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UNIQUENESS RESULTS ON DIFFERENCE PRODUCT OF ENTIRE FUNCTIONS

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Abstract. In this research article, we have studied the results of P. Sahoo and H. Karmakar, intending to determine, in any manner, whether it is possible to relax the nature of sharing by replacing the shift of non-constant transcendental entire functions of finite order with the product of shift. In this direction, we have investigated the uniqueness of shift difference polynomials of two entire functions when they share a non-zero polynomial with a finite weight and one being the Mobius transformation of the other satisfying $n \geq 2d - \sigma + 3$, and also when they share a small function with a finite weight satisfying $n \geq m + \sigma + 5$. We also investigate the same situation when the original functions f and g share the value zero counting multiplicities (CM) satisfying $n > 2(\Gamma_1 + d) - \sigma$. Our results in this paper extend and generalizes the previous results of P. Sahoo and H. Karmakar [Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), 2017, 52 (2), pp. 102-110].

1. Introduction and main results

Let $\mathbb{C}$ denote the complex plane and $f(z)$ a meromorphic function on $\mathbb{C}$. Here, we assume that the reader is familiar with the fundamental results and standard notations of Nevanlinna theory, found in Yang and Yi [31], L. Yang [34], and Hayman [11], such as $T(r, f), N(r, f)$, and $m(r, f)$ and so on. For any non-constant meromorphic function $h(z)$ we define $S(r, h) = o(T(r, h)), (r \to \infty, r \notin E)$, where E denotes any set of positive real numbers having finite linear measure. For a given value $a \in \mathbb{C} \cup \{\infty\}$ and two meromorphic functions $f(z)$ and $g(z)$, we say f and g share a IM (Ignoring multiplicity) if $f(z)$ and $g(z)$ have the same a-points; we say f and g share a CM (Counting multiplicity) if $f(z)$ and $g(z)$ have the same a-points with the same multiplicities.
We now recall the following definition from [13].

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Definition 1.1. [13] *Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_k(a, f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, then we say that f and g share the value a with weight k.*

The definition implies that if f and g share a value a with weight k, then z_0 is an a-points of f with multiplicity $m \leq k$ if and only if it is an a-ponts of g with multiplicity $m \leq k$, and z_0 is an a-points of f with multiplicity $m > k$ if and only if it is an a-ponts of g with multiplicity $n > k$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f and g share the value a with weight k. It is clear that if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p \leq k$. Also, note that f, g share the value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

We denote $\rho(f)$ for order of $f(z)$. $\rho_2(f)$ is hyper order of $f(z)$, defined as in [31] and some of the standard definitions that we have followed in this paper can be found in [13, 36].

It is well known that for $n \geq 2$ and for transcendental entire function f, $f^n(z)f(z+\eta) - P_0(z)$ has infinitely many zeros (see [18]). In 2011, X. M. Li, W. L. Li, H. X. Yi and Z. T. Wen, considering the sharing value problem and $n \geq 4$, proved that uniqueness of entire functions whose difference polynomials share a nonzero polynomial CM and share a meromorphic function of a smaller order (see [20]). In the same year X. Luo and W. C. Lin, dealt with the value distribution of difference products of entire functions and presented some result on two difference products of entire functions sharing one value with the same multiplicities (see [21]). Later, in 2016 W. L. Li. and X. M. Li., dealt with the uniqueness results on two difference products of entire functions sharing one value by considering that the functions share the value zero, counting multiplicities and investigate the situation where the difference products share a nonzero polynomial instead, by confining its degree and generalize the previous concerning results (see [19]).

Regarding Theorems 1.1 , 1.2 (see [20]) and 5 (see [19]), in 2017, P. Sahoo and H. Karmakar [27] proved the following theorems.

Theorem A [27]. Let f and g be two distinct transcendental entire functions of finite order, and let $P_0(\not\equiv 0)$ be a polynomial. Suppose that η is a non-zero complex constant, and $n \geq 4$ is an integer such that $2deg(P_0) < n+1$. Suppose that $f^n(z)f(z+\eta) - P_0(z)$ and $g^n(z)g(z+\eta) - P_0(z)$ share (0, 2). If $n \geq 4$ and $f^n(z)f(z+\eta)/P_0(z)$ is a Mobius transformation of $g^n(z)g(z+\eta)/P_0(z)$, or, if $n \geq 6$, then one of the following two cases hold.

(i) $f = tg$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$;
(ii) $f = e^Q$ and $g = te^{-Q}$, where P_0 reduces to a non-zero constant c, t is a constant such that $t^{n+1} = c^2$, and Q is a non-constant polynomial.

Theorem B [27]. Let f and g be two transcendental entire functions of finite order, and let $\alpha(\not\equiv 0, \infty)$ be a meromorphic function such that $\rho(\alpha) < \rho(f)$. Suppose that η is a non-zero complex number, and n and m are positive integers such that $n \geq m+6$. If $f^n(z)(f^m(z)-1)f(z+\eta)$ and $g^n(z)(g^m(z)-1)g(z+\eta)$ share $(\alpha, 2)$, then $f = tg$, where t is a constant satisfying $t^m = 1$.

Theorem C [27]. Let f and g be two transcendental entire functions of finite order such that f and g share 0 CM, and let $P_0(\not\equiv 0)$ be a polynomial. Suppose that η is a non-zero complex constant and n is an integer such that $deg(P_0) < n+1$. Assume that $P(f(z))f(z+\eta) - P_0$ and $P(g(z))g(z+\eta) - P_0$ share (0, 2). If $n > 2\Gamma_1 + 1$ and $P(f(z))f(z+\eta)/P_0$ is a Mobius transformation of $P(g(z))g(z+\eta)/P_0$, or if $n > 2\Gamma_2 + 1$, then one of the following two cases hold:

(i) $f = tg$, where $t^d = 1$;
(ii) $f = e^{\beta}$, $g = te^{-\beta}$, where P_0 reduces to a non-zero constant c, t is a constant such that $t^{n+1} = c^2$, and β is a non-constant polynomial.

A number of authors have shown their interest to find the uniqueness of entire and meromorphic functions whose differential polynomials share certain values or fixed points, and obtained some remarkable results (see [2, 10, 16, 26, 31]), and in recent years, the difference variant of the Nevanlinna theory has been established in [7, 8, 9, 23, 24]. Using these theories, some mathematicians in the world began to study the uniqueness questions of meromorphic functions sharing values with their shifts, and study the value distribution and uniqueness of differences and difference polynomials, and produced many fine works, for example, see ([5, 6, 12, 16, 17, 21, 25, 28, 29, 30, 35, 37]).

Regarding Theorems A, B and C, naturally, we pose the following question, which is the motivation of the present paper.

Question 1.1. Is it possible to relax the nature of sharing in some way by replacing shift with the product of a shift in Theorems A, B, and C ?

