Electronic Journal of Mathematical Analysis and Applications
Vol. 12(1) Jan. 2024, No. 13
ISSN: 2090-729X (online)
ISSN: 3009-6731(print)
http://ejmaa.journals.ekb.eg/

EXISTENCE AND UNIQUENESS OF COMMON FIXED POINT SOLUTION FOR INTEGRAL EQUATION VIA COMPLEX VALUED METRIC SPACE USING CLASS FUNCTION

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Abstract. The main aim of our manuscript is to present common fixed point solution for Urysohn integral equation 41, Our paper divided in to two folds, **Firstly** with the help of class function we established fixed point results for two pair of weakly compatible mapping satisfying the contractive condition of rational type in CVMS our work inspired from [15]. **Secondly** Common solution to the integral equation,

$$\omega(t) = \gamma_i(t) + \int_{\alpha}^{\beta} \chi_i(t, s, \omega(s))ds.$$

Where $\omega, \gamma_i \in C([\alpha, \beta], \mathbb{R}^n)$, $[\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}^n \to \mathbb{R}^n$, i=1,2,3,4 and $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$, $t \in [\alpha, \beta]$ and we use,

$$\triangle_i(\omega(t)) = \int_{\alpha}^{\beta} \chi_i(t, s, \omega(s))ds.$$

1. Introduction and background

In a single valued contractive type mapping with having completeness property admits a unique common fixed point, this is the statement of Banach [1] fixed point theorem later on which got lot more generalization which having useful application in geology, Biology, Mechanics, Economics, lead to mathematical model by non-linear integral and differential equation. One of the powerful tools in non-linear analysis which initiated a new era of research in metric space is Fixed point theory, In 2023 Shinde S.R. [15] introduced Complex valued approach to the system of non-linear second order Boundary value problem and multivalued mapping via

2010 *Mathematics Subject Classification.* 46N99, 47H10, 47H9, 30D35.
Key words and phrases. Complex valued metric space(CVMS), class function, Common fixed point, Urysohn integral equation, complete CVMS..
Submitted August 03, 2023. Revised December 18, 2023. Accepted Date March 18, 2024.

fixed point method subsequently H. Aytimur et. al. [7] given A Geometric Interpretation to Fixed point Theory on Sb-Metric Space and N. Ozgur et. al. [4] given some porperties on S- metric spaces with some topological aspects, due to a wide range of areas (like Numerical analysis, differential or integral equation) and its intensive applicability many researchers [2, 3, 20] contributed significantly to it. There are various techniques and numerous advances which have been focusing on the solution of non-linear integral equations here with the help of the class of Geraghty's real valued mapping Sintunavarat et. al.[6] generalized the class function which is helpful throughout our paper. Das and gupta [7] generalized banach contraction principle for rational type contractive condition under fixed point metric space. Many mathematicians [8, 9, 11, 14] successfully studied the different application of complex valued fixed point theory. In addition sopan shinde given [13, 12] a common solution to the Urysohn integral equation under CVMS. Here we used four self-mapping on a complex valued metric space and with the help of class function we developed a new contraction condition and proved that our result gives a unique common fixed point for all four functions at same time, to support our result we proved all possible condition Including some preposition and theorems. Finally, the application part of our result is to prove the uniqueness and existence of fixed points for the Urysohn integral equation as it is given below,

$$\omega(t) = \gamma_i(t) + \int_\alpha^\beta \chi_i(t, s, \omega(s))ds. \tag{1}$$

Where $\omega, \gamma_i \in C([\alpha, \beta], \mathbb{R}^n)$, $[\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}^n \to \mathbb{R}^n$, i=1,2,3,4 and $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$, $t \in [\alpha, \beta]$ At the end we showed that all four self-mapping gives a unique fixed point for the Urysohn integral equation under the contraction condition we developed.

In [15] author Shinde S. R. given Complex valued existence and fixed point mapping and their solution for second order nonlinear boundary value problem with the help of greens functions,

$$\begin{cases} \xi''(\hbar) = \Im(\hbar, \xi(\hbar), \xi'(\hbar)), & \text{if } \hbar \in [0, \mathsf{T}], \mathsf{T} > 0 \\ \xi(\hbar_1) = \xi_1, & \\ \xi(\hbar_2) = \xi_2, & \text{if } \hbar_1, \hbar_2 \in [0, \mathsf{T}]. \end{cases}$$

and subsequently in other fold of Application fixed point results for multivalued mapping in setting of $\mathcal{CVMSP}$ given without using notion of continuity.

