*Electronic Journal of Mathematical Analysis and Applications*
Vol. 12(1) Jan. 2024, No. 8
ISSN: 2090-729(online)
ISSN: 3009-6731(print)
https://ejmaa.journals.ekb.eg/

# CLASS OF MEROMORPHIC FUNCTIONS RELATRD TO NEW OPERATOR

A. M. YEHIA, S. M. MADIAN AND M. M. TARWAT

Abstract. In this paper, using a meromorphic univalent function of the form $f(z) = \frac{1}{z-\delta} + \sum\limits_{k=j}^{\infty} a_k (z-\delta)^k$, with missing coefficients which has a simple pole at the point $z = \delta,\ 0 \le \delta < 1$, that is it is defined in $U^* = U\setminus\{\delta\} = \{z : z \in \mathbb{C} \text{ and } 0 < |z-\delta| < 1\}$, we note that for $\delta = 0$ we obtain the class $\sum_j$ of meromorphic functions. We define a new operator analogue of that of Aouf et al. [2] for univalent analytic functions defined in the open unit dick $U$ and introduce new class of meromorphic univalent functions which for different values of its parameters many special new classes can be obtained from it. For this class of functions we obtained some of its properties such as coefficient estimates, distortion theorem, the modified Hadamard product of two functions in it and also for function which its coefficients are the sum of the squares of the coefficients of two functions. Also we obtained analogue results of these results for each of the subclass obtained from this class.

## 1. Introduction

Let $\sum_{j,\delta}$, $0 \le \delta < 1$ be the class of functions

$$f(z) = \frac{1}{z-\delta} + \sum_{k=j}^{\infty} a_k (z-\delta)^k,\ j \in \mathbb{N} = \{1, 2, ...\}, \tag{1}$$

which are analytic and univalent in $U^* = U\setminus\{\delta\} = \{z : z \in \mathbb{C} \text{ and } 0 < |z-\delta| < 1\}$.
We note that for $\delta = 0$ we obtain the class $\sum_j$ of meromorphic function given by (Pommerenke [9])

$$f(z) = \frac{1}{z} + \sum_{k=j}^{\infty} a_k z^k,\ j \in \mathbb{N}.$$

---

2010 *Mathematics Subject Classification.* : 30C45.
*Key words and phrases.* Meromorphic univalent functions, Hadamard products, distortion theorems, meromorphically starlike and convex functions.
Submitted Jan. 18, 2024. Revised Feb. 8, 2024.

For $g \in \sum_{j,\delta}$, given by

$$g(z) = \frac{1}{z-\delta} + \sum_{k=j}^{\infty} b_k (z-\delta)^k , \tag{2}$$

$$(f * g)(z) = \frac{1}{z-\delta} + \sum_{k=j}^{\infty} a_k b_k (z-\delta)^k = (g * f)(z)$$

is the Hadamard product (or convolution) of $f$ and $g$.

**Definition 1.** A function $f \in \sum_{j,\delta}$ is said to be meromorphically starlike of order $\gamma$, $(0 \le \gamma < 1$ and $0 \le \delta < 1)$ if

$$-\Re\left\{\frac{(z-\delta) f'(z)}{f(z)}\right\} > \gamma \tag{3}$$

and meromorphically convex of order $\gamma$ if

$$-\Re\left\{1 + \frac{(z-\delta) f''(z)}{f'(z)}\right\} > \gamma. \tag{4}$$

Denote by $\sum_{j,\delta}^{*}(\gamma)$ and $\sum_{j,\delta}^{c}(\gamma)$ the classes of such functions, respectively. We note that

$$f(z) \in \sum\nolimits_{j,\delta}^{c}(\gamma) \iff -(z-\delta) f'(z) \in \sum\nolimits_{j,\delta}^{*}(\gamma).$$