In this paper, our aim is to determine a potential response to question 1.1. The following are the main results of this paper:

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two distinct transcendental entire functions of finite order, and let $\mathcal{P}_0(\not\equiv 0)$ be a polynomial. Suppose that η is a non-zero complex constant, and $n \geq 2d-\sigma+3$ is an integer such that $2deg(\mathcal{P}_0) < n+\sigma$. Let γ be a non-negative integer such that $f^n(z)\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j} - \mathcal{P}_0$ and $g^n(z)\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j} - \mathcal{P}_0$ share $(0,\gamma)$. If $n \geq 2d-\sigma+3$ and $f^n(z)\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}/\mathcal{P}_0$ is a Mobius transformation of $g^n(z)\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}/\mathcal{P}_0$, or one of the following conditions holds:*

(1) $\gamma \geq 2$ *and* $n \geq \sigma+5$*;*
(2) $\gamma = 1$ *and* $n \geq \sigma+\frac{d}{2}+6$*;*
(3) $\gamma = 0$ *and* $n \geq \sigma+2d+8$,

then one of the following conclusions can be realized:

(a) $f=\tau g$, *where* $\tau \neq 1$, *is a constant satisfying* $\tau^{n+\sigma}=1$;
(b) $f=e^U$ *and* $g=\tau e^{-U}$, *where* $\mathcal{P}_0$ *reduces to a non-zero constant* κ, τ *is a constant such that* $\tau^{n+\sigma}=\kappa^2$ *and* U *is a non-constant polynomial.*

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and, let $\zeta(\not\equiv 0,\infty)$ be a meromorphic function such that $\rho(\zeta) < \rho(f)$. Suppose that η is a non-zero complex number, and n and m are positive integers such that $n \geq m+\sigma+5$. If $f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}$ and $g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}$ share $(\zeta,2)$, then $f\equiv\tau g$, where τ is a constant satisfying $\tau^m=1$.*

Theorem 1.3. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order such that $f(z)$ and $g(z)$ share 0 CM, and let $\mathcal{P}_0(\not\equiv 0)$ be a polynomial. Let $\mathcal{P}_n(z)$ be a non-zero polynomial of degree m. Let $\Gamma_1 = m_1+m_2$ and $\Gamma_2 = m_1+2m_2$, where m_1 and m_2 are the number of simple zeros and multiple zeros of $\mathcal{P}_n(z)$ respectively. Suppose that η is a non-zero complex constant and n is an integer such that $deg(\mathcal{P}_0) < n+\sigma$. Assume that $\mathcal{P}_n(f(z))\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j} - \mathcal{P}_0$ and $\mathcal{P}_n(g(z))\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j} - \mathcal{P}_0$ share $(0,\gamma)$. If $n > 2(\Gamma_1+d)-\sigma$ and $\mathcal{P}_n(f(z))\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}/\mathcal{P}_0$ is a Mobius transformation of $\mathcal{P}_n(g(z))\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}/\mathcal{P}_0$, or one of the following conditions holds:*

(1) $\gamma \geq 2$ *and* $n > 2\Gamma_2+\sigma$ *;*
(2) $\gamma = 1$ *and* $n > 2\Gamma_2+\frac{\Gamma_1+d}{2}+\sigma$ *;*
(3) $\gamma = 0$ *and* $n > 2\Gamma_2+4(\Gamma_1+d)+\sigma$,

then one of the following conclusions can be realized:

(a) $f=\tau g$, *where* $\tau^d=1$;
(b) $f=e^Q$ *and* $g=\tau e^{-Q}$, *where* $\mathcal{P}_0$ *reduces to a non-zero constant* κ, τ *is a constant such that* $\tau^{n+\sigma}=\kappa^2$ *and* Q *is a non-constant polynomial.*

Example 1.1. Let $f(z)=e^z$ and $g=\tau\frac{1}{e^{-z}}, \sigma=d=1$, and τ is a constant such that $\tau^{n+1}=1$, let η be any non-zero complex constant. Then for any given polynomial $\mathcal{P}_0$

such that $\mathcal{P}_0 \neq 0$ with $2deg(\mathcal{P}_0) < n+\sigma$, $f(z)^n f(z+\eta) - \mathcal{P}_0(z)$ and $g(z)^n g(z+\eta) - \mathcal{P}_0(z)$ share 1 CM. Here, f and g satisfy the conclusion (a) of Theorem 1.1.

Example 1.2. Let $f(z) = e^z, g(z) = \tau e^z$. Let c and τ are non-zero constants such that $e^c \neq 1, \tau^{n+m+1} = 1$ and let $d = \sigma = 1, \zeta = 1$, hence $\rho(\zeta) < \rho(f)$ and $n \geq m+6$. Then $f^n(f^m - 1)(f(z+c))$ and $g^n(g^m-1)(g(z+c))$ share 1 CM. Clearly, f and g satisfy the conclusion of Theorem 1.2.

Example 1.3. Let $\mathcal{P}_n(z) = z^4, f(z) = (z+2)^2(z+3)e^{(z-2)^2}, g(z) = (z+2)^2(z+3)e^{-(z-2)^2}, \mathcal{P}_0 = (z+2)^8(z+3)^4(z+4)^4(z+5), \sigma = 1, d = 1, s_1 = 1, \Gamma_1 = 2$. Thus, we have that f, g are of finite order 2 and f, g share 0 CM and $n = 4 > 3 = 2(\Gamma_1 + d) - \sigma$. Clearly, $\mathcal{P}_n(f)f(z+\eta) - \mathcal{P}_0$ and $\mathcal{P}_n(g)g(z+\eta) - \mathcal{P}_0$ share 1 CM. But, we get $f \not\equiv \tau g$ for a constant d such that $\tau^d = 1$.

Example 1.4. Let $f(z) = \frac{e^{2\pi i z/\eta}}{e^{e^{2\pi i z/\eta}}}$ and $g(z) = \frac{1}{e^{e^{2\pi i z/\eta}}}$, where η is a non-zero constant. Then it is easy to verify that $f(z)^n f(z+\eta)$ and $g(z)^n g(z+\eta)$ share 1 CM. But there does not exist a non-zero constant τ such that $f = \tau g$ or $fg = \tau$, where $\tau^{n+1} = 1$. This example shows that Theorem 1.1 is not true for infinite order entire functions.

Remark 1.1. When $\sigma = d = 1$ in Theorems 1.1, 1.2 and 1.3, these Theorems reduce to Theorems A, B and C and the results generalize and extend.

2. Auxiliary Lemmas

In this section, we will present some lemmas that will be used to prove the main results. Let $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions. Henceforth, we shall denote by $\mathcal{H}$ the following function.

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}-1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}-1}\right). \tag{2.1}$$

Lemma 2.1 [31]. Let $f(z)$ be a non-constant meromorphic function in the complex plane, and let

$$\mathcal{P}_n(f(z)) = a_n f^n(z) + a_{n-1}f^{n-1}(z) + ... + a_1 f(z) + a_0, \tag{2.2}$$

where $a_0, a_1, ..., a_n$ are constants and $a_n \neq 0$. Then $m(r, \mathcal{P}(f)) = nm(r,f) + O(1)$.

Lemma 2.2 [7]. Let $f(z)$ be a meromorphic function of finite order $\rho(f) < \infty$, and let $\eta(\neq 0)$ be a complex number. Then, for each $\epsilon > 0$, we have

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O(r^{\rho-1+\epsilon}).$$

Lemma 2.3 [7]. Let $f(z)$ be a meromorphic function of finite order $\rho(f) < \infty$, and let $\eta(\neq 0)$ be a complex number. Then, for each $\epsilon > 0$, we have

$$T(r, f(z+\eta)) = T(r,f) + O(r^{\rho-1+\epsilon}) + O(logr).$$

Lemma 2.4 [22]. Let f be a non-constant meromorphic function and let

$$R(f) = \sum_{i=0}^{n} a_i f^i / \sum_{j=0}^{m} b_j f^j$$

be an irreducible rational function in f with constant coefficients $\{a_i\}$ and $\{b_i\}$ where, $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r,f) + S(r,f), \text{where } d = max\{n, m\}.$$

Lemma 2.5. Let f be a transcendental entire function of finite order $\rho(\alpha) < \infty$, and let $\eta(\neq 0)$ be a complex number. Suppose that $\mathcal{F} = \mathcal{P}_n(f(z)) \prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}$, where $\mathcal{P}_n(z)$ is as in (2.2). Then

$$T(r, \mathcal{F}) = (n+\sigma)T(r,f) + O(r^{\rho-1+\epsilon}) + S(r,f).$$

Besides, we have $S(r,\mathcal{F}) = S(r,f)$.

