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COMMON FIXED POINT THEOREMS IN CONNECTION WITH TWO WEAKLY COMPATIBLE MAPPINGS IN MENGER SPACE WITH BICOMPLEX-VALUED METRIC

SARMILA BHATTACHARYYA, CHINMAY BISWAS, TANMAY BISWAS

Abstract. It is well-known that the fixed point theory plays a very important role in theory and applications. In 2017, Choi et al. [4] introduced the notion of bicomplex valued metric spaces (bi-CVMS) and established common fixed point results for weakly compatible mappings. On the other hand, in 1942, K. Menger [14] initiated the study of probabilistic metric spaces where he replaced the distance function $d(x, y)$ by distribution function $Fx, y(t)$, where the value of $Fx, y(t)$ is interpreted as the probability that the distance between x and y be less than t, $t > 0$. In this paper, we have used bicomplex-valued metric on a set. We have taken $Fx, y(t)$ as the probability that norm of the distance between x and y be less than t, i.e., $||d(x, y)|| < t$, $t > 0$ and initiated menger space with bicomplex valued metric. We also aim to prove certain common fixed point theorems for a pair of weakly compatible mappings satisfying (CLRg) or (E.A) property in this space.

1. Introduction, Definitions and Notations

In 2011, Azam et al. [1] introduced the notion of complex valued metric space (CVMS) as a generalization and extension of cone metric space and classical metric space. In 2017, Choi et al. [4] linked the concepts of bicomplex numbers and complex valued metric spaces and introduced the notion of bicomplex valued metric spaces (bi-CVMS). They established common fixed point results for weakly compatible mappings. For more characteristics in the direction of CVMS and bi-CVMS, we refer the researchers [1] to [13]. It is well known that the fixed point theory plays a very important role in theory and applications, in particular, whose importance comes from finding roots of algebraic equation and numerical analysis.

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On the other hand, in 1942 Menger [14] initiated the study of probabilistic metric spaces. He replaced the distance function $d(x, y)$, the distance between two point x and y by distribution function $Fx, y(t)$, where the value of $Fx, y(t)$ is interpreted as the probability that the distance between x, y, i.e $d(x, y)$ is less than t, $t > 0$.

Let $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$, and $\mathbb{N}$ be the sets of real numbers, nonnegative real numbers, complex numbers, and positive integers, respectively. The set of bicomplex numbers denoted by $\mathbb{C}_2$ is the first setting in an infinite sequence of multicomplex sets which are generalizations of the set of complex numbers $\mathbb{C} = \{z = x + iy|x, y \in \mathbb{R}$ and $i^2 = -1)$. Segre [17] set out the notion of bicomplex numbers, which is as follows :

$\mathbb{C}_2 = \{w = p_0 + i_1p_1 + i_2p_2 + i_1i_2p_3|p_k \in \mathbb{R},\ k = 0, 1, 2, 3\}$

Since each element w in $\mathbb{C}_2$ be written as

$w = p_0 + i_1p_1 + i_2(p_2 + i_1p_3)$

or $w = z_1 + i_2z_2$ $(z_1, z_2 \in \mathbb{C})$

we can also express $\mathbb{C}_2$ as

$\mathbb{C}_2 = \{w = z_1 + i_2z_2\ |z_1, z_2 \in \mathbb{C}\}$

where $z_1 = p_0 + i_1p_1$, $z_2 = p_2 + i_1p_3$ and i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$. The product of i_1 and i_2 defines a hyperbolic unit j such that $j^2 = 1$. The products of all units are commutative and satisfy

$i_1i_2 = j$, $i_1j = -\ i_2$, $i_2j = -1$.

