Journal of Fractional Calculus and Applications
Vol. 15(1) Jan. 2024, No.12
ISSN: 2090-5858.
ISSN 2090-584X (print)
http://jfca.journals.ekb.eg/

FRACTIONAL ORDER DIFFERENTIAL EQUATION WITH NONLOCAL INTEGRAL CONDITION

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ABSTRACT. This research paper focuses on investigating the solvability of the Riemann-Liouville differential equation with nonlocal integral condition, study the existence of solutions in the class of continuous functions, we use the technique of the Schauder fixed point Theorem. We drive sufficient conditions for a uniqueness and the continuous dependence on some functions. Additionally, we delve into the study of the Hyers–Ulam stability. Finally, we given an examples are provided to illustrate our results.

1. Introduction

The investigation of fractional functional equations has garnered significant attention in recent decades. The theory of fractional differential equations has found widespread applications in various domains, including astronomy, biology, economics, and others, as documented in references [1, 2, 7, 8, 15, 11, 13, 14, 17, 9, 10, 12]. Moreover, research efforts have also focused on addressing the challenges associated with solutions of fractional differential equations on both finite and infinite intervals.

The definition of the fractional derivative of the Riemann-Liouville type played an important role in the development of the theory of fractional derivatives and integrals and for its applications in pure mathematics.
However, the demands of modern technology require a certain revision of the well-established pure mathematical approach. Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions.

2010 *Mathematics Subject Classification.* 33A26, 34A08.
Key words and phrases. Riemann-Liouville, existence of solution, Hyers-Ulam stability, continuous dependence, fractional order.
Submitted January 9, 2024. Revised March 13, 2024.

In this paper we study the nonlocal problem of the fractional order differential equation,

$$ {}^{R}D^{\alpha}(x(t)-x(0)) = f(t, x(\phi(t))),\ t \in I,\ I = [0, T], \tag{1} $$

with nonlocal integral condition

$$ x(0) + \int_0^T h(s, x(s))ds = x_0, \tag{2} $$

where ${}^{R}D^{\alpha}$ is the refers to the fractional derivative of Riemann–Liouville of order $\alpha \in (0,\ 1)$. Our aim here is study the existence of solutions $x \in C(I)$. Moreover, the continuous dependence of the unique solution on the x_0 and on the functions f, g and ϕ will be proved. The Hyers – Ulam stability of the problem will be given.

The paper is organized as follows: Section 2 contains the solvability of the solutions $x \in C(I)$ by Schauder fixed point Theorem [16] and discuss some stability facts of the of the problem (1)-(2). Moreover, the Hyers – Ulam stability of our problem will be studied. Some examples in Section 3.

2. Existence of solution

Let $C = C(I)$, be the class of continuous functions with the standard norm

$$ \|x\| = \sup_{t \in I} |x(t)|. $$

Take into account the following assumptions:

(i) $\phi : I \to I$, $\phi(t) \leq t$; is continuous.

(ii) $h : I \times R \to R$ is Carathéodory functions [6] and there exist a bounded measurable function $a : I \to R$ and nonnegative constant b_1 such that

$$ |h(t,x)| \leq |a(t)| + b_1|x|\ \forall\ t \in I,\ x \in R. $$

(iii) $f : I \times R \to R$ is Carathéodory functions [6] and there exist a bounded measurable function $m : I \to R$ and nonnegative constant b_2 such that

$$ |f(t,x)| \leq |m(t)| + b_2|x|\ \forall\ t \in I,\ x \in R. $$

(iv) $(b_1\ T + \frac{b_2 T^{\alpha}}{\Gamma(\alpha+1)}) < 1$.

Now, the following lemma.

Lemma 2.1. *The problem (1)-(2) is equivalent to the integral equation*

$$ x(t) = x_0 - \int_0^T h(s, x(s))ds + I^{\alpha} f(t, x(\phi(t))),\ t \in I. \tag{3} $$