Proof. Noting that f is an entire function of finite order ρ, in view of Lemmas 2.1 and 2.2 and the standard Valiron-Mohon'ko theorem, we can write

$$\begin{aligned}(n+\sigma)T(r,f) &= T\left(r,\mathcal{P}_n(f(z))f^{\sigma}(z)\right) + S(r,f)\\ &= m\left(r, \frac{\mathcal{P}_n(f(z))f^{\sigma}(z)\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}{\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}\right) + S(r,f)\\ &\le m(r,\mathcal{F}) + m\left(r, \frac{f^{\sigma}(z)}{\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}\right) + O(r^{\rho-1+\epsilon}) + S(r,f)\\ &\le T(r,\mathcal{F}) + O(r^{\rho-1+\epsilon}) + S(r,f). \qquad (2.3)\end{aligned}$$

On the otherhand, by Lemmas 2.1 and 2.3 and the fact that f is a transcendental entire function of finite order, we obtain

$$\begin{aligned}T(r,\mathcal{F}) &\le T\left(r,\mathcal{P}_n(f(z))\right) + T\left(r, \prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + S(r,f)\\ &= nT(r,f) + T\left(r, \prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + S(r,f)\\ &\le (n+\sigma)T(r,f) + O(r^{\rho-1+\epsilon}) + S(r,f). \qquad (2.4)\end{aligned}$$

Now the result follows from (2.3) and (2.4). □

Lemma 2.6. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, $\eta(\neq 0)$ be a complex constant, $\zeta(z)$ be a small function of f and g, $\mathcal{P}_n(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ be a non-zero polynomial, where $a_0, a_1, ..., a_n(\neq 0)$ are complex constants, and let $n > \Gamma_1$ be an integer. If $\mathcal{P}_n(f(z))\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}$ and $\mathcal{P}_n(g(z))\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}$ share $\zeta(z)$ IM, then $\rho(f) = \rho(g)$.

Proof. Similar to the proof of Lemma 4 in [19], we can easily obtain the proof of Lemma 2.6. □

Lemma 2.7 [33]. Let $\mathcal{F}(z)$ and $\mathcal{G}(z)$ be two non-constant meromorphic functions such that $\mathcal{G}$ is a Mobius transformation of $\mathcal{F}$. Suppose that there exists a subset $I \subset \mathbb{R}^+$ with linear measure $mesI = +\infty$ such that for $r \in I$ and $r \to \infty$

$$\overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{G}) < (\lambda + 0(1))T(r,\mathcal{G}),$$

where $\lambda < 1$. If there exists a point $z_0 \in \mathbb{C}$ satisfying $\mathcal{F}(z_0) = \mathcal{G}(z_0) = 1$, then either $\mathcal{F} = \mathcal{G}$ or $\mathcal{F}\mathcal{G} = 1$.

Lemma 2.8 [34]. Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Then

$$N\left(r,\infty;\frac{f}{g}\right) - N\left(r,\infty;\frac{g}{f}\right) = N(r,\infty;f) + N(r,0;g) - N(r,\infty;g) - N(r,0;f).$$

Lemma 2.9. Let $f(z)$ be a transcendental entire function of finite order and $n \in \mathbb{N}$. Let $\mathcal{F} = f^n(z)\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}$, where $\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j} \not\equiv 0$. Then

$$nT(r,f) \le T(r,\mathcal{F}) - N\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + S(r,f).$$

Proof. Using Lemmas 2.2 and 2.8, and the first fundamental theorem of Nevanlinna, we obtain

$$\begin{aligned}
m(r, f^{n+\sigma+1}(z)) &= m\left(r, \frac{f^{\sigma+1}(z)\mathcal{F}(z)}{\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}\right)\\
&\leq m(r,\mathcal{F}(z)) + m(r, f(z)) + m\left(r, \frac{f^{\sigma}(z)}{\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}\right) + S(r,f)\\
&\leq m(r,\mathcal{F}(z)) + m(r, f(z)) + T\left(r, \frac{f^{\sigma}(z)}{\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}\right)\\
&\quad - N\left(r,\infty; \frac{f^{\sigma}(z)}{\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}\right) + S(r,f)\\
&\leq m(r,\mathcal{F}(z)) + m(r, f(z)) + T\left(r, \frac{\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}{f^{\sigma}(z)}\right)\\
&\quad - N\left(r,\infty; \frac{f^{\sigma}(z)}{\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}\right) + S(r,f)
\end{aligned}$$

$$\begin{aligned}
m(r, f^{n+\sigma+1}(z) &\leq m(r,\mathcal{F}(z)) + m(r, f(z)) + N\left(r,\infty; \frac{\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}{f^{\sigma}(z)}\right)\\
&\quad + m\left(r, \frac{\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}{f^{\sigma}(z)}\right) - N\left(r,\infty; \frac{f^{\sigma}(z)}{\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}\right)\\
&\quad + S(r,f)\\
&\leq m(r,\mathcal{F}(z)) + m(r, f(z)) + N(r,0; f^{\sigma}(z))\\
&\quad - N\left(r,0; \prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + S(r,f)
\end{aligned}$$

i.e.,

$$\begin{aligned}
m(r, f^{n+\sigma+1}(z)) \leq T(r,\mathcal{F}) + (\sigma+1)T(r,f) - N\left(r,0; \prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right)\\
+ S(r,f).
\end{aligned}$$

By Lemma 2.4, we obtain

$$\begin{aligned}
(n+\sigma+1)T(r,f) = m(r, f^{n+\sigma+1}) &\leq T(r,\mathcal{F}) + (\sigma+1)T(r,f)\\
&\quad - N\left(r,0; \prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + S(r,f),
\end{aligned}$$

i.e.,

$$nT(r,f) \leq T(r,\mathcal{F}) - N\left(r,0; \prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + S(r,f).$$

□

Lemma 2.10 [1]. If f, g be two non-constant meromorphic functions sharing $(1,1)$, then

$$2\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + N_E^{(2}(r,1;f) - \overline{N}_{f>2}(r,1;g) \leq N(r,1;g) - \overline{N}(r,1;g).$$

Lemma 2.11 [3]. If f, g be two non-constant meromorphic functions sharing $(1,1)$, then

$$\overline{N}_{f>2}(r,1;g) \leq \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_0(r,0;f') + S(r,f),$$

where $N_0(r,0;f')$ is the counting function of those zeros of f' which are not the zeros of $f(f-1)$.

Lemma 2.12 [32]. If f, g be two non-constant meromorphic functions sharing $(1,0)$ and $\mathcal{H} \not\equiv 0$, then

$$N_E^{1)}(r,1;f) \leq N(r,0;\mathcal{H}) + S(r,f) \leq N(r,\infty;\mathcal{H}) + S(r,f) + S(r,g).$$

Lemma 2.13 [3]. If f, g be two non-constant meromorphic functions such that they share $(1,0)$, then

$$\begin{aligned}&\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + N_E^{(2}(r,1;f) - \overline{N}_{f>1}(r,1;g) - \overline{N}_{g>1}(r,1;f)\\ &\leq N(r,1;g) - \overline{N}(r,1;g).\end{aligned}$$

Lemma 2.14 [3]. If f, g be share $(1,0)$, then

(i) $\overline{N}_L(r,1;f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f)$.