Let $u = u_1 + i_2u_2 \in \mathbb{C}_2$ and $v = v_1 + i_2v_2 \in \mathbb{C}_2$. A partial order relation $\precsim_{i_2}$ defined on $\mathbb{C}_2$, for details one may see [4]. A norm of a bicomplex number $w = z_1 + i_2z_2$ denoted by $||w||$ is defined by

$$||w|| = ||z_1 + i_2z_2|| = \left(|z_1|^2 + |z_2|^2\right)^{\frac{1}{2}}$$

which, upon choosing $w = p_0 + i_1p_1 + i_2p_2 + i_1i_2p_3$ $(p_k \in \mathbb{R},\ k = 0, 1, 2, 3)$, gives

$$||w|| = \left(p_0^2 + p_1^2 + p_2^2 + p_3^2\right)^{\frac{1}{2}}.$$

For details about bicomplex numbers, one may see [15]. For any two bicomplex numbers u, $v \in \mathbb{C}_2$, one can easily verify that $0 \precsim_{i_2} u \precsim_{i_2} v$ which implies $||u|| \leq ||v||$; $||u + v|| \leq ||u|| + ||v||$; $||\alpha u|| \leq \alpha||u||$ where α is non-negative real number. Choi et al. [4] have defined a bicomplex-valued metric as follows: Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}_2$ be a bicomplex-valued metric on X if it satisfies the following properties: For $x, y, z \in X$,

(M_1) $0 \precsim_{i_2} d(x, y)$ for all $x, y \in X$;

(M_2) $d(x, y) = 0$ if and only if $x = y$;

(M_3) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(M_4) $d(x, y) \precsim_{i_2} d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then (X, d) is called a bicomplex-valued metric space.

A sequence in a nonempty set X is a function $x : \mathbb{N} \to X$, which is expressed by its range set $\{x_n\}$ where $x(n) = x_n$ $(n \in \mathbb{N})$. A sequence $\{x_n\}$ in a bicomplex-valued metric space (X, d), is said to converge to $x \in X$ if and only if for any $0 \precsim_{i_2} \varepsilon \in \mathbb{C}_2$, there exists $n_0 \in \mathbb{N}$ depending on ε such that $d(x_n, x) \precsim_{i_2} \varepsilon$ for all $n > n_0$. It is denoted by $x_n \to x$ as $n \to \infty$. A sequence $\{x_n\}$ in a bicomplex-valued metric space (X, d) is said to be a Cauchy sequence if and only if for any $0 \precsim_{i_2} \varepsilon \in \mathbb{C}_2$, there exists $n_0 \in \mathbb{N}$ depending on ε such that $d(x_m, x_n) \precsim_{i_2} \varepsilon$ for all $m, n > n_0$. A bicomplex-valued metric space (X, d) is said to be complete if and only if every Cauchy sequence in X converges in X.

Let (X, d) be a metric space and $S, T : X \to X$ be two mappings. A point $x \in X$ is said to be a common fixed point of S and T if and only if

$$Sx = Tx = x.$$

Let (X, d) be a metric space. The self maps S and T on X are said to be commuting if $STx = TSx$ for all $x \in X$. The self maps S and T are said to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = l$ for some $l \in X$. The self-maps S and T are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$, that is, they commute at their coincidence point. The self-maps S and T are said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = l$ for some $l \in X$.

Suppose that (X, d) is a metric space and $f, g : X \to X$. Then f and g are said to satisfy the (CLRg) property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$$

for some $x \in X$ (see [18]). The property (CLRg) is seen to be stronger than the property (E.A).

In this paper, we have used bicomplex-valued metric on a non-empty set. Here we have taken $Fx, y(t)$ as the probability that norm of the distance between x and y be less than t, i.e.,$||d(x, y)|| < t, t > 0$ and initiated menger space with bicomplex-valued metric. Also we aim to prove certain common fixed point theorems for a pair of weakly compatible mappings satisfying (CLRg) (or (E.A)) property in this space. We have used some well-known definitions of probabilistic metric space and menger space which will still work for menger space with bicomplex-valued metric.

A probabilistic metric space (briefly PM space) (see [14, 16]) is an ordered pair (X, Γ), where X is a non-empty set of elements and Γ is a mapping of $X \times X$ into a collection Δ_+ of all distribution functions F (a distribution function F is a nondecreasing and left continuous mapping from the set of real numbers to $[0, 1]$ with $\inf F(t) = 0$ and $\sup F(t) = 1$). The value of F at $(x, y) \in X \times X$ will be denoted by $F_{x,y}$. The function $F_{x,y}$, $x, y \in X$, are assumed to satisfy the following conditions:

(a) $F_{x,y}(t) = 1$ for all $t > 0$, iff $x = y$,
(b) $F_{x,y}(0) = 0$,
(c) $F_{x,y}(t) = F_{y,x}(t)$,
(d) If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(t + s) = 1$ for all $x, y, z \in X$ and $s, t \geq 0$.

