - f\big(t, \lambda\phi'(t)y_s(\phi(t))\big)\big| + \big|f\big(t, \lambda\phi'(t)y_s(\phi(t))\big) - y_s(t)\big| \\
&\leq b\big|\lambda\phi'(t)y(\phi(t)) - \lambda\phi'(t)y_s(\phi(t))\big| + \delta \\
&= b\lambda\phi'(t)\big|y(\phi(t)) - y_s(\phi(t))\big| + \delta.
\end{aligned}$$

Hence

$$||y - y_s||_{L^1} \leq b\lambda ||y - y_s||_{L^1} + \delta T,$$

and

$$||y - y_s||_{L^1} \leq \frac{\delta T}{1 - b\lambda} = \epsilon^*.$$

Now

$$\begin{aligned}
|x(t) - x_s(t)| = \Big|x_0 &+ \int_0^{T-\tau} g\big(s, x(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^t y(s)ds \\
&- x_0 - \int_0^{T-\tau} g\big(s, x_s(s), y_s(s)\big)ds + \int_0^{\tau} y_s(s)ds - \int_0^t y_s(s)ds\Big| \\
&\leq \int_0^{T-\tau} \big|g\big(s, x(s), y(s)\big) - g\big(s, x_s(s), y_s(s)\big)\big|ds \\
&+ \int_0^{\tau} \big|y(s) - y_s(s)\big|ds + \int_0^t \big|y(s) - y_s(s)\big|ds \\
&\leq L\int_0^T \big[|x(s) - x_s(s)| + |y(s) - y_s(s)|\big]ds + 2||y - y_s||_{L^1} \\
&\leq LT||x - x_s||_C + (L + 2)||y - y_s||_{L^1}.
\end{aligned}$$

Hence

$$||x-x_s||_C \leq LT||x-x_s||_C + (L+2)\epsilon^*,$$

and

$$||x-x_s||_C \leq \frac{(L+2)\epsilon^*}{1-LT} = \epsilon.$$

As $LT < 1$, this implies that the problem (1)-(2) is Hyers-Ulam stable.

□

## 5. Continuous Dependence

Here, we investigate the second concept of stability, which is the continuous dependence of the unique solution to the problem (1)-(2) on its parameters. We explore whether minor variations in these parameters preserve the solution.

**Definition 5.2.** *The solution $x \in AC[0,T]$ of (1)-(2) depends continuously on the function $y \in L^1[0,T]$, if $\forall\ \epsilon > 0,\ \exists\ \delta(\epsilon) > 0$ such that*

$$||y-y^*||_{L^1} \leq \delta \Rightarrow ||x-x^*||_C \leq \epsilon,$$

*where $x^*$ represents the unique solution of the integral equation*

$$x^*(t) = x_0 + \int_0^{T-\tau} g\big(s, x^*(s), y^*(s)\big)ds - \int_0^{\tau} y^*(s)ds + \int_0^t y^*(s)ds,\ t \in [0,T]. \quad (7)$$

**Theorem 5.6.** *If the assumptions of Theorem 3.4 are met, then the solution $x \in AC[0,T]$ of (1)-(2) depends continuously on the function $y$.*

*Proof.* Suppose $x$ and $x^*$ are the two solutions of (3) and (7) respectively, then we obtain

$$\begin{aligned}
|x(t)-x^*(t)| &= \Big| x_0 + \int_0^{T-\tau} g\big(s, x(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^t y(s)ds \\
&\quad - x_0 - \int_0^{T-\tau} g\big(s, x^*(s), y^*(s)\big)ds + \int_0^{\tau} y^*(s)ds - \int_0^t y^*(s)ds \Big| \\
&\leq \int_0^{T-\tau} \big| g\big(s, x(s), y(s)\big) - g\big(s, x^*(s), y^*(s)\big) \big| ds \\
&\quad + \int_0^{\tau} |y(s)-y^*(s)|ds + \int_0^t |y(s)-y^*(s)|ds \\
&\leq L\int_0^T \big[|x(s)-x^*(s)| + |y(s)-y^*(s)|\big]ds + 2||y-y^*||_{L^1} \\
&\leq LT||x-x^*||_C + (L+2)||y-y^*||_{L^1}.
\end{aligned}$$