(ii) $\overline{N}_{f>1}(r,1;g) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + S(r,f)$.

(iii) $\overline{N}_{g>1}(r,1;f) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_0(r,0;g') + S(r,g)$.

Lemma 2.15 [15]. If f, g be two non-constant meromorphic functions that share $(1,0), (\infty,0)$ and $\mathcal{H} \not\equiv 0$, then

$$\begin{aligned}N(r,\infty;\mathcal{H}) \leq\ &\overline{N}(r,0;f|\geq 2) + \overline{N}(r,0;g|\geq 2) + \overline{N}_*(r,1;f,g) + \overline{N}_*(r,\infty;f,g)\\ &+ \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g') + S(r,f) + S(r,g),\end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Lemma 2.16 [14]. If $N\left(r,0;f^{(k)}|f\neq 0\right)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then

$$N\left(r,0;f^{(k)}|f\neq 0\right) \leq k\overline{N}(r,\infty;f) + N(r,0;f|<k) + k\overline{N}(r,0;f|\geq k) + S(r,f).$$

Lemma 2.17 [4]. Let $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions sharing $(1,2), (\infty,0)$ and $\mathcal{H} \neq 0$. Then the following assertions holds.

$$\begin{aligned}(i)\ \ T(r,\mathcal{F}) \leq\ &N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{G}) + \overline{N}_*(r,\infty;\mathcal{F},\mathcal{G})\\ &- m(r,1;\mathcal{G}) - N_E^{(3}(r,1;\mathcal{F}) - \overline{N}_L(r,1;\mathcal{G}) + S(r,\mathcal{F}) + S(r,\mathcal{G});\end{aligned}$$

$$\begin{aligned}(ii)\ \ T(r,\mathcal{G}) \leq\ &N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{G}) + \overline{N}_*(r,\infty;\mathcal{F},\mathcal{G})\\ &- m(r,1;\mathcal{F}) - N_E^{(3}(r,1;\mathcal{G}) - \overline{N}_L(r,1;\mathcal{F}) + S(r,\mathcal{F}) + S(r,\mathcal{G}).\end{aligned}$$

Lemma 2.18 [36]. Let $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions and let $\mathcal{H} \equiv 0$. If

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{G})}{T(r)} < 1,$$

where $T(r) = max\{T(r,\mathcal{F}), T(r,\mathcal{G})\}, r \in I$ and I is a set with infinite linear measure, then either $\mathcal{F} \equiv G$ or $\mathcal{F}\mathcal{G} \equiv 1$.

3. Proof of Theorems

Proof of Theorem 1.1.

Proof. Let $\mathcal{F} = \frac{f^n(z)\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}{\mathcal{P}_0(z)}$ and $\mathcal{G} = \frac{g^n(z)\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}}{\mathcal{P}_0(z)}$. Then $\mathcal{F}$ and $\mathcal{G}$ are two transcendental meromorphic functions that share $(1,\gamma)$. From Lemma 2.5, we get

$$T(r,\mathcal{F}) = (n+\sigma)T(r,f) + O\left(r^{\rho(f)+\epsilon-1}\right) + O(logr). \tag{3.1}$$

$$T(r,\mathcal{G}) = (n+\sigma)T(r,g) + O\left(r^{\rho(g)+\epsilon-1}\right) + O(logr). \tag{3.2}$$

Since f and g are of finite order, it follows from (3.1) and (3.2) that $\mathcal{F}$ and $\mathcal{G}$ are also of finite order. Moreover, from Lemma 2.6, we deduce that $\rho(f) = \rho(g) = \rho(\mathcal{F}) = \rho(\mathcal{G})$.

We now discuss the following two cases separately.

Case 1. Suppose that $\mathcal{F}$ is a Mobius transformation of $\mathcal{G}$. i.e.,

$$\mathcal{F} = \frac{A\mathcal{G}+B}{C\mathcal{G}+D}, \tag{3.3}$$

where A, B, C and D are complex constants satisfying $AD - BC \neq 0$. Let z_0 be a 1-point of $\mathcal{F}$. Since $\mathcal{F}, \mathcal{G}$ share $(1,\, \gamma)$, z_0 is also a 1-point of $\mathcal{G}$. Therefore, from (3.3), we obtain $A + B = C + D$, and hence (3.3) can be written as

$$\mathcal{F} - 1 = \frac{\mathcal{G}-1}{\alpha\mathcal{G}+\beta},$$

where $\alpha = C/(A-C)$ and $\beta = D/(A-C)$. From this we can say that $\mathcal{F}, \mathcal{G}$ share $(1,\, \infty)$. Now using the standard Valiron-Mohon'ko Lemma 2.4, we obtain from (3.3) that

$$T(r,\mathcal{F}) = T(r,\mathcal{G}) + O(logr).$$

Then from (3.1) and (3.2) and the fact that f and g are transcendental entire functions of finite order, we deduce

$$\frac{T(r,f)}{T(r,g)} \to 1, \quad \frac{T(r,\mathcal{F})}{T(r,f)} \to n+\sigma, \quad \text{as} \quad r\to\infty, \quad r \in I. \tag{3.4}$$

From Lemma 2.3 and the condition that f and g are transcendental entire functions, we have

$$\begin{aligned}\overline{N}(r,0;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{F}) &= \overline{N}(r,0;f^n(z)) + \overline{N}\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + O(logr)\\ &\leq (d+1)T(r,f) + O\left(r^{\rho(f)+\epsilon-1}\right) + O(logr),\end{aligned}$$

as $r \to \infty$ and $r \in I$. Similarly, we get

$$\begin{aligned}\overline{N}(r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{G}) &= \overline{N}(r,0;g^n(z)) + \overline{N}\left(r,0;\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}\right) + O(logr)\\ &\leq (d+1)T(r,g) + O\left(r^{\rho(g)+\epsilon-1}\right) + O(logr),\end{aligned}$$

as $r \to \infty$ and $r \in I$. Thus

$$\overline{N}(r,0;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{G}) \leq \frac{2(d+1)}{n+\sigma}T(r,\mathcal{F})(1+o(1)), \tag{3.5}$$

as $r \to \infty$ and $r \in I$. In view of Nevanlinna's second fundamental theorem, we obtain

$$\begin{aligned}T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,1;\mathcal{F}) + O(logr)\\ &\leq \overline{N}(r,0;f) + \overline{N}\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + \overline{N}(r,1;\mathcal{F}) + O(logr)\\ &\leq (d+1)T(r,f) + \overline{N}(r,1;\mathcal{F}) + O\left(r^{\rho(f)+\epsilon-1}\right) + O(logr),\end{aligned}$$

which together with (3.1) gives

$$(n+\sigma)T(r,f) \leq (d+1)T(r,f) + \overline{N}(r,1;\mathcal{F}) + S(r,f),$$

as $r \to \infty$ and $r \in I$. From this and the fact that $\mathcal{F}$ and $\mathcal{G}$ share (1, 2), we conclude that there exists a point $z_0 \in \mathbb{C}$ such that $\mathcal{F}(z_0) = \mathcal{G}(z_0) = 1$. Hence from (3.5), Lemma 2.7 and the condition $n \geq 2d - \sigma + 3$, we infer that either $\mathcal{F}\mathcal{G} = 1$ (or) $\mathcal{F} = \mathcal{G}$.

Now we consider the following subcases.