A triangle norm (briefly t-norm) (see [16]) is a mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies
(1) $\Delta(a, 1) = a$ for all $a \in [0, 1]$, $\Delta(0, 0) = 0$,
(2) $\Delta(a, b) = \Delta(b, a)$,
(3)$\Delta(c, d) > \Delta(a, b)$ for $c > a$, $d > b$,
(4) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$ for all $a, b, c \in [0, 1]$.

Three sample examples of continuous t-norms are the minimum t-norm $\Delta_M(x, y) = \min(x, y)$, the product t-norm $\Delta_p(x, y) = x.y$, the Lukasiecz t-norm $\Delta_L(x, y) = \max(x + y - 1, 0)$.

A menger space (see [16]) is a tripled (X, F, Δ) where (X, F) is a PM space and Δ is a t-norm such that the inequality

$$F_{x,z}(t+s) \geq \Delta(F_{x,y}(t), F_{y,z}(s))$$

holds for all $x, y, z \in X$ and $t, s \geq 0$.

A sequence $\{x_n\}$ in a menger space (X, F, Δ) is said to converge to a point x in X if for each $\varepsilon > 0$ and $\lambda \in (0,1)$, there exist is an integer $M(\varepsilon, \lambda) \in N$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ for all $n \geq M(\varepsilon, \lambda)$.

2. Main results

In this section, we established the main results of this paper.

Theorem 2.1. *Let X be a set of elements and (X, F, Δ_M) be a menger space with a bicomplex-valued metric, where Δ_M is the minimum t-norm. Let S and T be two self maps on X such that*

i. S and T are weakly compatible,

ii. S and T satisfy CLR_S property and

iii. $F_{Tx,Ty}(t) \geq \max\{F_{Sx,Sy}(\frac{t}{\alpha}),\ F_{Tx,Sy}(\frac{t}{\beta}),\ \alpha F_{Sx,Sy}(\frac{t}{\alpha}) + \beta F_{Tx,Sy}(\frac{t}{\beta})\}$ where α, β are real numbers with $\alpha > 0$, $\beta > 0$ and $0 < \alpha + \beta < 1$.

Then S and T have unique common fixed point.

Proof. Since S and T satisfy CLR_S property, there exists a sequence $\{x_n\}$ in X, such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Sa$$

for some $a \in X$.

Since $\lim_{n\to\infty} Sx_n = Sa$, for $\varepsilon = \frac{t}{\alpha} > 0$ and $\lambda \in (0,1)$, there exists $M_1(\varepsilon, \lambda) \in \mathbb{N}$, such that

$$F_{Sx_n,Sa}\left(\frac{t}{\alpha}\right) > 1 - \lambda, \text{ for all } n > M_1(\varepsilon, \lambda) \text{ and } \alpha \in (0,1).$$

Since $\lim_{n\to\infty} Tx_n = Sa$, for $\varepsilon = \frac{t}{\beta} > 0$ and $\lambda \in (0,1)$, there exists $M_2(\varepsilon, \lambda) \in \mathbb{N}$, such that

$$F_{Tx_n,Sa}\left(\frac{t}{\beta}\right) > 1 - \lambda, \text{ for all } n > M_2(\varepsilon, \lambda) \text{ and } \beta \in (0,1).$$

If we take

$$M = \max\{M_1(\varepsilon, \lambda), M_2(\varepsilon, \lambda)\},$$

then for all $n > M$ and $\alpha, \beta \in (0,1)$,

$$F_{Sx_n,Sa}\left(\frac{t}{\alpha}\right) > 1 - \lambda \text{ and } F_{Tx_n,Sa}\left(\frac{t}{\beta}\right) > 1 - \lambda.$$