The aim of our paper is to contribute and prove some common fixed point theorem by using class function with the help of four self-mapping and application given to existence and uniqueness of integral equations.

2. Preliminaries

Azam et.al. [5] introduce Complex valued metric space as follows,

Definition 2.1. *Assume partial order $\precsim$ defined on a complex number($\mathbb{C}$) as,*

$$\omega_1 \precsim \omega_2$$

if and only if Re. part of (ω_1) $\leq$ Re. part of (ω_2) ; Img. part of (ω_1) $\leq$ Img. part of (ω_2),

$$\omega_1 \leq \omega_2$$

(1) *Re. part of (ω_1)< Re. part of (ω_2) ; Img. part of (ω_1) < Img. part of (ω_2).*

(2) *Re. part of* (ω_1) = *Re. part of* (ω_2) ; *Img. part of* (ω_1) = *Img. part of* (ω_2).
(3) *Re. part of* (ω_1) < *Re. part of* (ω_2) ; *Img. part of* (ω_1) = *Img. part of* (ω_2).
(4) *Re. part of* (ω_1) = *Re. part of* (ω_2) ; *Img. part of* (ω_1) < *Img. part of* (ω_2).

Definition 2.2. [5] *Assume* Θ *nonempty & and self-mapping* $\eth : \Theta \to \Theta$ *said to be complex valued metric if* $\eth$ *holds following condition,*

(1) $0 \precsim \eth(\omega_1, \omega_2)$ *every* $\omega_1, \omega_2 \in \Theta$ *and* $\eth(\omega_1, \omega_2) = 0$ *if and only if* $\omega_1 = \omega_2$
(2) $\eth(\omega_1, \omega_2) = \eth(\omega_2, \omega_1)$ *every* $\omega_1, \omega_2 \in \Theta$
(3) $\eth(\omega_1, \omega_2) \precsim \eth(\omega_1, \rho) + \eth(\rho, \omega_2)$ *for every* $\omega_1, \omega_2, \rho \in \Theta$, *Then* $(\Theta, \eth)$ *is called complex valued metric space.*

Example 1. *Let* $\Theta = \mathbb{C}$ *& map* $\eth : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *defined as,* $\eth(\omega_1, \omega_2) = \mid \jmath_1 - \jmath_2 \mid + \iota \mid \imath_1 - \imath_2 \mid$ *Where,* $\omega_1 = \jmath_1 + \iota\imath_1$ $\omega_2 = \jmath_2 + \iota\imath_2$. *Then* $(\Theta, \eth)$ *be complex vlaued metric space.*

Example 2. *Let* $\Theta = \mathbb{C}$ *& mapping* $\eth : \Theta \times \Theta \to \mathbb{C}$ *which* $\eth(\omega_1, \omega_2) = e^{i\beta} \mid \omega_1 - \omega_2 \mid$ *Where,* $\beta \in [0, \frac{\pi}{2}]$, *Then* $(\Theta, \eth)$ *be complex vlaued metric space.*

Proposition 1. [5] *Let* $(\Theta, \eth)$ *be a CVMS. Then sequence* $\{\omega_r\}$ *in* Θ *Converges to* ω *if and only if* $\mid \eth(\omega_r, \omega) \mid \to 0$ *as* $r \to \infty$

Proposition 2. [5] *Assume* $(\Theta, \eth)$ *be a CVMS. Then sequence* $\{\omega_r\}$ *in* Θ *is a cauchy sequence if and only if* $\mid \eth(\omega_r, \omega_{r+s}) \mid \to 0$ *as* $r \to \infty$ *where* $s \in \mathbb{N}$.

Definition 2.3. [5] *Suppose* $\{\omega_r\}$ *be a sequence in a CVMS* $(\Theta, \eth)$ *and* $\omega \in \Theta$, *Then* $(\Theta, \eth)$ *is said to be a complete CVMS if every Cauchy sequence is convergent in* $(\Theta, \eth)$.