Proof. Let x be a solution of (1)-(2), then

$$ {}^{R}D^{\alpha}(x(t) - x(0)) = f(t, x(\phi(t))). $$

$$ \frac{d}{dt} I^{1-\alpha}(x(t) - x(0)) = f(t, x(\phi(t))). $$

Integrating we obtain

$$ I^{1-\alpha}(x(t) - x(0)) - I^{1-\alpha}(x(t) - x(0))\Big|_{t=0} = I f(t, x(\phi(t))). $$

and from the properties of the fractional calculus,[] we can obtain

$$I^{1-\alpha}(x(t)-x(0)) = If(t,x(\phi(t)))$$

and

$$\begin{aligned} I(x(t)-x(0)) &= I^{1+\alpha}f(t,x(\phi(t))) \\ (x(t)-x(0)) &= I^{\alpha}f(t,x(\phi(t))). \end{aligned}$$

then

$$x(t) = x(0) + I^{\alpha}y(t). \tag{4}$$

Substituting by (2) in (5), we obtain (3).
Conversely, let x be a solution of (3). Substituting by (2) in (3) we obtain

$$I^{1-\alpha}(x(t)-x(0)) = I^{1-\alpha}I^{\alpha}f(t,x(\phi(t))).$$

Differentiation we obtain

$$\frac{d}{ds}I^{1-\alpha}(x(t)-x(0)) = \frac{d}{ds}If(t,x(\phi(t))),$$

then

$${}^{R}D^{\alpha}(x(t)-x(0)) = f(t,x(\phi(t))).$$

This proves the equivalent between the problem (1)-(2) and (3). Now, we have the following existences theorem.

Theorem 2.1. *Assume that* $(i)-(iv)$ *be satisfied, then the integral equation (3) has at least one solution* $x \in C(I)$.

Proof. Define the set

$$Q_r = \{x \in C(I) : \|x\| \le r\},\ r \ = \ \left(|x_0| + \|a\|T + \frac{T^{\alpha}\ \|m\|}{\Gamma(\alpha+1)}\right) \div \left(1 - (b_1T + \frac{b_2T^{\alpha}}{\Gamma(\alpha+1)})\right)$$

and define the operator F by

$$Fx(t) = x_0 - \int_0^T h(s,x(s))ds + I^{\alpha}f(t,x(\phi(t))).$$

Now, let $x \in Q_r$, then

$$\begin{aligned} |Fx(t)| &= \left|x_0 - \int_0^T h(s,x(s))ds + I^{\alpha}f(t,x(\phi(t)))\right| \\ &\le |x_0| + \int_0^T |h(s,x(s))|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(\phi(s)))|ds \\ &\le |x_0| + \int_0^T (|a(s)| + b_1|x(s)|)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(|m(s)| + b_2|x(\phi(s))|)ds \\ &\le |x_0| + (\|a\| + b_1\|x\|)T + \frac{\|m\|T^{\alpha}}{\Gamma(\alpha+1)} + \frac{b_2T^{\alpha}}{\Gamma(\alpha+1)}\ \|x\| \\ &\le |x_0| + (\|a\| + b_1\ r)T + \frac{\|m\|T^{\alpha}}{\Gamma(\alpha+1)} + \frac{b_2T^{\alpha}}{\Gamma(\alpha+1)}\ r \ = \ r, \end{aligned}$$

then

$$\|Fx\| \ \le \ |x_0| + (\|a\| + b_1\ r)T + \frac{\|m\|T^{\alpha}}{\Gamma(\alpha+1)} + \frac{b_2T^{\alpha}}{\Gamma(\alpha+1)}\ r \ = \ r.$$

Hence the operator F maps the ball Q_r into itself and the class of functions $\{Fx\}$ is uniformly bounded on Q_r.
Now, let $x \in Q_r$ and $t_1,\ t_2 \in I$ such that $t_1 \leq t_2$ and $|t_2 - t_1| < \delta$, then we have

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= \Big| x_0 - \int_0^T h(s,x(s))ds + \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(\phi(s)))ds \\
&- x_0 + \int_0^T h(s,x(s))ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(\phi(s)))ds \Big| \\
&\leq \Big| \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(\phi(s)))ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(\phi(s)))ds \Big| \\
&\leq \Big| \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(\phi(s)))ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(\phi(s)))ds \\
&- \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(\phi(s)))ds \Big| \\
&\leq \int_0^{t_1} \frac{(t_2-s)^{1-\alpha} - (t_1-s)^{1-\alpha}}{\Gamma(\alpha)(t_1-s)^{1-\alpha}(t_2-s)^{1-\alpha}} |f(s,x(\phi(s)))|ds \\
&+ \int_{t_1}^{t_2} \frac{1}{\Gamma(\alpha)(t_2-s)^{1-\alpha}} |f(s,x(\phi(s)))|ds.
\end{aligned}$$