The classes $\sum_{0,\ell}^{*}(\gamma)$ and $\sum_{0,\ell}^{c}(\gamma)$ where introduced and studied by Acu and Owa [1], Aouf [2, 3, 4], Miller [5], Mogra et al. [6], Owa et al. [7] and Pommerenke [8].For $\lambda > 0$, $\ell \ge 0$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \sum_{j,\delta}$, we define the operator $I_{\delta}^{n}(\lambda, \ell) : \sum_{j,\delta} \longrightarrow \sum_{j,\delta}$ by,

$$I_{\lambda,\ell}^{0} f(z) = f(z),$$

$$I_{\lambda,\ell}^{1} f(z) = (1-\lambda) I_{\lambda,\ell}^{0} f(z) + \frac{\lambda}{\ell (z-\delta)^{\ell}} \left((z-\delta)^{\ell+1} I_{\lambda,\ell}^{0} f(z)\right)'$$

$$= (z-\delta)^{-1} + \sum_{k=j}^{\infty} \left[1 + \frac{\lambda (k+1)}{\ell}\right] a_k (z-\delta)^k = I_{\lambda,\ell} f(z),$$

$$I_{\lambda,\ell}^{2} f(z) = (1-\lambda) I_{\lambda,\ell} f(z) + \frac{\lambda}{\ell (z-\delta)^{\ell}} \left((z-\delta)^{\ell+1} I_{\lambda,\ell} f(z)\right)'$$

$$= (z-\delta)^{-1} + \sum_{k=j}^{\infty} \left[1 + \frac{\lambda (k+1)}{\ell}\right]^2 a_k (z-\delta)^k$$

and (in general)

$$I_{\lambda,\ell}^{n} f(z) = (z-\delta)^{-1} + \sum_{k=j}^{\infty} \left[1 + \frac{\lambda (k+1)}{\ell}\right]^n a_k (z-\delta)^k . \tag{5}$$

**Definition 2.** For $0 < \alpha \le 1$, $-1 \le A < B \le 1$ and $0 \le \gamma < 1$, we say $f \in \sum_{s,\lambda,\ell}^{n}(A, B, \gamma, \alpha, \delta)$ if it satisfies:

$$\left| \frac{\frac{(z-\delta)\left(I_{\lambda,\ell}^{n}f(z)\right)'}{I_{\lambda,\ell}^{n}f(z)}+1}{\frac{B\,(z-\delta)\left(I_{\lambda,\ell}^{n}f(z)\right)'}{I_{\lambda,\ell}^{n}f(z)}+\left[B+(A-B)\,(1-\gamma)\right]} \right| < \alpha \quad (z\in U^{*}) \tag{6}$$

and

$$f\in \sum\nolimits_{c,\lambda,\ell}^{n}(A,B,\gamma,\alpha,\delta) \iff -(z-\delta)\,f'(z)\in \sum\nolimits_{s,\lambda,\ell}^{n}(A,B,\gamma,\alpha,\delta). \tag{7}$$

Let $\sum_{j,\delta}^{+}$ be subclass of $\sum_{j,\delta}$ consisits of the functions

$$f(z)=\frac{1}{z-\delta}-\sum_{k=j}^{\infty}a_{k}\,(z-\delta)^{k},\ \ a_{k}\geq 0 \tag{8}$$

and

$$\sum\nolimits_{s,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)=\sum\nolimits_{s,\lambda,\ell}^{n}(A,B,\gamma,\alpha,\delta)\cap \sum\nolimits_{j,\delta}^{+},$$

$$\sum\nolimits_{c,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)=\sum\nolimits_{c,\lambda,\ell}^{n}(A,B,\gamma,\alpha,\delta)\cap \sum\nolimits_{j,\delta}^{+}.$$

We note that:

(i) $\sum_{s,\lambda,\ell}^{n,+}(-1,1,0,\alpha,\delta)=\sum_{s,\lambda,\ell}^{n,+}(\alpha,\delta)=\left\{ f: \left| \frac{\frac{(z-\delta)\left(I_{\lambda,\ell}^{n}f(z)\right)'}{I_{\lambda,\ell}^{n}f(z)}+1}{\frac{(z-\delta)\left(I_{\lambda,\ell}^{n}f(z)\right)'}{I_{\lambda,\ell}^{n}f(z)}-1} \right| <\alpha \right\};$