Definition 2.4. [6] *The notion of class function given by, Sintunavarat et.al.[17] and define as,* $\Im = \{\Upsilon : \mathbb{C}_+ \to [0, 1) : \mu_n \subseteq \mathbb{C}_+\}$ *with having condition,* $\Upsilon(\mu_n) \to 1$ *which gives* $|\mu_n| \to 0$. *In our paper we use,* $\mathbb{C}_+ = \{\mu \in \mathbb{C} : \mu \succsim 0\}$ *which is extension of Geraghty's real valued mapping.*

Example 3. *Suppose* $\Upsilon_1, \Upsilon_2 : \mathbb{C}_+ \to [0, 1]$ *and* $\Upsilon_2(\mu) = k$, *for every* $\mu \in \mathbb{C}_+$, *where* $k \in [0, 1]$ $\Upsilon_1(\mu) = \frac{1}{1+k|\mu|}$, *for every* $\mu \in \mathbb{C}_+$, *where* $k \in (0, \infty)$ *then* $\Upsilon_1, \Upsilon_2 \in \Im$.

Definition 2.5. [21] *Let* Θ *be complete complex valued metric space then every pair of self-mapping i.e.* ζ, $\psi : \Theta \to \Theta$ *said to be weakly compatible if they commute at their coincidence point i.e.* $\mu \in \Theta$ *with* $\zeta\mu = \psi\mu$ *which implies* $\psi\zeta\mu = \zeta\psi\mu$

Definition 2.6. *Assume* ζ, $\psi : \Theta \to \Theta$ *and* Θ *non-empty, having point* $\mu \in \Theta$,
(1) If μ *is a coincidence point of* ζ *and* ψ *then* $\zeta\mu = \psi\mu$.
(2) If μ *is a common fixed point of* ζ *and* ψ *then* $\zeta\mu = \psi\mu = \mu$.
(3) and If μ *is a fixed point of* ψ *then* $\psi\mu = \mu$.

In our article, we introduce some new fixed point function under complex valued metric space $(\Theta, \eth)$ using class function with weakly compatible property and along with we proved common fixed point theorem gives fixed point for Urysohn type integral equations.

3. Main Results

Theorem 3.1. *If ζ,ψ,ξ,φ are four self mapping on a complex valued metric space (Θ,ð) and η_i:$\mathbb{C}_+\to[0,1)$, $\forall$ i =1,2,3,4 be given mapping suppose we have following condition,*
Step I)

$$\frac{ð(\zeta_\mu,\psi_\nu)}{ð(\xi_\mu,\varphi_\nu)} \leq \eta_1(ð(\xi_\mu,\varphi_\nu)).[1+\eta_2(ð(\xi_\mu,\varphi_\nu)).[1+ð(\varphi_\nu,\psi_\nu)]]-[\frac{\eta_3(ð(\xi_\mu,\varphi_\nu)).ð(\psi_\nu,\varphi_\nu)}{1+ð(\psi_\nu,\varphi_\nu)}]-$$

$$[\frac{\eta_4(ð(\xi_\mu,\varphi_\nu)).ð(\psi_\nu,\varphi_\nu)}{ð(\psi_\nu,\varphi_\nu)+max[1,[\frac{ð(\varphi_\nu,\psi_\mu)}{ð(\xi_\mu,\varphi_\nu).ð(\xi_\mu,\psi_\nu)}],[\frac{ð(\xi_\mu,\varphi_\mu)}{ð(\xi_\mu,\varphi_\nu).ð(\xi_\mu,\psi_\nu)}]]}] \quad (2)$$

Step II)

$$\eta_1(\mu)(1+\eta_2(\mu))+\eta_3(\mu)+\eta_4(\mu)<1, \eta_1(\mu)(1+\eta_2(\mu))<1 \quad (3)$$

Step III) The mapping Υ: $\mathbb{C}_+\to[0,1)$ define as,

$$\Upsilon(\mu)=[\frac{\eta_1(\mu).(1+\eta_2(\mu))}{1-[\mu.\eta_1(\mu).\eta_2(\mu)]}]-[\frac{\eta_3(\mu)+\eta_4(\mu)}{1-[\mu.\eta_1(\mu).\eta_2(\mu)]}], \forall\mu\in\mathbb{C}_+ \quad (4)$$

If $\zeta(\Theta)\subseteq\varphi(\Theta)$ and $\psi(\Theta)\subseteq\xi(\Theta)$ having properties of class function with (ζ,ξ) & (ψ,φ) both weakly compatible then ζ,ψ,ξ,φ admits unique fixed point in Θ.