Thus

$$||x-x^*||_C \leq LT||x-x^*||_C + (L+2)\delta.$$

Hence

$$||x-x^*||_C \leq \frac{(L+2)\delta}{1-LT} = \epsilon.$$

As $LT < 1$, then the solution of (1)-(2) depends continuously on $y$. □

**Definition 5.3.** *The solution $y \in L^1[0,T]$ of the functional equation* (4) *depends continuously on the function $f$ and parameter $\lambda$, $if\ \forall\ \epsilon > 0,\ \exists\ \delta(\epsilon) > 0$ such that*

$$\max\big\{|f(t,x) - f^*(t,x)|,\ |\lambda - \lambda^*|\big\} \leq \delta \Rightarrow ||y - y^*||_{L^1} \leq \epsilon,$$

*where $y^*$ represents the unique solution of the functional equation*

$$y^*(t) = f^*\big(t, \lambda^* \phi'(t) y^*(\phi(t))\big),\ t \in [0,T]. \tag{8}$$

**Theorem 5.7.** *If the assumptions of Theorem 3.3 are met, then the solution $y \in L^1[0,T]$ of* (4) *depends continuously on the function $f$ and parameter $\lambda$.*

*Proof.* Suppose $y$ and $y^*$ are the two solutions of (4) and (8) respectively, then we obtain

$$\begin{aligned}
|y(t) - y^*(t)| &= \big|f\big(t, \lambda\phi'(t)y(\phi(t))\big) - f^*\big(t, \lambda^*\phi'(t)y^*(\phi(t))\big)\big| \\
&\leq \big|f\big(t, \lambda\phi'(t)y(\phi(t))\big) - f^*\big(t, \lambda\phi'(t)y(\phi(t))\big)\big| \\
&+ \big|f^*\big(t, \lambda\phi'(t)y(\phi(t))\big) - f^*\big(t, \lambda^*\phi'(t)y^*(\phi(t))\big)\big| \\
&\leq \delta + b\big|\lambda\phi'(t)y(\phi(t)) - \lambda^*\phi'(t)y^*(\phi(t))\big| \\
&= \delta + b\phi'(t)|\lambda y(\phi(t)) - \lambda y^*(\phi(t)) + \lambda y^*(\phi(t)) - \lambda^* y^*(\phi(t))| \\
&\leq \delta + b\lambda\phi'(t)|y(\phi(t)) - y^*(\phi(t))| + b|\lambda - \lambda^*|\phi'(t)|y^*(\phi(t))|.
\end{aligned}$$

Thus

$$||y - y^*||_{L^1} \leq \delta T + b\lambda||y - y^*||_{L^1} + b\delta||y^*||_{L^1}.$$

Hence

$$||y - y^*||_{L^1} \leq \frac{\delta T + b\delta r_1}{1 - b\lambda} = \epsilon.$$

As $b\lambda < 1$, then the solution of (4) depends continuously on $f$ and $\lambda$. □

We now have the following corollary derived from Theorem 5.6.

**Corollary 5.1.** *Let the assumptions of Theorem 5.6 be valid, then the solution $x \in AC[0,T]$ of* (1)-(2) *depends continuously on the function $f$ and parameter $\lambda$.*

**Definition 5.4.** *The solution $x \in AC[0,T]$ of* (1)-(2) *depends continuously on the initial data $x_0$ and the function $g$, $if\ \forall\ \epsilon > 0,\ \exists\ \delta(\epsilon) > 0$ such that*

$$\max\big\{|x_0 - x_0^*|,\ |g(t,x,y) - g^*(t,x,y)|\big\} \leq \delta \Rightarrow ||x - x^*||_C \leq \epsilon,$$