This means that the class of functions $\{Fx\}$ is equicontinuous on Q_r and by Arzela-Ascoli Theorem [16] the class of functions $\{Fx\}$ is relatively compact, then the operator F is compact.
Now, let $\{x_n\} \subset Q_r$, and $x_n \to x$, then

$$Fx_n(t) = x_0 - \int_0^T h(s,x_n(s))ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x_n(\phi(s)))ds$$

and

$$\lim_{n\to\infty} Fx_n(t) = x_0 - \lim_{n\to\infty} \int_0^T h(s,x_n(s))ds + \lim_{n\to\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x_n(\phi(s)))ds.$$

Applying Lebesgue dominated convergence Theorem [16], then from our assumptions we get

$$\begin{aligned}
\lim_{n\to\infty} Fx_n(t) &= x_0 - \int_0^T h(s,\ \lim_{n\to\infty} x_n(s))ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,\ \lim_{n\to\infty} x_n(\phi(s)))ds \\
&= x_0 - \int_0^T h(s,x(s))ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(\phi(s)))ds = Fx(t).
\end{aligned}$$

This means that $Fx_n(t) \to Fx(t)$. Hence the operator F is continuous.
Now, by Schauder fixed point Theorem [16] there exists at least one solution $x \in C(I)$ of (3). Consequently there exists at least one solution $x \in C(I)$ of the problem (1)-(2).

2.1. **Uniqueness of the solution.** Now, replace the assumption (ii) and (iii) by $(ii)^*$ and $(iii)^*$ as follows:

- $(ii)^*$ $h : I \times R \to R$ is measurable in $t \in I\ \forall x \in R$ and satisfies Lipschitz condition,

$$|h(t,x) - h(t,y)| \leq b_1\ |x-y|\ \forall\ t \in I,\ x,y \in R. \tag{5}$$

$(iii)^*$ $f : I \times R \rightarrow R$ is measurable in $t \in I \ \forall x \in R$ and satisfies Lipschitz condition,

$$|f(t,x) - f(t,y)| \leq b_2 \ |x-y| \ \forall \ t \in I, \ x,y \in R. \tag{6}$$

So, we have the following Lemma.

Lemma 2.2. *The assumption $(ii)^*$ and $(iii)^*$ implies the assumption (ii) and (iii).*

Proof. From equation (5), let $y = 0$, then we have

$$|h(t,x)| - |h(t,0)| \leq |h(t,x) - h(t,0)| \leq b_1|x|,$$

$$|h(t,x)| \leq |h(t,0)| + b_1|x|$$

and

$$|h(t,x)| \leq |a(t)| + b_1|x|, \ where \ |a(t)| = \sup_{t\in I} |h(t,0)|.$$

Also, from equation (6), then we have

$$|f(t,x)| \leq |m(t)| + b_2|x|, \ where \ |m(t)| = \sup_{t\in I} |f(t,0)|.$$

Theorem 2.2. *Let the assumptions (i), $(ii)^*$, $(iii)^*$ and (iv) be satisfied, then the solution of integral equation (3) is unique.*

Proof. Let $x_1, \ x_2$ be two solutions in Q_r of (3), then

$$\begin{aligned}
|x_2(t) - x_1(t)| &= \Big| x_0 - \int_0^T h(s,x_2(s))ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x_2(\phi(s)))ds \\
&- x_0 + \int_0^T h(s,x_1(s))ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x_1(\phi(s)))ds \Big| \\
&\leq \int_0^T \Big| h(s,x_2(s)) - h(s,x_1(s)) \Big| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f(s,x_2(\phi(s))) - f(s,x_1(\phi(s))) \Big| ds \\
&\leq b_1 \int_0^T |x_2(s) - x_1(s)|ds + b_2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_2(\phi(s)) - x_1(\phi(s))|ds \\
&\leq b_1 \|x_2 - x_1\| \ T + b_2 \ \frac{T^\alpha}{\Gamma(\alpha+1)} \|x_2 - x_1\|.
\end{aligned}$$

Hence

$$\|x_2 - x_1\| \left(1 - \left(b_1 \ T + \ \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\right)\right) \quad \leq \quad 0,$$

then $x_1 = \ x_2$ and the solution of (3) is unique. Consequently the solution of the problem (1) and (2) is unique.