Proof. In our proof we play with two sequence $\{\mu_n\}$ and $\{\nu_n\}$ with assuming $\mu_0 \in\Theta$ and we have $\zeta(\Theta)\subseteq\varphi(\Theta)$ and $\psi(\Theta)\subseteq\xi(\Theta)$ with following rule,

$$\zeta\mu_{2n} = \varphi\mu_{2n+1}=\nu_{2n+1} \text{ and } \psi\mu_{2n+1}=\xi\nu_{2n+2}=\nu_{2n+2}\ , \forall n\in\mathbb{N}$$

we have equation (1),

$$\frac{ð(\zeta_\mu,\psi_\nu)}{ð(\xi_\mu,\varphi_\nu)} \leq \eta_1(ð(\xi_\mu,\varphi_\nu)).[1+\eta_2(ð(\xi_\mu,\varphi_\nu)).[1+ð(\varphi_\nu,\psi_\nu)]]-\eta_3(ð(\xi_\mu,\varphi_\nu))[\frac{ð(\psi_\nu,\varphi_\nu)}{1+ð(\psi_\nu,\varphi_\nu)}]-$$

$$\eta_4(ð(\xi_\mu,\varphi_\nu))[\frac{ð(\psi_\nu,\varphi_\nu)}{ð(\psi_\nu,\varphi_\nu)+max[1,[\frac{ð(\varphi_\nu,\psi_\mu)}{ð(\xi_\mu,\varphi_\nu).ð(\xi_\mu,\psi_\nu)}],[\frac{ð(\xi_\mu,\varphi_\mu)}{ð(\xi_\mu,\varphi_\nu).ð(\xi_\mu,\psi_\nu)}]}] \quad (5)$$

we use here, μ=μ_{2n} and ν=μ_{2n+1} then we get,

$$\frac{ð(\zeta\mu_{2n},\psi\mu_{2n+1})}{ð(\xi\mu_{2n},\varphi\mu_{2n+1})} \leq \eta_1(ð(\xi\mu_{2n},\varphi\mu_{2n+1})).[1+\eta_2(ð(\xi\mu_{2n},\varphi\mu_{2n+1})).[1+ð(\varphi\mu_{2n+1},\psi\mu_{2n+1})]]$$

$$-\eta_3(ð(\xi\mu_{2n},\varphi\mu_{2n+1}))[\frac{ð(\psi\mu_{2n+1},\varphi\mu_{2n+1})}{1+ð(\psi\mu_{2n+1},\varphi\mu_{2n+1})}]-\eta_4(ð(\xi\mu_{2n},\varphi\mu_{2n+1}))$$

$$[\frac{ð(\psi\mu_{2n+1},\varphi\mu_{2n+1})}{ð(\psi\mu_{2n+1},\varphi\mu_{2n+1})+max[1,[\frac{ð(\varphi\mu_{2n+1},\psi\mu_{2n})}{ð(\xi\mu_{2n},\varphi\mu_{2n+1}).ð(\xi\mu_{2n},\psi\mu_{2n+1})}],[\frac{ð(\xi\mu_{2n},\varphi\mu_{2n})}{ð(\xi\mu_{2n},\varphi\mu_{2n+1}).ð(\xi\mu_{2n},\psi\mu_{2n+1})}]}] \quad (6)$$

$$\frac{|ð(\nu_{2n+1},\nu_{2n+2})|}{|ð(\nu_{2n},\nu_{2n+1})|} \leq \eta_1(ð(\nu_{2n},\nu_{2n+1})).[1+\eta_2(ð(\nu_{2n},\nu_{2n+1})).[1+|ð(\nu_{2n+1},\nu_{2n+2})|]]$$

$$-\eta_3(ð(\nu_{2n},\nu_{2n+1}))[\frac{|ð(\nu_{2n+2},\nu_{2n+1})|}{1+|ð(\nu_{2n+2},\nu_{2n+1})|}]$$

$$-\eta_4(ð(\nu_{2n},\nu_{2n+1}))[\frac{|ð(\nu_{2n+2},\nu_{2n+1})|}{|ð(\nu_{2n+2},\nu_{2n+1})|+max[1,[\frac{|ð(\nu_{2n+1},\nu_{2n+1})|}{|ð(\nu_{2n},\nu_{2n+1}).ð(\nu_{2n},\nu_{2n+2})|}],[\frac{|ð(\nu_{2n},\nu_{2n})|}{|ð(\nu_{2n},\nu_{2n+1}).ð(\nu_{2n},\nu_{2n+2})|}]}]$$