*where $x^*$ represents the unique solution of the integral equation*

$$x^*(t) = x_0^* + \int_0^{T-\tau} g^*\big(s, x^*(s), y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^t y(s)ds,\ t \in [0,T]. \tag{9}$$

**Theorem 5.8.** *If the assumptions of Theorem 3.4 are met, then the solution $x \in AC[0,T]$ of* (1)-(2) *depends continuously on the initial data $x_0$ and the function $g$.*

*Proof.* suppose $x$ and $x^*$ are the two solutions of (3) and (9) respectively, then we obtain

$$\begin{aligned}
|x(t)-x^*(t)| &= \Big|x_0 + \int_0^{T-\tau} g\big(s,x(s),y(s)\big)ds - \int_0^{\tau} y(s)ds + \int_0^{t} y(s)ds \\
&\quad - x_0^* - \int_0^{T-\tau} g^*\big(s,x^*(s),y(s)\big)ds + \int_0^{\tau} y(s)ds - \int_0^{t} y(s)ds\Big| \\
&\le |x_0 - x_0^*| + \int_0^{T-\tau} \big|g\big(s,x(s),y(s)\big) - g^*\big(s,x^*(s),y(s)\big)\big|ds \\
&\le \delta + \int_0^{T} \big|g\big(s,x(s),y(s)\big) - g^*\big(s,x(s),y(s)\big)\big|ds \\
&+ \int_0^{T} \big|g^*\big(s,x(s),y(s)\big) - g^*\big(s,x^*(s),y(s)\big)\big|ds \\
&\le \delta + \delta T + L\int_0^{T} \big|x(s)-x^*(s)\big|ds \\
&\le \delta + \delta T + LT||x-x^*||_C.
\end{aligned}$$

Thus

$$||x-x^*||_C \le (1+T)\delta + LT||x-x^*||_C.$$

Hence

$$||x-x^*||_C \le \frac{(1+T)\delta}{1-LT} = \epsilon.$$

As $LT<1$, then the solution of (1)-(2) depends continuously on $x_0$ and $g$. □

## 6. Special cases and examples

In this section, we present various special cases and examples to further illustrate the concepts discussed in the preceding sections.

**Corollary 6.2.** *Suppose the assumptions of Theorem 2.2 are met with $\phi(t)=\gamma t$, where $\gamma \in (0,1]$, then the equation*

$$\frac{dx(t)}{dt} = f\Big(t, \lambda \frac{d}{dt} x\big(\gamma t\big)\Big), \ a.e.\ t \in (0,T],$$

*subject to the internal integro-differential condition (2), possesses at least one solution $x \in AC[0,T]$.*
*Consequently, under the assumptions of Theorem 3.4, it possesses a unique solution $x \in AC[0,T]$.*

**Corollary 6.3.** *Suppose the assumptions of Theorem 2.2 are met with $\phi(t)=t^{\beta}$, where $\beta \ge 1$, then the equation*

$$\frac{dx(t)}{dt} = f\Big(t, \lambda \frac{d}{dt} x\big(t^{\beta}\big)\Big), \ a.e.\ t \in (0,1],$$

*subject to the condition (2), possesses at least one solution $x \in AC[0,1]$.*
*Consequently, under the assumptions of Theorem 3.4, it possesses a unique solution $x \in AC[0,1]$.*

**Example 1.** *Given the functional differential equation*

$$\frac{dx(t)}{dt} = \frac{3e^{-2t}}{t+3} + \frac{1}{7}\frac{d}{dt}x\Big(\frac{1}{2}(t+1)\Big), \ a.e.\ t \in (0,1], \tag{10}$$

*with condition*

$$x(\tau) = 1 + \int_0^{1-\tau} \Big( \big(\frac{s}{5}\big)^4 + \frac{1}{7}\Big(x(s) + \frac{dx(s)}{ds}\Big)\Big) ds, \ \tau \in (0,1). \tag{11}$$