Now from (iii), we get

$$\begin{aligned} & F_{Tx_n,Ta}(t) \\ \geq\ & \max\left\{F_{Sx_n,Sa}\left(\frac{t}{\alpha}\right), F_{Tx_n,Sa}\left(\frac{t}{\beta}\right), \alpha F_{Sx_n,Sa}\left(\frac{t}{\alpha}\right) + \beta F_{Tx_n,Sa}\left(\frac{t}{\beta}\right)\right\} \\ >\ & \max\{1-\lambda,\ 1-\lambda,\ \alpha(1-\lambda) + \beta(1-\lambda)\} \\ =\ & \max\{1-\lambda,\ 1-\lambda,\ (\alpha+\beta)(1-\lambda)\} \\ =\ & 1-\lambda. \end{aligned}$$

Which implies that the sequence $\{Tx_n\}$ converges to Ta,

$$\text{i.e., } \lim_{n\to\infty} Tx_n = Ta,$$
$$\text{i.e., } Sa = Ta.$$

Again S and T are weakly compatible.
Therefore,

$$TTa = TSa = STa = SSa.$$

Now, we show that Ta is a common fixed point of S and T,

$$\text{i.e., } TTa = STa = Ta,$$

it is enough to show that $TTa = Ta$.

Replacing x, y by x_n, Ta respectively in condition (iii), we have

$$\begin{aligned}&F_{Tx_n,TTa}(t)\\ \geq\ & \max\left\{F_{Sx_n,STa}\left(\frac{t}{\alpha}\right), F_{Tx_n,STa}\left(\frac{t}{\beta}\right), \alpha F_{Sx_n,STa}\left(\frac{t}{\alpha}\right) + \beta F_{Tx_n,STa}\left(\frac{t}{\beta}\right)\right\}.\end{aligned}$$

Now,

$$\begin{aligned} F_{Tx_n,TTa}(t) &\geq \Delta_M\left\{F_{Tx_n,Ta}(t_1), F_{Ta,TTa}(t_2)\right\}, (t_1+t_2=t)\\ &= \min\left\{F_{Tx_n,Ta}(t_1), F_{Ta,TTa}(t_2)\right\}, (t_1+t_2=t).\end{aligned}$$

Since $\lim_{n\to\infty} Tx_n = Ta$, we can choose n large enough such that $F_{Tx_n,Ta}(t_1) \geq F_{Ta,TTa}(t_2)$. So, $F_{Tx_n,TTa}(t) \geq F_{Ta,TTa}(t_2)$. Using same min norm we can show that $F_{Sx_n,STa}\left(\frac{t}{\alpha}\right) \geq F_{Sa,STa}\left(\frac{t_2}{\alpha}\right)$ and $F_{Tx_n,STa}\left(\frac{t}{\beta}\right) \geq F_{Ta,STa}\left(\frac{t_2}{\beta}\right)$. Hence,

$$\begin{aligned}&F_{Tx_n,TTa}(t)\\ \geq\ & F_{Ta,TTa}(t_2)\\ \geq\ & \max\left\{F_{Sa,STa}\left(\frac{t_2}{\alpha}\right), F_{Ta,STa}\left(\frac{t_2}{\beta}\right), \alpha F_{Sa,STa}\left(\frac{t_2}{\alpha}\right) + \beta F_{Ta,STa}\left(\frac{t_2}{\beta}\right)\right\},\end{aligned}$$

$$\begin{aligned}&\text{i.e., } F_{Ta,TTa}(t_2)\\ \geq\ & \max\left\{F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right), F_{Ta,TTa}\left(\frac{t_2}{\beta}\right), \alpha F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right) + \beta F_{Ta,TTa}\left(\frac{t_2}{\beta}\right)\right\},\end{aligned}$$

$$\text{i.e., } F_{Ta,TTa}(t_2) \geq F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right) \text{ and } F_{Ta,TTa}(t_2) \geq F_{Ta,TTa}\left(\frac{t_2}{\beta}\right).$$

Since $\alpha, \beta \in (0,1)$,

$$\frac{t_2}{\alpha} > t_2 \text{ and } \frac{t_2}{\beta} > t_2.$$

So,

$$F_{Ta,TTa}(t_2) \leq F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right) \text{ and } F_{Ta,TTa}(t_2) \leq F_{Ta,TTa}\left(\frac{t_2}{\beta}\right).$$

$$\text{which implies } F_{Ta,TTa}(t_2) = F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right) = F_{Ta,TTa}\left(\frac{t_2}{\beta}\right).$$