2.2. **Continuous dependence.**

Theorem 2.3. *Let the assumptions of Theorem 2.2 be satisfied for f, f^*, g and g^*. Then the unique solution $x \in C(I)$ depends continuously on the functions f and g in the sense that*

$$\forall \epsilon > 0, \ \exists \ \delta(\epsilon) \ such \ that$$

$$max \ \{ \ |x_0 - x_0^*|, \ |h(t,x(t)) - h^*(t,x(t))|, \ |f(t,x(t)) - f^*(t,x(t))|, \ |\phi - \phi^*| \ \} < \delta, then \ \|x - x^*\| < \epsilon.$$

where x^* be a solution of

$$x^*(t) = x_0^* - \int_0^T h^*(s, x_2^*(s))ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f^*(s, x^*(\phi^*(s)))ds.$$

Proof.

$$\begin{aligned}
|x(t) - x^*(t)| &= \Big| x_0 - \int_0^T h(s, x(s))ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\phi(s)))ds \\
&- x_0^* + \int_0^T h^*(s, x^*(s))ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f^*(s, x^*(\phi^*(s)))ds \Big| \\
&\leq |x_0 - x_0^*| + \int_0^T |h(s, x(s)) - h^*(s, x(s))|ds + \int_0^T |h^*(s, x(s)) - h^*(s, x^*(s))|ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f(s, x(\phi(s))) - f^*(s, x(\phi(s))) \Big| ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f^*(s, x(\phi(s))) - f^*(s, x^*(\phi(s))) \Big| ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| f^*(s, x^*(\phi(s))) - f^*(s, x^*(\phi^*(s))) \Big| ds \\
&\leq \delta + \delta\ T + b_1\ \|x - x^*\|\ T\ + \delta\ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\ \|x - x^*\| \\
&+ b_2\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x^*(\phi(s)) - x^*(\phi^*(s))|ds \\
&\leq \delta + \delta\ T + b_1\ \|x - x^*\|\ T\ + \delta\ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\ \|x - x^*\| \\
&+ \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\ \epsilon^*.
\end{aligned}$$

Hence

$$\|x - x^*\| \quad \leq \quad \frac{\left(1 + T + \frac{T^\alpha}{\Gamma(\alpha+1)}\right)\delta + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\ \epsilon^*}{1 - \left(b_1\ T + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\right)} = \epsilon.$$

2.3. **Hyers-Ulam stability.**

Definition 2.1. [5] *Let the solution $x \in C(I)$ of the problem (1)-(2) be exists, then the problem (1)-(2) is Hyers - Ulam stable if $\forall \epsilon > 0$, $\exists\ \delta(\epsilon)$ such that for any $\delta-$approximate solution x_s satisfies,*

$$\Big|\ {}^R D^\alpha (x_s(t) - x_s(0)) - f(t, x_s(\phi(t))) \Big| < \delta, \tag{7}$$

implies $\|x - x_s\| < \epsilon$.

Theorem 2.4. *Let the assumptions of Theorem 2.2 be satisfied, then the problem (1)-(2) is Hyers - Ulam stable.*

Proof. From (7), we have

$$\begin{aligned} -\delta &\leq {}^R D^{\alpha}(x_s(t)-x_s(0)) - f(t,x_s(\phi(t))) \leq \delta \\ -\delta^* = -\delta\, I^{\alpha} &\leq x_s(t)-x_s(0) - I^{\alpha} f(t,x_s(\phi(t))) \leq \delta\, I^{\alpha} = \delta^* \\ -\delta^* &\leq x_s(t) - \left(x_0 - \int_0^T h(s,x_s(s))ds + I^{\alpha} f(t,x_s(\phi(t)))\right) \leq \delta^*. \end{aligned}$$

Now,

$$\begin{aligned} |x(t)-x_s(t)| &= \left| x_0 - \int_0^T h(s,x(s))ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(\phi(s)))ds - x_s(t) \right| \\ &\leq \left| x_0 - \int_0^T h(s,x(s))ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(\phi(s)))ds \right. \\ &- \left. x_0 + \int_0^T h(s,x_s(s))ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x_s(\phi(s)))ds \right| \\ &+ \left| x_s(t) - (x_0 - \int_0^T h(s,x_s(s))ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x_s(\phi(s)))ds) \right| \\ &\leq \int_0^T |h(s,x(s)) - h(s,x_s(s))|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,x(\phi(s))) - f(s,x_s(\phi(s)))|ds \\ &+ \delta^* \\ &\leq b_1 \int_0^T |x(s)-x_s(s)|ds + b_2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(\phi(s)) - x_s(\phi(s))|ds + \delta^*, \end{aligned}$$

then

$$\|x-x_s\| \leq b_1\|x-x_s\|\, T + \frac{b_2 T^{\alpha}}{\Gamma(\alpha+1)} \|x-x_s\| + \delta^*.$$