Subcase 1.1. $\mathcal{F} \equiv \mathcal{G}$. Then we get

$$f^n(z)\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j} \equiv g^n(z)\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}. \tag{3.6}$$

Let $h(z) = f(z)/g(z)$. Then we deduce that

$$h^{n+1} \equiv \frac{f}{\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}.\frac{\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}}{g}. \tag{3.7}$$

If h is not a constant, then we have

$$\begin{aligned}(n+1)T(r,h) = T(r,h^{n+1}) &\leq T\left(r, \frac{f}{\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}\right) + T\left(r, \frac{\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}}{g}\right)\\ &\leq (\sigma+1)\left[T(r,f)+T(r,g)\right] + S(r,f) + S(r,g).\end{aligned}$$

Combining above inequality with $T(r,h) = T\left(r, \frac{f}{g}\right) = T(r,f)+T(r,g)+S(r,f)+S(r,g)$, we obtain $(n-\sigma)\left[T(r,f)+T(r,g)\right] \leq S(r,f)+S(r,g)$, which is impossible. Therefore h is a constant, then substituting $f = gh$ into (3.6), we have $h^{n+\sigma} = 1$. Therefore $f = \tau g$, where τ is a constant with $\tau^{n+\sigma} = 1$, which is the conclusion (a).

Subcase 1.2. Suppose $\mathcal{F}\mathcal{G} \equiv 1$. Then we get

$$f^n(z)\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}.g^n(z)\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j} \equiv \mathcal{P}_0^2(z). \tag{3.8}$$

From (3.8) and the condition that f and g are transcendental entire functions, one can immediately say that both f and g have at most finitely many zeros. So, we may write

$$f(z) = \mathcal{P}_1(z)e^{\alpha(z)}, \quad g(z) = \mathcal{P}_2(z)e^{\beta(z)}, \tag{3.9}$$

where $\mathcal{P}_1, \mathcal{P}_2, \alpha, \beta$ are polynomials, and α, β are non-constants. Substituting (3.9) in (3.8), we obtain

$$(\mathcal{P}_1\mathcal{P}_2)^n e^{n(\alpha+\beta)+\sum_{j=1}^{d}\left(\alpha(z+\eta_j)+\beta(z+\eta_j)\right)\mu_j}\prod_{j=1}^{d}\mathcal{P}_1(z+\eta_j)^{\mu_j}.\mathcal{P}_2(z+\eta_j)^{\mu_j} \equiv \mathcal{P}_0^2(z), \tag{3.10}$$

for all $z \in \mathbb{C}$. To avoid a contradiction, from (3.10) we must have

$$n(\alpha(z)+\beta(z)) + \sum_{j=1}^{d}\left(\alpha(z+\eta_j)+\beta(z+\eta_j)\right)\mu_j = A, \tag{3.11}$$

for all $z \in \mathbb{C}$, where A is a constant. Let $\alpha(z)+\beta(z) = w(z)$. Then (3.11) can be written as

$$nw(z) + \sum_{j=1}^{d} w(z+\eta_j)\mu_j = A, \tag{3.12}$$

for all $z \in \mathbb{C}$. From (3.12), we must have $w = B$, where B is a constant, and therefore, we have

$$\beta = B - \alpha. \tag{3.13}$$

Keeping in view of (3.13), (3.9) can be written as

$$f(z) = \mathcal{P}_1(z)e^{\alpha(z)}, \quad g(z) = \mathcal{P}_2(z)e^{B}e^{-\alpha(z)}. \tag{3.14}$$

Now (3.10) can be written as

$$(\mathcal{P}_1\mathcal{P}_2)^n \prod_{j=1}^{d} \mathcal{P}_1(z+\eta_j)^{\mu_j}.\mathcal{P}_2(z+\eta_j)^{\mu_j} \equiv e^{A}\mathcal{P}_0^2(z). \tag{3.15}$$

If $\mathcal{P}_1\mathcal{P}_2$ is not a constant, then the degree of the left side of (3.15) is at least $n+\sigma$. But the condition $2deg(\mathcal{P}_0) < n+\sigma$ implies that the degree of the right side of (3.15) is less than $n+\sigma$, which is a contradiction. Thus $\mathcal{P}_1\mathcal{P}_2$ and $\mathcal{P}_0$ reduces to non-zero constants.

Since $\mathcal{P}_1, \mathcal{P}_2$ are both polynomials and their product is constant, each of them must be constant. Therefore, (3.14) can be written as

$$f(z) = e^{U}, \quad g(z) = e^{B}e^{-U}, \tag{3.16}$$

where U is a non-constant polynomial. Using the above forms of f and g and keeping in mind that $\mathcal{P}_0$ is a constant, say κ^2, (3.6) reduces to

$$e^{B(n+\sigma)} \equiv \kappa^2.$$

Set $e^B = \tau$. Then (3.16) can be written as

$$f(z) = e^{U}, \;\; g(z) = \tau e^{-U}, \text{where } \tau \text{ is a constant such that } \tau^{n+\sigma} = \kappa^2,$$

which is the conclusion (b).
Case 2. Suppose $n \geq \sigma + 5$.
Since $f^n(z)\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j} - \mathcal{P}_0(z)$ and $g^n(z)\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j} - \mathcal{P}_0(z)$ share $(0,\gamma)$, it follows that $\mathcal{F}$ and $\mathcal{G}$ share $(1,\gamma)$.
Let $\mathcal{H} \not\equiv 0$. First suppose $\gamma \geq 2$.
Using Lemmas 2.12 and 2.15, we obtain

$$\begin{aligned}\overline{N}(r,1;\mathcal{F}) &= N(r,1;\mathcal{F}| = 1) + \overline{N}(r,1;\mathcal{F}| \geq 2) \leq N(r,\infty;\mathcal{H}) + \overline{N}(r,1;\mathcal{F} \geq 2)\\ &\leq \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;\mathcal{G}| \geq 2) + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + \overline{N}(r,1;\mathcal{F}| \geq 2)\\ &\quad + \overline{N}_0(r,0;\mathcal{F}') + \overline{N}_0(r,0;\mathcal{G}') + S(r,\mathcal{F}) + S(r,\mathcal{G}). \end{aligned} \tag{3.17}$$

Keeping in view of the above observation and Lemma 2.16, we see that

$$\begin{aligned}&\overline{N}_0(r,0;\mathcal{G}') + \overline{N}(r,1;\mathcal{F}| \geq 2) + \overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\ &\leq \overline{N}_0(r,0;\mathcal{G}') + \overline{N}(r,1;\mathcal{F}| \geq 2) + \overline{N}(r,1;\mathcal{F}| \geq 3) + S(r,\mathcal{F})\\ &\leq \overline{N}_0(r,0;\mathcal{G}') + \overline{N}(r,1;\mathcal{G}| \geq 2) + \overline{N}(r,1;\mathcal{G}| \geq 3) + S(r,\mathcal{F}) + S(r,\mathcal{G})\\ &\leq \overline{N}_0(r,0;\mathcal{G}') + N(r,1;\mathcal{G}) - \overline{N}(r,1;\mathcal{G}) + S(r,\mathcal{F}) + S(r,\mathcal{G})\\ &\leq N_0(r,0;\mathcal{G}'|\mathcal{G} \neq 0) \leq \overline{N}(r,0;\mathcal{G}) + S(r,\mathcal{G}). \end{aligned} \tag{3.18}$$