This is possible only if

$$\begin{aligned} F_{Ta,TTa}(t) &= 1, \text{ for all } t > 0,\\ \text{i.e., } Ta &= TTa.\end{aligned}$$

Therefore,

$$Ta = TTa = STa.$$

Hence Ta is the common fixed point of S and T.
To show the uniqueness, if possible, let there exists $b \in X$ such that $Sb = Tb = b$.
Now,

$$\begin{aligned}
& F_{Ta,b}(t) \\
= \;& F_{Ta,Tb}(t) \\
\geq \;& \max\left\{F_{Sa,Sb}\left(\frac{t}{\alpha}\right), F_{Ta,Sb}\left(\frac{t}{\beta}\right), \alpha F_{Sa,Sb}\left(\frac{t}{\alpha}\right) + \beta F_{Ta,Sb}\left(\frac{t}{\beta}\right)\right\} \\
= \;& \max\left\{F_{Ta,Tb}\left(\frac{t}{\alpha}\right), F_{Ta,Tb}\left(\frac{t}{\beta}\right), \alpha F_{Ta,Tb}\left(\frac{t}{\alpha}\right) + \beta F_{Ta,Tb}\left(\frac{t}{\beta}\right)\right\}.
\end{aligned}$$

Which implies

$$F_{Ta,Tb}(t) \geq F_{Ta,Tb}\left(\frac{t}{\alpha}\right) \text{ and } F_{Ta,Tb}(t) \geq F_{Ta,Tb}\left(\frac{t}{\beta}\right).$$

Since $\alpha, \beta \in (0,1)$, $\frac{t}{\alpha} > t$ and $\frac{t}{\beta} > t$.
Therefore,

$$F_{Ta,Tb}(t) \leq F_{Ta,Tb}\left(\frac{t}{\alpha}\right) \text{ and } F_{Ta,Tb}(t) \leq F_{Ta,Tb}\left(\frac{t}{\beta}\right).$$

So,

$$F_{Ta,Tb}(t) = F_{Ta,Tb}\left(\frac{t}{\alpha}\right) = F_{Ta,Tb}\left(\frac{t}{\beta}\right).$$

Which implies $F_{Ta,Tb}(t) = 1$, for all $t > 0$, i.e., $F_{Ta,b}(t) = 1$, for all $t > 0$ which implies $Ta = b$.

Hence Ta is unique common fixed point of S and T. □

Corollary 2.1. *Let (X, F, Δ_M) be a menger space with a bicomplex-valued metric. Let S and T be two weakly compatible self maps on X such that*

i. S and T satisfy CLR_T property,

ii. $TX \subset SX$ and

iii. $F_{Tx,Ty}(t) \geq \max\{F_{Sx,Sy}(\frac{t}{\alpha}),\ F_{Tx,Sy}(\frac{t}{\beta}),\ \alpha F_{Sx,Sy}(\frac{t}{\alpha}) + \beta F_{Tx,Sy}(\frac{t}{\beta})\}$ where α, β are real numbers with $\alpha > 0$, $\beta > 0$ and $0 < \alpha + \beta < 1$, then S and T have unique common fixed point.

Proof. Since S and T satisfy CLR_T property, there exists a sequence $\{x_n\}$ in X, such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tb,$$

for some $b \in X$.

Since $TX \subset SX$, $Tb = Sa$ for some $a \in X$. Thus

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Sa,$$

for some $a \in X$. Which means S and T satisfy CLR_S property.

Hence by Theorem 2.1, S and T have unique common fixed point. □

Corollary 2.2. *Let (X, F, Δ_M) be a menger space with bicomplex-valued metric and S, $T : X \to X$ be two weakly compatible mappings defined on X such that*
i. S and T satisfy (E.A) property,
ii. SX is a complete subspace of X and
iii. $F_{Tx,Ty}(t) \geq \max\{F_{Sx,Sy}(\frac{t}{\alpha}),\ F_{Tx,Sy}(\frac{t}{\beta}),\ \alpha F_{Sx,Sy}(\frac{t}{\alpha}) + \beta F_{Tx,Sy}(\frac{t}{\beta})\}$ where α, β are real numbers with $\alpha > 0$, $\beta > 0$ and $0 < \alpha + \beta < 1$, then S and T have unique common fixed point.