*The functional equation is expressed as*

$$y(t) = \frac{3e^{-2t}}{t+3} + \frac{1}{14} y\big(\frac{1}{2}(t+1)\big), \ t \in [0,1]. \tag{12}$$

*Set*

$$f\big(t, \lambda\phi'(t)y(\phi(t))\big) = \frac{3e^{-2t}}{t+3} + \frac{1}{14} y\big(\frac{1}{2}(t+1)\big),$$

*and*

$$g\big(t, x(t), y(t)\big) = \big(\frac{t}{5}\big)^4 + \frac{1}{7}\big(x(t) + y(t)\big).$$

*We have* $\phi(t) = \frac{1}{2}(t+1) \leq 1$, $\phi'(t) = \frac{1}{2}$, $\lambda = \frac{1}{7}$, $x_0 = 1$ *and* $T = 1$.
*Hence,* $f(t,0) = \frac{3e^{-2t}}{t+3} \in L^1[0,1]$, $g(t,0,0) = \big(\frac{t}{5}\big)^4 \in L^1[0,1]$,
*and*

$$\big|f(t,x) - f(t,y)\big| \leq \frac{1}{14}|x-y|,$$
$$\big|g(t,x,y) - g(t,w,z)\big| \leq \frac{1}{7}\big(|x-w| + |y-z|\big),$$

*then* $b = \frac{1}{14}$, $L = \frac{1}{7}$, $b\lambda \approx 0.01020408 < 1$, *and* $LT \approx 0.14285714 < 1$.
*Hence, according to Theorem 3.4, the solution* $x \in AC[0,1]$ *of* (10)-(11) *is unique.*

**Example 2.** *Given the functional differential equation*

$$\frac{dx(t)}{dt} = \frac{t}{t+1} + \frac{1}{9}\frac{d}{dt}x\Big(\frac{1}{3}t\Big), \ a.e.\ t \in (0,3], \tag{13}$$

*with condition*

$$x(\tau) = 2 + \int_0^{3-\tau} \Big( \frac{s}{s^2+1} + \frac{1}{9}\Big(x(s) + \frac{dx(s)}{ds}\Big)\Big) ds, \ \tau \in (0,3). \tag{14}$$

*The functional equation is expressed as*

$$y(t) = \frac{t}{t+1} + \frac{1}{27} y\big(\frac{1}{3}t\big), \ t \in [0,3]. \tag{15}$$

*Set*

$$f\big(t, \lambda\phi'(t)y(\phi(t))\big) = \frac{t}{t+1} + \frac{1}{27} y\big(\frac{1}{3}t\big),$$

*and*

$$g\big(t, x(t), y(t)\big) = \frac{t}{t^2+1} + \frac{1}{9}\big(x(t) + y(t)\big).$$

*We have* $\phi(t) = \frac{1}{3}t$, $\gamma = \frac{1}{3}$, $\lambda = \frac{1}{9}$, $x_0 = 2$ *and* $T = 3$.
*Hence,* $f(t,0) = \frac{t}{t+1} \in L^1[0,3]$, $g(t,0,0) = \frac{t}{t^2+1} \in L^1[0,3]$,

*and*

$$\big|f(t,x)-f(t,y)\big| \le \frac{1}{27}|x-y|,$$
$$\big|g(t,x,y)-g(t,w,z)\big| \le \frac{1}{9}\big(|x-w|+|y-z|\big),$$

*then* $b=\frac{1}{27}$, $L=\frac{1}{9}$, $b\lambda \approx 0.00411523 < 1$, *and* $LT \approx 0.33333333 < 1$.
*Hence, according to Corollary 6.2, the solution* $x \in AC[0,3]$ *of* (13)-(14) *is unique.*

**Example 3.** *Given the functional-differential equation*

$$\frac{dx(t)}{dt} = \frac{1}{9-t^3} + \frac{1}{3}\frac{d}{dt}x\big(t^3\big), \ a.e.\ t \in (0,1], \tag{16}$$