$$\frac{|\eth(\nu_{2n+1},\nu_{2n+2})|}{|\eth(\nu_{2n},\nu_{2n+1})|} \leq \eta_1(\eth(\nu_{2n},\nu_{2n+1})).[1+\eta_2(\eth(\nu_{2n},\nu_{2n+1})).[1+|\eth(\nu_{2n+1},\nu_{2n+2})|]]-\eta_3(\eth(\nu_{2n},\nu_{2n+1})) - \eta_4(\eth(\nu_{2n},\nu_{2n+1})) \quad (7)$$

$$\frac{|\eth(\nu_{2n+1},\nu_{2n+2})|}{|\eth(\nu_{2n},\nu_{2n+1})|} \leq \eta_1(\eth(\nu_{2n},\nu_{2n+1}))+[\eta_1(\eth(\nu_{2n},\nu_{2n+1})).\eta_2(\eth(\nu_{2n},\nu_{2n+1})).[1+|\eth(\nu_{2n+1},\nu_{2n+2})|]] - \eta_3(\eth(\nu_{2n},\nu_{2n+1})) - \eta_4(\eth(\nu_{2n},\nu_{2n+1}))$$

$$\frac{|\eth(\nu_{2n+1},\nu_{2n+2})|}{|\eth(\nu_{2n},\nu_{2n+1})|} \leq \eta_1(\eth(\nu_{2n},\nu_{2n+1}))+[\eta_1(\eth(\nu_{2n},\nu_{2n+1})).\eta_2(\eth(\nu_{2n},\nu_{2n+1}))]+[\eta_1(\eth(\nu_{2n},\nu_{2n+1}))]. [\eta_2(\eth(\nu_{2n},\nu_{2n+1}))].|\eth(\nu_{2n+1},\nu_{2n+2})| - \eta_3(\eth(\nu_{2n},\nu_{2n+1})) - \eta_4(\eth(\nu_{2n},\nu_{2n+1}))$$

$$\frac{|\eth(\nu_{2n+1},\nu_{2n+2})|}{|\eth(\nu_{2n},\nu_{2n+1})|} - [[\eta_1(\eth(\nu_{2n},\nu_{2n+1}))].[\eta_2(\eth(\nu_{2n},\nu_{2n+1}))].|\eth(\nu_{2n+1},\nu_{2n+2})|] \leq \eta_1(\eth(\nu_{2n},\nu_{2n+1}))+ + [\eta_1(\eth(\nu_{2n},\nu_{2n+1})).\eta_2(\eth(\nu_{2n},\nu_{2n+1}))] - \eta_3(\eth(\nu_{2n},\nu_{2n+1})) - \eta_4(\eth(\nu_{2n},\nu_{2n+1}))$$

$$\frac{|\eth(\nu_{2n+1},\nu_{2n+2})|}{|\eth(\nu_{2n},\nu_{2n+1})|}.[1-[\eta_1(\eth(\nu_{2n},\nu_{2n+1}))].[\eta_2(\eth(\nu_{2n},\nu_{2n+1}))].|(\eth(\nu_{2n},\nu_{2n+1}))|] \leq \eta_1(\eth(\nu_{2n},\nu_{2n+1}))+ + [\eta_1(\eth(\nu_{2n},\nu_{2n+1})).\eta_2(\eth(\nu_{2n},\nu_{2n+1}))] - \eta_3(\eth(\nu_{2n},\nu_{2n+1})) - \eta_4(\eth(\nu_{2n},\nu_{2n+1}))$$

$$\frac{|\eth(\nu_{2n+1},\nu_{2n+2})|}{|\eth(\nu_{2n},\nu_{2n+1})|} \leq \frac{\eta_1(\eth(\nu_{2n},\nu_{2n+1})).[1+\eta_2(\eth(\nu_{2n},\nu_{2n+1}))]}{[1-[\eta_1(\eth(\nu_{2n},\nu_{2n+1}))].[\eta_2(\eth(\nu_{2n},\nu_{2n+1}))].|(\eth(\nu_{2n},\nu_{2n+1}))|]} - [\frac{[\eta_3(\eth(\nu_{2n},\nu_{2n+1}))+\eta_4(\eth(\nu_{2n},\nu_{2n+1}))]}{[1-[\eta_1(\eth(\nu_{2n},\nu_{2n+1}))].[\eta_2(\eth(\nu_{2n},\nu_{2n+1}))].|(\eth(\nu_{2n},\nu_{2n+1}))|]}]$$

with the help of equation (3) we can write,

$$\frac{|\eth(\nu_{2n+1},\nu_{2n+2})|}{|\eth(\nu_{2n},\nu_{2n+1})|} \leq \Upsilon(\eth(\nu_{2n},\nu_{2n+1})) \quad (8)$$