Using (3.17) and (3.18), Lemmas 2.2, 2.9 and applying second fundamental theorem of Nevanlinna to $\mathcal{F}$, we obtain

$$\begin{aligned}
nT(r,f) &\leq T(r,\mathcal{F}) - N\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + S(r,f)\\
&\leq \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,1;\mathcal{F}) - \overline{N}(r,0;\mathcal{F}')\\
&\quad - N\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + S(r,f)\\
&\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) - N\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + S(r,f) + S(r,g)\\
&\leq N_2\left(r,0;f^n\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + N_2\left(r,0;g^n\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}\right)\\
&\quad - N\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + S(r,f) + S(r,g)\\
&\leq 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + N\left(r,0;\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}\right) + S(r,f) + S(r,g)\\
&\leq 2T(r,f) + (2+\sigma)T(r,g) + S(r,f) + S(r,g). \qquad (3.19)
\end{aligned}$$

Similarly, we have

$$nT(r,g) \leq 2T(r,g) + (2+\sigma)T(r,f) + S(r,f) + S(r,g). \qquad (3.20)$$

Combining (3.19) and (3.20), we get

$$(n-\sigma-4)[T(r,f)+T(r,g)] \leq S(r,f) + S(r,g), \qquad (3.21)$$

which contradicts with $n \geq \sigma + 5$.
When $\gamma = 1$, keeping in view of Lemmas 2.10, 2.11, 2.12, 2.15 and 2.16, we obtain

$$\begin{aligned}
\overline{N}(r,1;\mathcal{F}) &= N(r,1;\mathcal{F}|=1) + \overline{N}_L(r,1;\mathcal{F}) + \overline{N}_L(r,1;\mathcal{G}) + \overline{N}_E^{(2}(r,1;\mathcal{F})\\
&\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;\mathcal{G}|\geq 2) + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + \overline{N}_L(r,1;\mathcal{F})\\
&\quad + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;\mathcal{F}) + \overline{N}_0(r,0;\mathcal{F}') + \overline{N}_0(r,0;\mathcal{G}') + S(r,\mathcal{F})\\
&\quad + S(r,\mathcal{G})\\
&\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;\mathcal{G}|\geq 2) + 2\overline{N}_L(r,1;\mathcal{F}) + 2\overline{N}_L(r,1;G)\\
&\quad + \overline{N}_E^{(2}(r,1;\mathcal{F}) + \overline{N}_0(r,0;\mathcal{F}') + \overline{N}_0(r,0;\mathcal{G}') + S(r,\mathcal{F}) + S(r,\mathcal{G})\\
&\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;\mathcal{G}|\geq 2) + \overline{N}_{\mathcal{F}>2}(r,1;\mathcal{G}) + N(r,1;\mathcal{G})\\
&\quad - \overline{N}(r,1;\mathcal{G}) + \overline{N}_0(r,0;\mathcal{F}') + \overline{N}_0(r,0;\mathcal{G}') + S(r,\mathcal{F}) + S(r,\mathcal{G})\\
&\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;\mathcal{G}|\geq 2) + N(r,0;\mathcal{G}'|\mathcal{G}\neq 0) + \frac{1}{2}\overline{N}(r,0;\mathcal{F})\\
&\quad + \overline{N}_0(r,0;\mathcal{F}') + S(r,\mathcal{F}) + S(r,\mathcal{G})\\
&\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \frac{1}{2}\overline{N}(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + \overline{N}_0(r,0;\mathcal{F}') + S(r,\mathcal{F})\\
&\quad + S(r,\mathcal{G}). \qquad (3.22)
\end{aligned}$$

Using (3.22), Lemmas 2.2, 2.9 and applying second fundamental theorem of Nevanlinna to $\mathcal{F}$, we obtain

$$\begin{aligned}
nT(r,f) &\leq T(r,\mathcal{F}) - N\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + S(r,f)\\
&\leq \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,1;\mathcal{F}) - \overline{N}(r,0;\mathcal{F}') - N\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right)\\
&\quad + S(r,f)\\
&\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + \frac{1}{2}\overline{N}(r,0;\mathcal{F}) - N\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right)\\
&\quad + S(r,f) + S(r,g)\\
&\leq 2\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + 2\overline{N}(r,0;g)\\
&\quad + N\left(r,0;\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}\right) + S(r,f) + S(r,g)\\
&\leq \left(\frac{5+d}{2}\right)T(r,f) + (2+\sigma)T(r,g) + S(r,f) + S(r,g). \qquad (3.23)
\end{aligned}$$

In a similar manner, we may obtain

$$nT(r,g) \leq \left(\frac{5+d}{2}\right)T(r,g) + (2+\sigma)T(r,f) + S(r,f) + S(r,g). \qquad (3.24)$$

Combining (3.23) and (3.24), we obtain

$$\left(n - \left(\frac{2\sigma+d+9}{2}\right)\right)[T(r,f)+T(r,g)] \leq S(r,f) + S(r,g),$$

which is a contradiction with $n \geq \sigma + \frac{d}{2} + 6$.
When $\gamma = 0$, using Lemmas 2.12, 2.13, 2.14, 2.15 and 2.16, we obtain

$$\begin{aligned}
\overline{N}(r,1;\mathcal{F}) &= N(r,1;\mathcal{F}|=1) + \overline{N}_L(r,1;\mathcal{F}) + \overline{N}_L(r,1;\mathcal{G}) + \overline{N}_E^{(2}(r,1;\mathcal{F})\\
&\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;\mathcal{G}|\geq 2) + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + \overline{N}_L(r,1;\mathcal{F})\\
&\quad + \overline{N}_L(r,1;\mathcal{G}) + \overline{N}_E^{(2}(r,1;\mathcal{F}) + \overline{N}_0(r,0;\mathcal{F}') + \overline{N}_0(r,0;\mathcal{G}') + S(r,\mathcal{F})\\
&\quad + S(r,\mathcal{G})\\
&\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;\mathcal{G}|\geq 2) + 2\overline{N}_L(r,1;\mathcal{F}) + 2\overline{N}_L(r,1;\mathcal{G})\\
&\quad + \overline{N}_E^{(2}(r,1;\mathcal{F}) + \overline{N}_0(r,0;\mathcal{F}') + \overline{N}_0(r,0;\mathcal{G}') + S(r,\mathcal{F}) + S(r,\mathcal{G})\\
&\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;\mathcal{G}|\geq 2) + \overline{N}_L(r,1;\mathcal{F}) + \overline{N}_{\mathcal{F}>1}(r,1;\mathcal{G})\\
&\quad + \overline{N}_{\mathcal{G}>1}(r,1;\mathcal{F}) + N(r,1;\mathcal{G}) - \overline{N}(r,1;\mathcal{G}) + \overline{N}_0(r,0;\mathcal{F}') + \overline{N}_0(r,0;\mathcal{G}')\\
&\quad + S(r,\mathcal{F}) + S(r,\mathcal{G})\\
&\leq \overline{N}(r,0;\mathcal{F}|\geq 2) + \overline{N}(r,0;\mathcal{G}|\geq 2) + N(r,0;\mathcal{G}'|\mathcal{G}\neq 0) + 2\overline{N}(r,0;\mathcal{F})\\
&\quad + \overline{N}(r,0;\mathcal{G}) + \overline{N}_0(r,0;\mathcal{F}') + S(r,\mathcal{F}) + S(r,\mathcal{G})\\
&\leq N_2(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}_0(r,0;\mathcal{F}')\\
&\quad + S(r,\mathcal{F}) + S(r,\mathcal{G}). \qquad (3.25)
\end{aligned}$$