(ii) $\sum_{c,\lambda,\ell}^{n,+}(-1,1,\gamma,\alpha,\delta)=\sum_{c,\lambda,\ell}^{n,+}(\gamma,\alpha,\delta)=\left\{ f: \left| \frac{\frac{(z-\delta)\left(I_{\lambda,\ell}^{n}f(z)\right)''}{\left(I_{\lambda,\ell}^{n}f(z)\right)'}+2}{\frac{(z-\delta)\left(I_{\lambda,\ell}^{n}f(z)\right)''}{\left(I_{\lambda,\ell}^{n}f(z)\right)'}+2\gamma} \right| <\alpha \right\};$

(iii) $\sum_{c,\lambda,\ell}^{n,+}(-1,1,\gamma,\alpha,0)=\sum_{c,\lambda,\ell}^{n,+}(\gamma,\alpha)=\left\{ f: \left| \frac{\frac{z\left(I_{\lambda,\ell}^{n}f(z)\right)''}{\left(I_{\lambda,\ell}^{n}f(z)\right)'}+2}{\frac{z\left(I_{\lambda,\ell}^{n}f(z)\right)''}{\left(I_{\lambda,\ell}^{n}f(z)\right)'}+2\gamma} \right| <\alpha \right\};$

(iv) $\sum_{c,\lambda,\ell}^{n,+}(-1,1,0,\alpha,0)=\sum_{c,\lambda,\ell}^{n,+}(\alpha)=\left\{ f: \left| \frac{\frac{z\left(I_{\lambda,\ell}^{n}f(z)\right)''}{\left(I_{\lambda,\ell}^{n}f(z)\right)'}+2}{\frac{z\left(I_{\lambda,\ell}^{n}f(z)\right)''}{\left(I_{\lambda,\ell}^{n}f(z)\right)'}} \right| <\alpha \right\};$

## 2. Main results

We shall assume that, $f$ defined by (8), $0<\alpha\leq 1$, $0\leq\gamma,\delta<1$, $\lambda>0$, $-1\leq A<B\leq 1$, $\ell\geq 0$, $n\in\mathbb{N}_0$ and $z\in U^*$.

**Theorem 1.** The function $f\in\sum_{s,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)$ if and only if

$$\sum_{k=j}^{\infty}\{(k+1)+\alpha\left[(Bk+A)+(B-A)\gamma\right]\}\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_k\leq(B-A)\alpha(1-\gamma). \quad (9)$$

**Proof.** Assume that (9) holds. Then

$$\left|(z-\delta)\left(I_{\lambda,\ell}^{n}f(z)\right)'+I_{\lambda,\ell}^{n}f(z)\right|-\alpha\left|B(z-\delta)\left(I_{\lambda,\ell}^{n}f(z)\right)'+\left[B+(A-B)(1-\gamma)\right]I_{\lambda,\ell}^{n}f(z)\right|$$

$$=\left|(z-\delta)\left(\frac{1}{(z-\delta)}-\sum_{k=j}^{\infty}\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_k(z-\delta)^k\right)'+\left(\frac{1}{(z-\delta)}-\sum_{k=j}^{\infty}\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_k(z-\delta)^k\right)\right|-$$

$$\alpha\left|B(z-\delta)\left(\frac{1}{(z-\delta)}-\sum_{k=j}^{\infty}\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_k(z-\delta)^k\right)'+\left(\frac{1}{(z-\delta)}-\sum_{k=1}^{\infty}\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_k(z-\delta)^k\right)(B+(A-B)(1-\gamma))\right|$$

$$=\left|\sum_{k=1}^{\infty}(k+1)\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_k(z-\delta)^k\right|-\alpha\left|(B-A)(1-\gamma)(z-\delta)^{-1}+\sum_{k=1}^{\infty}\left[(Bk+A)+(B-A)\gamma\right]\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_k(z-\delta)^k\right|$$

$$\leq\sum_{k=j}^{\infty}\{(k+1)-\alpha\left[(Bk+A)+(B-A)\gamma\right]\}\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_k-(B-A)\alpha(1-\gamma)$$

$$\leq 0.$$

Hence, by the maximum modulus theorem, we have $f\in\sum_{s,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)$. Conversely, let $f\in\sum_{s,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)$. Then

$$\left|\frac{\dfrac{(z-\delta)\left(I_{\lambda,\ell}^{n}f(z)\right)'}{I_{\lambda,\ell}^{n}f(z)}+1}{\dfrac{B(z-\delta)\left(I_{\lambda,\ell}^{n}f(z)\right)'}{I_{\lambda,\ell}^{n}f(z)}+\left[B+(A-B)(1-\gamma)\right]}\right|<\alpha,$$

that is

$$\frac{\left|\sum\limits_{k=j}^{\infty}(k+1)\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_{k}(z-\delta)^{k}\right|}{\left|(B-A)(1-\gamma)(z-\delta)^{-1}-\sum\limits_{k=j}^{\infty}\left[(Bk+A)+(B-A)\gamma\right]\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_{k}(z-\delta)^{k}\right|}<\alpha.$$