Simillarly for all n$\in$**N** one obtained,

$$\frac{|\eth(\nu_{2n},\nu_{2n+1})|}{|\eth(\nu_{2n-1},\nu_{2n})|} \leq \Upsilon(\eth(\nu_{2n-1},\nu_{2n})) \quad (9)$$

$\forall$ n$\in$**N** consequently,

$$\frac{|\eth(\nu_{n},\nu_{n+1})|}{|\eth(\nu_{n-1},\nu_{n})|} \leq \Upsilon(\eth(\nu_{n-1},\nu_{n})) \quad (10)$$

Finally we write,

$$|\eth(\nu_n,\nu_{n+1})| \leq \Upsilon(\eth(\nu_{n-1},\nu_n)).|\eth(\nu_{n-1},\nu_n)| \leq |\eth(\nu_{n-1},\nu_n)| \quad (11)$$

Here we observed that the sequence,$\{|\eth(\nu_n,\nu_{n+1})|\}$ for every n$\in$ **N** $\setminus\{1\}$ is monotone decreasing and bounded below and gives, $\{|\eth(\nu_n,\nu_{n+1})|\}$ n$\in$ **N** $\setminus\{1\}$ which tending to ℓ and $\ell\geq 0$.
Our final claim is, $\ell = 0$
To get our proof done on contrary suppose that $\ell\neq 0$, then possible cases are remain ($\ell < 0$) which never hold. That is we consider that $\ell > 0$ and apply as $n \to \infty$ in inequalitie (10)

$$\lim_{n\to\infty} |\eth(\nu_n,\nu_{n+1})| \leq \lim_{n\to\infty} \Upsilon(\eth(\nu_{n-1},\nu_n)).|\eth(\nu_{n-1},\nu_n)| \leq \lim_{n\to\infty} |\eth(\nu_{n-1},\nu_n)| \quad (12)$$

$$1 \leq \lim_{n\to\infty} \Upsilon(\eth(\nu_{n-1},\nu_n)) \leq 1 \quad (13)$$

But $\Upsilon \in \Im$, which gives $|\eth(\nu_{n-1},\nu_n)| \to 0$ that is $\ell = 0$. Which is contradict to our assumption for $\ell > 0$

$$\lim_{n\to\infty} |\eth(\nu_{n-1},\nu_n)| = 0 \quad (14)$$

Next we neet to work that $\{\nu_n\}$ is a cauchy sequence or not. That mean On contrary assume $\{\nu_{2n}\}$ is not a cauchy sequence, then their is $c \in \mathbb{C}$ with $c > 0$ and $\exists\ 2m_k > 2n_k \geq k$, $\forall$k$\in$ **N** such that,

$$\eth(\nu_{2nk}, \nu_{2mk}) \gtrsim c \tag{15}$$

Now looking after n_k, we choose smallest integer m_k in such way it $2m_k > 2n_k \geq k$ holds (14)

$$\eth(\nu_{2nk}, \nu_{2mk-2}) \prec c \tag{16}$$

By triangle inequality and eqution no. (14)(15) we write,

$$c \precsim \eth(\nu_{2nk}, \nu_{2mk}) \precsim \eth(\nu_{2nk}, \nu_{2mk-2}) + \eth(\nu_{2mk-2}, \nu_{2mk-1}) + \eth(\nu_{2mk-1}, \nu_{2mk}) \tag{17}$$

$$\prec c + \eth(\nu_{2mk-2}, \nu_{2mk-1}) + \eth(\nu_{2mk-1}, \nu_{2mk})$$

$$|c| \prec |\eth(\nu_{2nk}, \nu_{2mk})| \prec |c| + |\eth(\nu_{2mk-2}, \nu_{2mk-1})| + |\eth(\nu_{2mk-1}, \nu_{2mk})| \tag{18}$$