Using (3.25), Lemmas 2.2, 2.9 and applying second fundamental theorem of Nevanlinna to $\mathcal{F}$, we obtain

$$\begin{aligned}
nT(r,f) &\leq T(r,\mathcal{F}) - N\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + S(r,f)\\
&\leq \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,1;\mathcal{F}) - \overline{N}(r,0;\mathcal{F}') - N\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right)\\
&\quad + S(r,f)\\
&\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + 2\overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G})\\
&\quad - N\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + S(r,f) + S(r,g)\\
&\leq 4\overline{N}(r,0;f) + 2\overline{N}\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) + 3\overline{N}(r,0;g)\\
&\quad + N\left(r,0;\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}\right) + \overline{N}\left(r,0;\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}\right) + S(r,f)\\
&\quad + S(r,g)\\
&\leq (2d+4)T(r,f) + (\sigma+d+3)T(r,g) + S(r,f) + S(r,g). \qquad (3.26)
\end{aligned}$$

In a similar manner, we may obtain

$$nT(r,g) \leq (2d+4)T(r,g) + (\sigma+d+3)T(r,f) + S(r,f) + S(r,g). \qquad (3.27)$$

Combining (3.26) and (3.27), we obtain

$$(n-\sigma-3d-7)[T(r,f)+T(r,g)] \leq S(r,f) + S(r,g),$$

which is a contradiction with $n \geq \sigma + 3d + 8$.
Thus $\mathcal{H} \equiv 0$. Then by integration we obtain (3.3). Therefore, the result follows from Case 1. This completes the proof of theorem. □

Proof of Theorem 1.2.

Proof. Let $\mathcal{F}_1 = \frac{f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}{\zeta(z)}$ and $\mathcal{G}_1 = \frac{g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}}{\zeta(z)}$. Then $\mathcal{F}_1$ and $\mathcal{G}_1$ are two transcendental meromorphic functions that share $(1,2)$. Noting that $\rho(\zeta) < \rho(f)$, from Lemma 2.5, we see that

$$T(r,\mathcal{F}_1) = (n+m+\sigma)T(r,f) + O(r^{\rho(f)+\epsilon-1}) + O(r^{\rho(\zeta)+\epsilon}),$$

$$T(r,\mathcal{G}_1) = (n+m+\sigma)T(r,g) + O(r^{\rho(g)+\epsilon-1}) + O(r^{\rho(\zeta)+\epsilon}). \qquad (3.28)$$

From (3.28), we get

$$\begin{aligned}
&\rho(\mathcal{F}_1) \leq max\{\rho(f),\rho(\zeta)\}, \quad \rho(f) \leq max\{\rho(\mathcal{F}_1),\rho(\zeta)\},\\
&\rho(\mathcal{G}_1) \leq max\{\rho(g),\rho(\zeta)\}, \quad \rho(g) \leq max\{\rho(\mathcal{G}_1),\rho(\zeta)\}. \qquad (3.29)
\end{aligned}$$

Using (3.29) and the fact that $\rho(\zeta) < \rho(f)$, we obtain

$$\rho(\mathcal{F}_1) = \rho(f). \qquad (3.30)$$

Now, using Nevanlinna's second fundamental theorem, we can write

$$\begin{aligned}T(r,\mathcal{F}_1) &\leq \overline{N}(r,0;\mathcal{F}_1)+\overline{N}(r,\infty;\mathcal{F}_1)+\overline{N}(r,1;\mathcal{F}_1)+S(r,f)\\ &\leq \overline{N}(r,0;f)+\overline{N}\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right)+\overline{N}(r,1;f^m)+\overline{N}(r,1;\mathcal{G}_1)\\ &\quad+O\left(r^{\rho(\zeta)+\epsilon}\right)+S(r,f)\\ &\leq (\sigma+m+1)T(r,f)+T(r,\mathcal{G}_1)+O\left(r^{\rho(f)+\epsilon-1}\right)+O\left(r^{\rho(\zeta)+\epsilon}\right)+S(r,f).\end{aligned} \tag{3.31}$$

Similarly, we get

$$T(r,\mathcal{G}_1) \leq (\sigma+m+1)T(r,g)+T(r,\mathcal{F}_1)+O\left(r^{\rho(g)+\epsilon-1}\right)+O\left(r^{\rho(\zeta)+\epsilon}\right)+S(r,g). \tag{3.32}$$

From (3.28), (3.30), and (3.31) and the condition $\rho(\zeta)<\rho(f)<\infty$, we see that

$$\rho(F_1) \leq \rho(\mathcal{G}_1), \tag{3.33}$$

and from (3.29), (3.30), and (3.32) and the condition $\rho(\zeta)<\rho(f)<\infty$, we see that

$$\rho(\mathcal{G}_1) = \rho(g). \tag{3.34}$$

Also, from (3.28), (3.30), and (3.32)-(3.34) and the condition $\rho(\zeta)<\rho(f)<\infty$, we see that

$$\rho(\mathcal{G}_1) \leq \rho(\mathcal{F}_1). \tag{3.35}$$

Combining (3.28) and (3.33)-(3.35) we obtain

$$\rho(f)=\rho(g)=\rho(\mathcal{F}_1)=\rho(\mathcal{G}_1). \tag{3.36}$$

Suppose that $\mathcal{H}\not\equiv 0$. Then using Lemmas 2.3 and 2.17 we can write

$$\begin{aligned}T(r,\mathcal{F}_1)+T(r,\mathcal{G}_1) &\leq 2N_2(r,0;\mathcal{F}_1)+2N_2(r,0;\mathcal{G}_1)+2\overline{N}(r,\infty;\mathcal{F}_1)+2\overline{N}(r,\infty;\mathcal{G}_1)\\ &\quad+2\overline{N}_*(r,\infty;\mathcal{F}_1,\mathcal{G}_1)+S(r,\mathcal{F}_1)+S(r,\mathcal{G}_1)\\ &\leq 4\overline{N}(r,0;f)+4\overline{N}(r,0;g)+2N(r,1;f^m)+2N(r,1;g^m)\\ &\quad+2N\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right)+2N\left(r,0;\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}\right)\\ &\quad+S(r,f)+S(r,g)\\ &\leq (2\sigma+2m+4)\{T(r,f)+T(r,g)\}+O\left(r^{\rho(f)-1+\epsilon}\right)\\ &\quad+O\left(r^{\rho(g)-1+\epsilon}\right)+S(r,f)+S(r,g).\end{aligned} \tag{3.37}$$

Therefore, from (3.28) and (3.37), we obtain

$$\begin{aligned}(n-\sigma-m-4)\{T(r,f)+T(r,g)\} &\leq O\left(r^{\rho(f)-1+\epsilon}\right)+O\left(r^{\rho(g)-1+\epsilon}\right)+S(r,f)\\ &\quad+S(r,g),\end{aligned}$$

yielding a contradiction with the assumption that $n \geq \sigma + m + 5$. Thus we must have $\mathcal{H} \equiv 0$. Taking into account that

$$\begin{aligned}
&\overline{N}(r,0;\mathcal{F}_1) + \overline{N}(r,\infty;\mathcal{F}_1) + \overline{N}(r,0;\mathcal{G}_1) + \overline{N}(r,\infty;\mathcal{G}_1) \\
&\leq \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,1;f^m) + \overline{N}(r,1;g^m) + \overline{N}\left(r,0;\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}\right) \\
&\quad + \overline{N}\left(r,0;\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}\right) + S(r,f) + S(r,g) \\
&\leq (\sigma+m+1)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g) \\
&\leq \frac{2\sigma+2m+2}{n+m+\sigma}\ T(r),
\end{aligned}$$

where, $T(r) = max\{T(r,\mathcal{F}_1), T(r,\mathcal{G}_1)\}$, by Lemma 2.18, we deduce that either $\mathcal{F}_1 \equiv \mathcal{G}_1$ (or) $\mathcal{F}_1\mathcal{G}_1 \equiv 1$. Let $\mathcal{F}_1\mathcal{G}_1 \equiv 1$. Then we have

$$f^n(z)\left(f^m(z)-1\right)\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j} g^n(z)\left(g^m(z)-1\right)\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j} \equiv \zeta^2. \tag{3.38}$$