*with condition*

$$x(\tau) = \frac{1}{3} + \int_0^{1-\tau} \left(3s^2+2s+1+\frac{1}{3}\left(x(s)+\frac{dx(s)}{ds}\right)\right)ds, \ \tau \in (0,1). \tag{17}$$

*The functional equation is expressed as*

$$y(t) = \frac{1}{9-t^3} + t^2 y\big(t^3\big), \ t \in [0,1]. \tag{18}$$

*Set*

$$f\big(t,\lambda\phi'(t)y(\phi(t))\big) = \frac{1}{9-t^3} + t^2 y\big(t^3\big),$$

*and*

$$g\big(t,x(t),y(t)\big) = 3t^2+2t+1+\frac{1}{3}\big(x(t)+y(t)\big).$$

*We have* $\phi(t)=t^3$, $\beta=3$, $\lambda=\frac{1}{3}$, $x_0=\frac{1}{3}$ *and* $T=1$.
*Hence,* $f(t,0)=\frac{1}{9-t^3} \in L^1[0,1]$, $g(t,0,0)=3t^2+2t+1 \in L^1[0,1]$,
*and*

$$\big|f(t,x)-f(t,y)\big| \le |x-y|,$$
$$\big|g(t,x,y)-g(t,w,z)\big| \le \frac{1}{3}\big(|x-w|+|y-z|\big),$$

*then* $b=1$, $L=\frac{1}{3}$, $b\lambda = LT \approx 0.33333333 < 1$.
*Hence, according to Corollary 6.3, the solution* $x \in AC[0,1]$ *of* (16)-(17) *is unique.*

## 7. Conclusion

This study establishes the existence and uniqueness of solutions $x \in AC[0,T]$ for the nonlocal problem (1)-(2) under specific assumptions by converting it to a fixed point problem. Our study exposes the Hyers-Ulam stability, which provides insights into the resilience of the problem to disturbances. Most significantly, it reveals the continuous dependence of the unique solution on certain factors. Furthermore, To demonstrate the applicability of our work, we provided a range of instances and special cases. In summary, this paper is a great resource for researchers exploring the existence and stability of nonlocal problems of functional differential equations. It suggests a promising avenue for future research, especially looking into how Hyers-Ulam stability applies to the problem.

## References


[1] A. Boucherif, R. Precup, *On the nonlocal initial value problem for first order differential equations*, Fixed Point Theory **4 2** (2003), 205–212.

[2] R. F. Brown, *A topological introduction to nonlinear analysis (2nd edition)*, Birkhäuser Boston, MA, (2004).

[3] T. A. Burton, *Stability by fixed point theory for functional differential equations*, Dover Publications, New York, (2006).

[4] T. A. Burton, T. Furumochi, *Asymptotic behavior of solutions of functional differential equations by fixed point theorems*, Dyn. Syst. Appl. **11 4** (2002), 499–520.

[5] C. Chena, L. Liua, Q. Donga, *Existence and Hyers-Ulam stability for boundary value problems of multi-term Caputo fractional differential equations*, Filomat **37 28** (2023), 9679–9692.

[6] C. Corduneanu, Y. Li, M. Mahdavi, *Functional differential equations: advances and applications*, John Wiley & Sons, New Jersey, (2016).

[7] K. Deimling, *Nonlinear functional analysis*, Dover Publications, New York, (2010).

[8] B. C. Dhage, *A fixed point theorem in Banach algebras involving three operators with applications*, Kyungpook Math. J. **44 1** (2004), 145–155.

[9] N. Dunford, J. T. Schwartz, *Linear operators (part I: general theory)*, John Wiley & Sons, New Jersey, (1988).

[10] A. M. A. El-Sayed, A. A. A. Alhamali, E. M. A. Hamdallah, *Analysis of a Fractional-Order Quadratic Functional Integro-Differential Equation with Nonlocal Fractional-Order Integro-Differential Condition*, Axioms **12 8** (2023), 788.