Since $\Re\{f(z)\}\leq|f(z)|$ for all $z$, we have

$$\Re\left\{\frac{\sum\limits_{k=j}^{\infty}(k+1)\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_{k}(z-\delta)^{k}}{(B-A)(1-\gamma)(z-\delta)^{-1}-\sum\limits_{k=j}^{\infty}\left[(Bk+A)+(B-A)\gamma\right]\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_{k}(z-\delta)^{k}}\right\}<\alpha. \tag{10}$$

Choosing $z$ real so that $\frac{(z-\delta)\left(I_{\lambda,\ell}^{n}f(z)\right)'}{I_{\lambda,\ell}^{n}f(z)}+1$ is real. Then upon clearing the denominator in (10) and letting $z-\delta\longrightarrow 1^{-}$, we have

$$\frac{\sum\limits_{k=j}^{\infty}(k+1)\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_{k}}{(B-A)(1-\gamma)-\sum\limits_{k=j}^{\infty}\left[(Bk+A)+(B-A)\gamma\right]\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_{k}}\leq\alpha.$$

That is

$$\sum_{k=j}^{\infty}\left\{(k+1)+\alpha\left[(Bk+A)+(B-A)\gamma\right]\right\}\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_{k}\leq(B-A)\alpha(1-\gamma),$$

which is the required condition.

From Theorem 1 and (7). we have the following corollary.

**Corollary 1.** The function $f\in\sum_{c,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)$ if and only if

$$\sum_{k=j}^{\infty}k\left\{(k+1)+\alpha\left[(Bk+A)+(B-A)\gamma\right]\right\}\left[1+\frac{\lambda(k+1)}{\ell}\right]^{n}a_{k}\leq(B-A)\alpha(1-\gamma). \tag{11}$$

**Theorem 2.** If $f\ \in\sum_{s,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)$, then

$$\frac{1}{|z-\delta|}-\frac{\alpha(B-A)(1-\gamma)}{\left\{(j+1)+\alpha\left[(Bj+A)+(B-A)\gamma\right]\right\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^{n}}|z-\delta|^{j}\leq|f(z)|$$

$$\leq\frac{1}{|z-\delta|}+\frac{\alpha(B-A)(1-\gamma)}{\left\{(j+1)+\alpha\left[(Bj+A)+(B-A)\gamma\right]\right\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^{n}}|z-\delta|^{j}, \tag{12}$$

and

$$\frac{1}{|z-\delta|^{2}}-\frac{j(B-A)\alpha(1-\gamma)}{\left\{(j+1)+\alpha\left[(Bj+A)+(B-A)\gamma\right]\right\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^{n}}|z-\delta|^{j-1}\leq\left|f'(z)\right|$$

$$\leq\frac{1}{|z-\delta|^{2}}+\frac{j\alpha(B-A)(1-\gamma)}{\left\{(j+1)+\alpha\left[(Bj+A)+(B-A)\gamma\right]\right\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^{n}}|z-\delta|^{j-1}. \tag{13}$$

The bounds in (12) and (13) are attained for

$$f(z)=\frac{1}{(z-\delta)}-\frac{(B-A)\,\alpha\,(1-\gamma)}{\{(j+1)+\alpha\,[(Bj+A)+(B-A)\,\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^{n}}(z-\delta)^{j}. \qquad (14)$$