Proof. Since S and T satisfy (E.A) property, there exists a sequence $\{x_n\}$ in X, such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some $t \in X$.
Since SX is a complete subspace of X, $t = Sa$ for some $a \in X$.
Hence

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Sa,$$

for some $a \in X$.
Which means S and T satisfy CLR_S property.
Hence by Theorem 2.1, S and T have unique common fixed point. □

Theorem 2.2. *Let (X, F, Δ_M) be a menger space with bicomplex-valued metric and S, $T : X \to X$ be two weakly compatible mappings defined on X such that*
i. S and T satisfy CLR_S property and
ii. $F_{Tx,Ty}(t) \geq \max\{F_{Sx,Sy}(\frac{t}{\alpha}),\ F_{Tx,Sy}(\frac{t}{\alpha})\}$ where $x, y \in X$ and $\alpha \in (0, 1)$. Then S and T have unique common fixed point.

Proof. Since S and T satisfy CLR_S property, there exists a sequence $\{x_n\}$ in X, such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Sa,$$

for some $a \in X$.
Hence for $t > 0$, $\lambda \in (0, 1)$, there exists $M(t, \lambda) \in \mathbb{N}$, such that

$$F_{Sx_n,Sa}\left(\frac{t}{\alpha}\right) > 1 - \lambda \text{ and } F_{Tx_n,Sa}\left(\frac{t}{\alpha}\right) > 1 - \lambda, \text{ for all } n > M(t, \lambda).$$

So,

$$\begin{aligned} F_{Tx_n,Ta}(t) &\geq \max\left\{F_{Sx_n,Sa}\left(\frac{t}{\alpha}\right),\ F_{Tx_n,Sa}\left(\frac{t}{\alpha}\right)\right\} \\ &> \max\{1 - \lambda,\ 1 - \lambda\} \\ &= 1 - \lambda. \end{aligned}$$

$$\begin{aligned} \text{i.e., } \lim_{n\to\infty} Tx_n &= Ta, \\ \text{which implies } Sa &= Ta. \end{aligned}$$

Since S and T are weakly compatible,

$$TTa = TSa = STa = SSa.$$

Now, replacing x, y by x_n, Ta respectively in condition (ii), we have

$$F_{Tx_n,TTa}(t) \geq \max\left\{F_{Sx_n,STa}\left(\frac{t}{\alpha}\right),\ F_{Tx_n,STa}\left(\frac{t}{\alpha}\right)\right\}.$$

Now,

$$\begin{aligned} F_{Tx_n,TTa}(t) &\geq \Delta_M\left\{F_{Tx_n,Ta}\left(\frac{t_1}{\alpha}\right), F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right)\right\}, (t_1+t_2=t) \\ &= \min\left\{F_{Tx_n,Ta}\left(\frac{t_1}{\alpha}\right), F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right)\right\}, (t_1+t_2=t). \end{aligned}$$

Since, $\lim_{n\to\infty} Tx_n = Ta$, we can choose n so large such that

$$F_{Tx_n,Ta}\left(\frac{t_1}{\alpha}\right) \geq F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right).$$

Therefore,

$$F_{Tx_n,TTa}(t) \geq F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right).$$

Using same min norm, we get

$$F_{Sx_n,STa}\left(\frac{t}{\alpha}\right) \geq F_{Sa,STa}\left(\frac{t_2}{\alpha}\right) \text{ and } F_{Tx_n,STa}\left(\frac{t}{\alpha}\right) \geq F_{Ta,STa}\left(\frac{t_2}{\alpha}\right).$$

Therefore,

$$\begin{aligned} F_{Tx_n,TTa}(t) &\geq F_{Ta,TTa}(t_2) \\ &\geq \max\left\{F_{Sa,STa}\left(\frac{t_2}{\alpha}\right), F_{Ta,STa}\left(\frac{t_2}{\alpha}\right)\right\} \\ &= \max\left\{F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right), F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right)\right\} \\ &= F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right), \end{aligned}$$

$$\text{which implies } F_{Ta,TTa}(t_2) \geq F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right).$$