Noting that f and g are transcendental entire functions of finite order, it is easily seen from the above equality that $\overline{N}(r,0;f) = S(r,f)$, $\overline{N}(r,1;f) = S(r,f)$ and $\overline{N}(r,\infty;f) = S(r,f)$, for $r \in I$ and $r \to \infty$, where $I \subset (0,+\infty)$ is a subset of infinite linear measure. Thus, we obtain

$$T(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,\infty;f) = S(r,f),$$

for $r \in I$ and $r \to \infty$, which is meaningless. Thus, we must have $\mathcal{F}_1 \equiv \mathcal{G}_1$, and hence

$$f^n(z)\left(f^m(z)-1\right)\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j} \equiv g^n(z)\left(g^m(z)-1\right)\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}. \tag{3.39}$$

Set $\tau = \frac{f}{g}$. If τ is not a constant, from (3.39), we have

$$g^m(z) = \frac{\tau^n(z)\prod_{j=1}^{d}\tau(z+\eta_j)^{\mu_j} - 1}{\tau^{n+m}(z)\prod_{j=1}^{d}\tau(z+\eta_j)^{\mu_j} - 1}. \tag{3.40}$$

If 1 is a Picard value of $\tau^{n+m}(z)\prod_{j=1}^{d}\tau(z+\eta_j)^{\mu_j}$, applying the Nevanlinna second fundamental theorem, we get

$$\begin{aligned}
T\left(r,\tau^{n+m}(z)\prod_{j=1}^{d}\tau(z+\eta_j)^{\mu_j}\right) &\leq \overline{N}\left(r,\infty;\tau^{n+m}(z)\prod_{j=1}^{d}\tau(z+\eta_j)^{\mu_j}\right) \\
&\quad + \overline{N}\left(r,0;\tau^{n+m}(z)\prod_{j=1}^{d}\tau(z+\eta_j)^{\mu_j}\right) + S(r,\tau) \\
&\leq (2d+2)T(r,\tau) + S(r,\tau). \qquad (3.41)
\end{aligned}$$

On the otherhand, combining the standard Valiron-Mohon'ko theorem, we get

$$\begin{aligned}
(n+m+\sigma)T(r,\tau) &= T(r,\tau^{n+m+\sigma}) + S(r,h) \\
&\leq T\left(r,\tau^{n+m}(z)\prod_{j=1}^{d}\tau(z+\eta_j)^{\mu_j}\right) + T\left(r,\prod_{j=1}^{d}\tau(z+\eta_j)^{\mu_j}\right) \\
&\leq (2d+3)T(r,\tau) + S(r,h).
\end{aligned}$$

Therefore, 1 is not a Picard exceptional value of $\tau^{n+m}(z)\prod_{j=1}^{d}\tau(z+\eta_j)^{\mu_j}$. Thus $\exists\ z_0$ such that $\tau^{n+m}(z_0)\prod_{j=1}^{d}\tau(z_0+\eta_j)^{\mu_j} = 1$, by (3.39), we have $\tau^{n+m}(z_0)\prod_{j=1}^{d}\tau(z_0+\eta_j)^{\mu_j} = 1$.

Hence $\tau_0^m = 1$, and

$$\overline{N}\left(r,0;\tau^{n+m}(z)\prod_{j=1}^{d}\tau(z+\eta_j)^{\mu_j}-1\right) \le \overline{N}(r,0;\tau^m-1) \le mT(r,\tau)+S(r,\tau). \tag{3.42}$$

From the above inequality and by the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T\left(r,\tau^{n+m}(z)\prod_{j=1}^{d}\tau(z+\eta_j)^{\mu_j}\right) &\le \overline{N}\left(r,\infty;h^{n+m}(z)\prod_{j=1}^{d}\tau(z+\eta_j)^{\mu_j}\right) \\ &\quad + \overline{N}\left(r,0;\tau^{n+m}(z)\prod_{j=1}^{d}\tau(z+\eta_j)^{\mu_j}-1\right) \\ &\quad + S(r,h) \\ &\le (m+2d+2)T(r,\tau)+S(r,\tau), \end{aligned} \tag{3.43}$$

which is a contradiction with $n \ge m+\sigma+5$. Therefore τ is not a constant. Substituting $f = g\tau$ into (3.39), we can get

$$\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}\left(g^{n+m}(z)(\tau^{n+m+\sigma}-1)+g^n(z)(\tau^{n+\sigma}-1)\right)=0. \tag{3.44}$$

Since g is an entire function, we have $\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j} \ne 0$. Thus

$$g^{n+m}(z)(\tau^{n+m+\sigma}-1)+g^n(z)(\tau^{n+\sigma}-1)=0. \tag{3.45}$$

If $\tau^{n+\sigma} \ne 1$, by (3.45) we can deduce $T(r,g) = S(r,g)$, which contradicts with a transcendental function g. So $\tau^{n+\sigma} = 1$. We can also deduce that $\tau^{n+m+\sigma} = 1$. Then $\tau^m = 1$. This completes the proof of theorem. □

Proof of Theorem 1.3.

Proof. Let $\mathcal{F}_2 = \frac{\mathcal{P}_n(f(z))\prod_{j=1}^{d} f(z+\eta_j)^{\mu_j}}{\mathcal{P}_0(z)}$ and $\mathcal{G}_2 = \frac{\mathcal{P}_n(g(z))\prod_{j=1}^{d} g(z+\eta_j)^{\mu_j}}{\mathcal{P}_0(z)}$. Then $\mathcal{F}_2$ and $\mathcal{G}_2$ are two transcendental meromorphic functions that share $(1,\gamma)$. Applying arguments similar to those used in the proof of Theorem 1.1, we can get a Theorem 1.3. Here, we omit the details. □

4. Conclusion

Nevanlinna Theory is a powerful quantitative tool used to study the growth and behaviour of entire and meromorphic functions on the complex plane. It has a wide range of applications within and outside function theory. By understanding the properties of these functions is essential for solving difference-differential equations, analyzing complex systems, and studying mathematical physics phenomena.

P. Sahoo and H. Karmakar [27] proved that uniqueness results when two difference polynomials of entire functions share a nonzero polynomial or a small function with a finite weight $(0,2)$. They also investigate the situation when the original functions share 0 CM. In this paper, we investigates the same situation for $(0,\gamma)$, where $\gamma = 0, \gamma = 1, \gamma \ge 2$ for the product of shift operator; hence, the results extend and generalize. Related to our results, we also point out possible examples that show the conclusions of all the theorems actually hold.

We pose the following open questions to the readers:

Open questions:

(1) What happens to condition n if we study meromorphic functions by using weakly weighted sharing, truncated weighted sharing, or partial sharing?

(2) Keeping all the assumptions of Theorems 1.1-1.3, what can be said about the relation between two non-constant entire functions f and $\mathcal{L}$ if

$$\left(f^n\mathcal{P}(f)\prod_{j=1}^{d}f(z+\eta_j)^{\mu_j}\right)^{(k)} \quad \text{and} \quad \left(\mathcal{L}^n\mathcal{P}(\mathcal{L})\prod_{j=1}^{d}\mathcal{L}(z+\eta_j)^{\mu_j}\right)^{(k)}$$

share the same polynomial, where $\mathcal{L}$ is a $\mathcal{L}$-Function ?

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