Hence

$$\|x-x_s\| \leq \frac{\delta^*}{1-\left(b_1\, T + \frac{b_2 T^{\alpha}}{\Gamma(\alpha+1)}\right)} = \epsilon.$$

3. Examples

Example 1.
Taking into account the equation

$${}^R D^{\frac{1}{2}}(x(t)-x(0)) = \frac{t}{2} + \frac{1}{4}|x(\frac{t}{3})|, \ t \in (0,\ 1], \tag{8}$$

with nonlocal integral condition

$$x(0) + \int_0^1 (\frac{s}{3} + \frac{1}{6}|x(s)|)ds = 1. \tag{9}$$

Set

$$\begin{aligned} f(t,x(t)) &= \frac{t}{2} + \frac{1}{4}|x(\frac{t}{3})|, \\ h(t,x(t)) &= \frac{t}{3} + \frac{1}{6}|x(t)|. \end{aligned}$$

Putting

$$\begin{aligned}\|a\| &= \frac{1}{3},\ \|m\| \ = \frac{1}{2},\\ b_1 &= \frac{1}{6},\ b_2 = \frac{1}{4},\\ x_0 &= 1,\ \alpha = \ \frac{1}{2},\\ \phi \ &= \ \frac{1}{2},\ T = 1.\end{aligned}$$

We can find that

$$(b_1\ T \ + \frac{b_2 T^{\alpha}}{\Gamma(\alpha+1)}) = 0.4487614584 < 1$$

and

$$\begin{aligned} r &= \left(|x_0| + \|a\|T + \frac{T^{\alpha}\ \|m\|}{\Gamma(\alpha+1)}\right) \div \left(1 - (b_1 T + \frac{b_2 T^{\alpha}}{\Gamma(\alpha+1)})\right)\\ r &= 3.325843886.\end{aligned}$$

Then the Riemann-Liouville differential equation with nonlocal integral condition (8)-(9) has at least one solution $x \in C(I)$.

Example 2.

Consider the problem

$$^{R}D^{\frac{1}{2}}(x(t) - x(0)) = \frac{sin\ t}{3} + \frac{1}{2}|x(\frac{t}{5})|,\ t \in (0,\ 1], \tag{10}$$

with nonlocal integral condition

$$x(0) + \int_0^1 (\frac{cos\ s}{5} + \frac{1}{6}|x(s)|)ds = 1. \tag{11}$$

Set

$$\begin{aligned} f(t, x(t)) &= \frac{sin\ t}{3} + \frac{1}{2}|x(\frac{t}{5})|,\\ h(t, x(t)) &= \frac{cos\ t}{5} + \frac{1}{6}|x(t)|.\end{aligned}$$

Putting

$$\begin{aligned}\|a\| &= \frac{1}{5},\ \|m\| \ = \frac{1}{3},\\ b_1 &= \frac{1}{6},\ b_2 = \frac{1}{2},\\ x_0 &= 1,\ \alpha = \ \frac{1}{2},\\ \phi \ &= \ \frac{1}{5},\ T = 1.\end{aligned}$$

We can find that

$$(b_1\ T \ + \frac{b_2 T^{\alpha}}{\Gamma(\alpha+1)}) = 0.7308562502 < 1.$$

Moreover, we have

$$\begin{aligned} r &= \left(|x_0| + \|a\|T + \frac{T^\alpha \ \|m\|}{\Gamma(\alpha+1)}\right) \div \left(1 - (b_1 T + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)})\right) \\ &= 5.697079477. \end{aligned}$$

It is clear that all assumptions of Theorem 2.1 are satisfied. Hence there exist at least one solution $x \in C(I)$ of (10)-(11).

4. Conclusions

In this investigation, the Riemann-Liouville differential equation with nonlocal integral condition have been established on finite interval. We proved the existences of the solutions $x \in C(I)$ of the problem (1)-(2), by applying the technique Schauder fixed point Theorem. Next, we studied the continuous dependence of the unique solution on the x_0 and on the functions f, g and ϕ. Moreover, we thoroughly investigated the Hyers–Ulam stability of our problem. Finally, we given an examples are provided to illustrate our results.

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