**Proof.** First of all, for $\sum_{s,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)$, it follows from (9) that

$$\sum_{k=j}^{\infty}a_k\le\frac{\alpha\,(B-A)\,(1-\gamma)}{\{(j+1)+\alpha\,[(Bj+A)+(B-A)\,\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^{n}},$$

which, in view of (8) and for $|z-\delta|=r$, yields

$$|f(z)|\ge\frac{1}{r}-r^{j}\sum_{k=j}^{\infty}a_k$$

$$\ge\frac{1}{|z-\delta|}-\frac{\alpha\,(B-A)\,(1-\gamma)}{\{(j+1)+\alpha\,[(Bj+A)+(B-A)\,\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^{n}}|z-\delta|^{j},$$

and

$$|f(z)|\le\frac{1}{r}+r^{j}\sum_{k=j}^{\infty}a_k$$

$$\le\frac{1}{|z-\delta|}+\frac{\alpha\,(B-A)\,(1-\gamma)}{\{(j+1)+\alpha\,[(Bj+A)+(B-A)\,\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^{n}}|z-\delta|^{j}.$$

Next, we see from (9) that

$$\frac{\{(j+1)+\alpha\,[(Bj+A)+(B-A)\,\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^{n}}{j}\sum_{k=j}^{\infty}ka_k$$

$$\le\sum_{k=j}^{\infty}\{(k+1)+\alpha\,[(Bk+A)+(B-A)\,\gamma]\}\left[1+\frac{\lambda\,(k+1)}{\ell}\right]^{n}a_k\le(B-A)\,\alpha\,(1-\gamma),$$

then

$$\sum_{k=j}^{\infty}ka_k\le\frac{j\alpha\,(B-A)\,(1-\gamma)}{\{(j+1)+\alpha\,[(Bj+A)+(B-A)\,\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^{n}},$$

which, again in view of (8), yields

$$\left|f'(z)\right|\ge\frac{1}{|z-\delta|^{2}}-|z-\delta|^{j-1}\sum_{k=j}^{\infty}ka_k$$

$$\ge\frac{1}{|z-\delta|^{2}}-\frac{j\alpha\,(B-A)\,(1-\gamma)}{\{(j+1)+\alpha\,[(Bj+A)+(B-A)\,\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^{n}}|z-\delta|^{j-1},$$

and

$$\left|f'(z)\right|\le\frac{1}{|z-\delta|^{2}}+|z-\delta|^{j-1}\sum_{k=j}^{\infty}ka_k$$

$$\le\frac{1}{|z-\delta|^{2}}+\frac{j\alpha\,(B-A)\,(1-\gamma)}{\{(j+1)+\alpha\,[(Bj+A)+(B-A)\,\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^{n}}|z-\delta|^{j-1}.$$

Finally, it is easy to see that the bounds in (12) and (13) are attained for $f$ given by (14).

**Corollary 2.** If $f \in \sum_{c,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)$, then

$$\frac{1}{|z-\delta|} - \frac{\alpha(B-A)(1-\gamma)}{j\{(j+1)+\alpha[(Bj+A)+(B-A)\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^n}|z-\delta|^j \le |f(z)|$$

$$\le \frac{1}{|z-\delta|} + \frac{\alpha(B-A)(1-\gamma)}{j\{(j+1)+\alpha[(Bj+A)+(B-A)\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^n}|z-\delta|^j, \tag{15}$$

and

$$\frac{1}{|z-\delta|^2} - \frac{\alpha(B-A)(1-\gamma)}{\{(j+1)+\alpha[(Bj+A)+(B-A)\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^n}|z-\delta|^{j-1} \le \left|f'(z)\right|$$

$$\le \frac{1}{|z-\delta|^2} + \frac{(B-A)\alpha(1-\gamma)}{\{(j+1)+\alpha[(Bj+A)+(B-A)\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^n}|z-\delta|^{j-1}. \tag{16}$$

The bounds in (15) and (16) are attained for the function $f$ given by

$$f(z) = \frac{1}{(z-\delta)} - \frac{\alpha(B-A)(1-\gamma)}{j\{(j+1)+\alpha[(Bj+A)+(B-A)\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^n}(z-\delta)^j.$$

Let $f_1$ and $f_2$ defined by

$$f_i(z) = \frac{1}{z-\delta} + \sum_{k=j}^{\infty} a_{k,i}(z-\delta)^k \qquad (i=1,2). \tag{17}$$