As $\alpha \in (0,1)$, $\frac{t}{\alpha} > t$, so,

$$F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right) \geq F_{Ta,TTa}(t_2),$$

$$\text{which implies } F_{Ta,TTa}(t_2) = F_{Ta,TTa}\left(\frac{t_2}{\alpha}\right).$$

This is possible only if

$$\begin{aligned} F_{Ta,TTa}(t) &= 1, \text{ for all } t > 0, \\ \text{i.e., } Ta &= TTa, \end{aligned}$$

$$\text{which implies } Ta = TTa = STa.$$

Hence Ta is the common fixed point of S and T.

If possible, let there exists $b \in X$ such that $Sb = Tb = b$.

Therefore,

$$\begin{aligned} F_{Ta,b}(t) &= F_{Ta,Tb}(t) \\ &\geq \max\left\{F_{Sa,Sb}\left(\frac{t}{\alpha}\right), F_{Ta,Sb}\left(\frac{t}{\alpha}\right)\right\} \\ &= \max\left\{F_{Ta,Tb}\left(\frac{t}{\alpha}\right), F_{Ta,Tb}\left(\frac{t}{\alpha}\right)\right\}. \end{aligned}$$

Which implies

$$F_{Ta,Tb}(t) \geq F_{Ta,Tb}\left(\frac{t}{\alpha}\right).$$

Since $\alpha \in (0,1)$, $\frac{t}{\alpha} > t$, so

$$F_{Ta,Tb}(t) \leq F_{Ta,Tb}\left(\frac{t}{\alpha}\right).$$

Hence

$$F_{Ta,Tb}(t) = F_{Ta,Tb}\left(\frac{t}{\alpha}\right).$$

Which implies $F_{Ta,Tb}(t) = 1$, for all $t > 0$, i.e., $F_{Ta,b}(t) = 1$, for all $t > 0$ which implies $Ta = b$.

Hence Ta is unique common fixed point of S and T. □

The following Corollary can be easily obtained in the line of Corollary 2.1, so we omit its proof.

Corollary 2.3. *Let (X, F, Δ_M) be a menger space with bicomplex-valued metric and S, $T : X \to X$ be two weakly compatible mappings defined on X such that*
i. S and T satisfy CLR_T property,
ii. $TX \subset SX$ and
iii. $F_{Tx,Ty}(t) \geq \max\{F_{Sx,Sy}(\frac{t}{\alpha}),\ F_{Tx,Sy}(\frac{t}{\alpha})\}$ where $x, y \in X$ and $\alpha \in (0,1)$. Then S and T have unique common fixed point.

We state the following Corollary without its proof, as it can be easily obtained from Corollary 2.2.

Corollary 2.4. *Let (X, F, Δ_M) be a menger space with bicomplex-valued metric and S, $T : X \to X$ be two weakly compatible mappings defined on X such that*
i. S and T satisfy (E.A) property,
ii. SX is a complete subspace of X and
iii. $F_{Tx,Ty}(t) \geq \max\{F_{Sx,Sy}(\frac{t}{\alpha}),\ F_{Tx,Sy}(\frac{t}{\alpha})\}$ where $x, y \in X$ and $\alpha \in (0,1)$. Then S and T have unique common fixed point.

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Sarmila Bhattacharyya
Department of Mathematics, Netaji Mahavidyalaya,
P.O.- Arambagh, Dist.-Hooghly, PIN-712601, West Bengal, India.
Email address: sarmilabhattacharyya1982@gmail.com

Chinmay Biswas
Department of Mathematics, Nabadwip Vidyasagar College,
Nabadwip, Dist.- Nadia, PIN-741302, West Bengal, India.
Email address: chinmay.shib@gmail.com

Tanmay Biswas
Rajbari, Rabindrapally, R. N. Tagore Road,
P.O. Krishnagar, P.S.-Katwali, Dist.-Nadia, PIN- 741101, West Bengal, India.
Email address: tanmaybiswas_math@rediffmail.com