[11] A. M. A. El-Sayed, A. A. A. Alhamali, E. M. A. Hamdallah, H. R. Ebead, *Qualitative Aspects of a Fractional-Order Integro-Differential Equation with a Quadratic Functional Integro-Differential Constraint*, Fractal Fract. **7 12** (2023), 835.

[12] A. M. A. El-Sayed, M. M. S. Ba-Ali, E. M. A. Hamdallah, *An Investigation of a Nonlinear Delay Functional Equation with a Quadratic Functional Integral Constraint*, Mathematics **11 21** (2023), 4475.

[13] A. M. A. El-Sayed, W. G. El-Sayed, S. I. Nasim, *On the solvability of a delay tempered-fractal differential equation*, J. Fract. Calc. Appl. **15 1** (2020), 1–15.

[14] A. M. A. El-Sayed, E. Hamdallah, H. R. Ebead, *Positive nondecreasing solutions of a self-refereed differential equation with two state-delay functions*, Adv. Math. Sci. J. **9 12** (2020), 10357–10365.

[15] A. M. A. El-Sayed, M. S. Mohamed, K. M. O. Msaik, *On some boundary-value problems of functional integro-differential equations with nonlocal conditions*, Malaya J. Mat. **5 2** (2017), 186–193.

[16] T. Faria, L. T. Magalhães, *Normal forms for retarded functional differential equations with parameters and applications to Hopf bifurcation*, J. Differential Equations **122 2** (1995), 181–200.

[17] A. Granas, J. Dugundji, *Fixed point theory*, Springer, New York, (2003).

[18] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27 4** (1941), 222–224.

[19] S. M. Jung, *Hyers-Ulam-Rassias stability of functional equations in nonlinear analysis*, Springer, New York, (2011).

[20] H. Khan, C. Tunç, A. Khan, *Stability results and existence theorems for nonlinear delay-fractional differential equations with $\varphi_p^*$-operator*, J. Appl. Anal. Comput. **10 2** (2020), 584–597.

[21] J. Klamka, *Schauder's fixed-point theorem in nonlinear controllability problems*, Control Cybernet. **29 1** (2000), 153–165.

[22] V. Kolmanovskii, A. Myshkis, *Introduction to the theory and applications of functional differential equations*, Springer, Dordrecht, (2013).

[23] G. A. Monteiro, M. Tvrdý, *Generalized linear differential equations in a Banach space: Continuous dependence on a parameter*, Discrete Contin. Dyn. Syst. **33 1** (2013), 283–303.

[24] A. B. Muravnik, *Nonlocal problems and functional-differential equations: theoretical aspects and applications to mathematical modelling*, Math. Model. Nat. Phenom. **14 6** (2019), 601.

[25] K. A. Murphy, *stimation of time-and state-dependent delays and other parameters in functional differential equations*, SIAM J. Appl. Math. **50 4** (1990), 972–1000.

[26] G. M. N'Guérékata, *A Cauchy problem for some fractional abstract differential equation with non local conditions*, Nonlinear Anal. **70 5** (2009), 1873–1876.
[27] S. L. Ross, *Differential equations (3rd edition)*, John Willey & Sons, New York, (1984).
[28] J. Wang, M. Fec, Y. Zhou, *Fractional order iterative functional differential equations with parameter*, Appl. Math. Model. **37 8** (2013), 6055–6067.
[29] E. Zhang, *Functional differential equation models in epidemiology with theoretical and numerical studies*, The University of Wisconsin-Madison, (1992).

Ahmed M. A. El-Sayed
Faculty of Science, Alexandria University, Alexandria, Egypt
*Email address*: amasayed@alexu.edu.eg

Moamen O. Radwan
Faculty of Science, Alexandria University, Alexandria, Egypt
*Email address*: sci.moamenosama2018@alexu.edu.eg

Hanna R. Ebead
Faculty of Science, Alexandria University, Alexandria, Egypt
*Email address*: HanaaRezqalla@alexu.edu.eg