**Theorem 3.** Let $f_i \in \sum_{s,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)$ $(i=1,2)$. Then $(f_1 * f_2)(z) \in \sum_{s,\lambda,\ell}^{n,+}(A,B,\beta,\alpha,\delta)$, where

$$\beta = 1 - \frac{\alpha(B-A)(1-\gamma)^2(1+\alpha B)(j+1)}{\{(j+1)+\alpha[(Bj+A)+(B-A)\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^n + (B-A)^2\alpha^2(1-\gamma)^2}.$$

The result is sharp for the functions $f_i$ $(i=1,2)$ given by

$$f_i(z) = \frac{1}{z-\delta} - \sum_{k=j}^{\infty}\frac{\alpha(B-A)(1-\gamma)}{\{(j+1)+\alpha[(Bj+A)+(B-A)\gamma]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^n}(z-\delta)^j \qquad (i=1,2) \tag{18}$$

**Proof.** Employing method of Schild and Silverman [10], we need to find the largest $\beta$ such that

$$\sum_{k=j}^{\infty}\frac{\{(k+1)+\alpha[(Bk+A)+(B-A)\beta]\}\left[1+\frac{\lambda(k+1)}{\ell}\right]^n}{\alpha(B-A)(1-\beta)}a_{k,1}a_{k,2} \le 1.$$

Indeed, since $f_i \in \sum_{s,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)$ $(i=1,2)$, then

$$\sum_{k=j}^{\infty}\frac{\{(k+1)+\alpha[(Bk+A)+(B-A)\gamma]\}\left[1+\frac{\lambda(k+1)}{\ell}\right]^n}{\alpha(B-A)(1-\gamma)}a_{k,i} \le 1 \quad (i=1,2). \tag{19}$$

Now, by the Cauchy-Schwarz inequality, we find from (19) that

$$\sum_{k=j}^{\infty}\frac{\{(k+1)+\alpha[(Bk+A)+(B-A)\gamma]\}\left[1+\frac{\lambda(k+1)}{\ell}\right]^n}{\alpha(B-A)(1-\gamma)}\sqrt{a_{k,1}a_{k,2}} \le 1. \tag{20}$$

So, we need only to show that

$$\frac{\{(k+1)+\alpha\left[(Bk+A)+(B-A)\beta\right]\}}{(1-\beta)}a_{k,1}a_{k,2}$$
$$\leq \frac{\{(k+1)+\alpha\left[(Bk+A)+(B-A)\gamma\right]\}}{(1-\gamma)}\sqrt{a_{k,1}a_{k,2}} \quad (k\geq j),$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{\{(k+1)+\alpha\left[(Bk+A)+(B-A)\gamma\right]\}(1-\beta)}{\{(k+1)+\alpha\left[(Bk+A)+(B-A)\beta\right]\}(1-\gamma)} \quad (k\geq j).$$

Hence, we may prove that

$$\frac{\alpha(B-A)(1-\gamma)}{\{(k+1)+\alpha\left[(Bk+A)+(B-A)\gamma\right]\}} \leq \frac{\{(k+1)+\alpha\left[(Bk+A)+(B-A)\gamma\right]\}(1-\beta)}{\{(k+1)+\alpha\left[(Bk+A)+(B-A)\beta\right]\}(1-\gamma)} \quad (k\geq j). \tag{21}$$

It follows from (21) that

$$\beta = 1-\frac{(B-A)\alpha(1-\gamma)^2(1+\alpha B)(k+1)}{\{(k+1)+\alpha\left[(Bk+A)+(B-A)\gamma\right]\}\left[1+\frac{\lambda(k+1)}{\ell}\right]^n+\alpha^2(B-A)^2(1-\gamma)^2} \quad (k\geq j). \tag{22}$$

Let

$$\Phi(k) = 1-\frac{\alpha(B-A)(1-\gamma)^2(1+\alpha B)(k+1)}{\{(k+1)+\alpha\left[(Bk+A)+(B-A)\gamma\right]\}\left[1+\frac{\lambda(k+1)}{\ell}\right]^n+\alpha^2(B-A)^2(1-\gamma)^2} \quad (k\geq j),$$

we see that $\Phi(k)$ is an increasing function of $k$ $(k\geq j)$. Therefore, we conclude from (22) that

$$\beta \leq \Phi(j) = 1-\frac{\alpha(B-A)(1-\gamma)^2(1+\alpha B)(j+1)}{\{(j+1)+\alpha\left[(Bj+A)+(B-A)\gamma\right]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^n+\alpha^2(B-A)^2(1-\gamma)^2}.$$

**Corollary 3.** Let each of the functions $f_i$ $(i=1,2)$ defined by (17)

be in the class $\sum_{c,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)$. Then $(f_1 * f_2)(z) \in \sum_{c,\lambda,\ell}^{n,+}(A,B,\eta,\alpha,\delta)$, where

$$\eta = 1-\frac{\alpha(B-A)(1-\gamma)^2(1+\alpha B)(j+1)}{j\{(j+1)+\alpha\left[(Bj+A)+(B-A)\gamma\right]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^n+\alpha^2(B-A)^2(1-\gamma)^2}.$$

The result is sharp for the functions $f_i$ $(i=1,2)$ given by

$$f_i(z) = \frac{1}{(z-\delta)}-\frac{\alpha(B-A)(1-\gamma)}{j\{(j+1)+\alpha\left[(Bj+A)+(B-A)\gamma\right]\}\left[1+\frac{\lambda(j+1)}{\ell}\right]^n}(z-\delta)^j \quad (i=1,2). \tag{23}$$

**Theorem 4.** Let $f_i$ $(i=1,2)$ defined by (17) be in the class $\sum_{c,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)$. Then the function $h(z)$ defined by

$$h(z) = \frac{1}{(z-\delta)}+\sum_{k=j}^{\infty}\left(a_{k,1}^2+a_{k,2}^2\right)(z-\delta)^k \tag{24}$$

belongs to the class $\sum_{s,\lambda,\ell}^{n,+}(A,B,\zeta,\alpha,\delta)$ where

$$\zeta = 1 - \frac{2\alpha(B-A)(1-\gamma)^2(1+\alpha B)(j+1)}{\{(j+1)+\alpha[(Bj+A)+(B-A)\gamma]\}^2\left[1+\frac{\lambda(j+1)}{\ell}\right]^n + 2\alpha^2(B-A)^2(1-\gamma)^2}.$$

The result is sharp for $f_i$ $(i=1,2)$ given by (20).

**Proof.** Noting that

$$\sum_{k=j}^{\infty}\left[\frac{\{(k+1)+\alpha[(Bk+A)+(B-A)\gamma]\}\left[1+\frac{\lambda(k+1)}{\ell}\right]^n}{\alpha(B-A)(1-\gamma)}\right]^2(a_{k,j})^2$$
$$\leq\left[\sum_{k=j}^{\infty}\frac{\{(k+1)+\alpha[(Bk+A)+(B-A)\gamma]\}\left[1+\frac{\lambda(k+1)}{\ell}\right]^n}{\alpha(B-A)(1-\gamma)}a_{k,j}\right]^2\leq 1,$$

for $f_i\in\sum_{s,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)$ $(i=1,2)$, we have

$$\sum_{k=j}^{\infty}\frac{\{(k+1)+\alpha[(Bk+A)+(B-A)\gamma]\}^2\left[1+\frac{\lambda(k+1)}{\ell}\right]^{2n}}{2\alpha^2(B-A)^2(1-\gamma)^2}\left[(a_{k,1})^2+(a_{k,2})^2\right]\leq 1.$$

Thus we need to find the largest $\zeta$ such that

$$\frac{\{(k+1)+\alpha[(Bk+A)+(B-A)\zeta]\}}{(1-\zeta)}$$
$$\leq\frac{\{(k+1)+\alpha[(Bk+A)+(B-A)\gamma]\}^2\left[1+\frac{\lambda(k+1)}{\ell}\right]^n}{2\alpha(B-A)(1-\gamma)^2}\quad(k\geq j),$$

that is, that

$$\zeta = 1-\frac{2\alpha(B-A)(1-\gamma)^2(1+\alpha B)(k+1)}{\{(k+1)+\alpha[(Bk+A)+(B-A)\gamma]\}^2\left[1+\frac{\lambda(k+1)}{\ell}\right]^n+2\alpha^2(B-A)^2(1-\gamma)^2}\quad(k\geq j).\tag{25}$$

Let

$$\varphi(k) = 1-\frac{2\alpha(B-A)(1-\gamma)^2(1+\alpha B)(k+1)}{\{(k+1)+\alpha[(Bk+A)+(B-A)\gamma]\}^2\left[1+\frac{\lambda(k+1)}{\ell}\right]^n+2\alpha^2(B-A)^2(1-\gamma)^2}\,(k\geq j),$$

we observe that $\varphi(k)$ is an increasing function of $k$ $(k\geq j)$. Therefore, we conclude from (25) that

$$\zeta\leq\varphi(j) = 1-\frac{2\alpha(B-A)(1-\gamma)^2(1+\alpha B)(j+1)}{\{(j+1)+\alpha[(Bj+A)+(B-A)\gamma]\}^2\left[1+\frac{\lambda(j+1)}{\ell}\right]^n+2\alpha^2(B-A)^2(1-\gamma)^2}.$$

**Corollary 4.** Let $f_i\in\sum_{c,\lambda,\ell}^{n,+}(A,B,\gamma,\alpha,\delta)$ $(i=1,2)$. Then $h(z)$ defined by (24) belongs to the class $\sum_{c,\lambda,\ell}^{n,+}(A,B,\rho,\alpha,\delta)$, where

$$\rho = 1-\frac{2\alpha(B-A)(1-\gamma)^2(1+\alpha B)(j+1)}{j\{(j+1)+\alpha[(Bj+A)+(B-A)\gamma]\}^2\left[1+\frac{\lambda(j+1)}{\ell}\right]^n+\alpha^2(B-A)^2(1-\gamma)^2}.$$

The result is sharp for the functions $f_1$ and $f_2$ given by (23).

**Acknowledgements**
The authors wish to thank Prof. Dr. M. K. Aouf for his kind encouragement and help in the preparation of this paper and the referees for their valuable comments and helpful suggestions.

## References

[1] M. Acu and S. Own. On some subclass of univalent functions, J. Inequality in Pure and Appl. Math, 6, $1-6$, 2005.
[2] M. K. Aouf, A certain subclass of meromorphic starlike functions with positive coefficients, Rend Mat. Appl., 9 , no. 7, $225-235$, 1989.
[3] M. K. Aouf, The generalized hypergeometric function and associatedfamiles of meromorphically multivalent functions, J. Math. Ineq. 3, no. 1, $43-62$, 2009.
[4] M. K. Aouf, H. E. Darwish and H. M. Srivastava, A general class ofmeromorphically univalent functions, PanAmer. Math. J., 7, no. 3, $69-84$, 1997.
[5] J. E. Miller, Convex meromorphic mapping and related functions, Proc. Amer. Math. Soc., 25, $220-228$, 1970.
[6] M. L. Mogra, T. R. Reddy, and O. P. Juneja, Meromorphic univalent functions with positive coefficients, Bull. Austral. Math. Soc., 32, $161-176$, 1985.
[7] S. Owa, M. Nunokawa and H. Saitoh, Generalization of certain sub classes of meromorphic univalent functions, J. Fac. Sci. Tech. Kinki Univ., 25, $21-27$, 1989.
[8] Ch. Pommerenke, On meromorphic starlike functions, Pacific J. Math., 13, $221-235$, 1963.
[9] Ch. Pommerenke, $\ddot{U}$ber einige Klassen meromorfer schlichter Funktion Math. Zeitschr., 78, $263-284$, 1962.
[10] A. Schild and H. Silverman, Convolution of univalent functions with negative coefficients, Ann. Math Curie-Sklodowska Sect. A, 29, $99-107$, 1975.

A. M. Yehia, Dept. of Math. and Computer Sci., Faculty of Science, Beni Suef, University, Egypt.
*Email address*: abdelrahmnyehia@gmail.com

S. M. Madian, Basic Science Dept., Higher Institute for, Engineering and, Technology, New Damietta, Egypt
*Email address*: samar_math@yahoo.com

M. M. Tharwat, Dept. of Math. and Computer Sci., Faculty of Science, Beni Suef, University, Egypt.
*Email address*: zahraa26@yahoo.com