apply (13) and use $\lim_{k\to\infty}$,

$$|c| \leq \lim_{k\to\infty} |\eth(\nu_{2nk}, \nu_{2mk})| < |c|$$

$$\lim_{k\to\infty} |\eth(\nu_{2nk}, \nu_{2mk})| = |c| \tag{19}$$

again by triangle inequality,

$$|\eth(\nu_{2nk}, \nu_{2mk})| \leq |\eth(\nu_{2nk}, \nu_{2mk+1})| + |\eth(\nu_{2mk+1}, \nu_{2mk})|$$

$$|\eth(\nu_{2nk}, \nu_{2mk})| \leq |\eth(\nu_{2nk}, \nu_{2mk})| + |\eth(\nu_{2mk}, \nu_{2mk+1})| + |\eth(\nu_{2mk+1}, \nu_{2mk})|$$

apply k$\to\infty$,

$$\lim_{k\to\infty} |\eth(\nu_{2nk}, \nu_{2mk})| \leq \lim_{k\to\infty} |\eth(\nu_{2nk}, \nu_{2mk+1})| + \lim_{k\to\infty} |\eth(\nu_{2mk+1}, \nu_{2mk})| \leq \lim_{k\to\infty} |\eth(\nu_{2nk}, \nu_{2mk})|$$

$$+ \lim_{k\to\infty} |\eth(\nu_{2mk}, \nu_{2mk+1})| + \lim_{k\to\infty} |\eth(\nu_{2mk+1}, \nu_{2mk})|$$

$$|c| \leq \lim_{k\to\infty} |\eth(\nu_{2nk}, \nu_{2mk+1})| \leq |c|$$

$$\lim_{k\to\infty} |\eth(\nu_{2nk}, \nu_{2mk+1})| = |c| \tag{20}$$

Now,

$$\eth(\nu_{2nk}, \nu_{2mk+1}) \precsim \eth(\nu_{2nk}, \nu_{2nk+1}) + \eth(\nu_{2nk+1}, \nu_{2mk+2}) + \eth(\nu_{2mk+2}, \nu_{2mk+1})$$

$$\eth(\nu_{2nk}, \nu_{2mk+1}) \precsim \eth(\nu_{2nk}, \nu_{2nk+1}) + \eth(\zeta\mu_{2nk}, \psi\mu_{2mk+1}) + \eth(\nu_{2mk+2}, \nu_{2mk+1}) \tag{21}$$

apply Step(I), to the $\eth(\zeta\mu_{2nk}, \psi\mu_{2mk+1})$ from above equality,

$$|\eth(\nu_{2nk}, \nu_{2mk+1})| \precsim |\eth(\nu_{2nk}, \nu_{2nk+1})| + \Upsilon(\eth(\nu_{2nk}, \nu_{2mk+1})).|\eth(\nu_{2nk}, \nu_{2mk+1})| + 2|\eth(\nu_{2mk+2}, \nu_{2mk+1})| \tag{22}$$

Here we can write $|\eth(\nu_{2mk+2}, \nu_{2mk+1})|$ as,

$$|\eth(\nu_{2mk+2}, \nu_{2mk+1})| = \frac{|\eth(\nu_{2mk+2}, \nu_{2mk+1})|.|\eth(\nu_{2nk}, \nu_{2mk+1})|}{1 + |\eth(\nu_{2nk}, \nu_{2mk+1})|}$$

$$= \frac{[|\eth(\nu_{2nk}, \nu_{2nk+1})| + |\eth(\nu_{2nk+1}, \nu_{2mk+1})|].|\eth(\nu_{2mk+2}, \nu_{2mk+1})|}{1 + |\eth(\nu_{2nk}, \nu_{2mk+1})|}$$

$$= \frac{|\eth(\nu_{2nk}, \nu_{2nk+1})|.|\eth(\nu_{2mk+2}, \nu_{2mk+1})|}{1 + |\eth(\nu_{2nk}, \nu_{2mk+1})|} + \frac{[\eth(\nu_{2nk+1}, \nu_{2mk+1})|.|\eth(\nu_{2mk+2}, \nu_{2mk+1})|}{1 + |\eth(\nu_{2nk}, \nu_{2mk+1})|}$$

By using equality (21),

$$|\eth(\nu_{2nk}, \nu_{2mk+1})| \precsim |\eth(\nu_{2nk}, \nu_{2nk+1})| + \Upsilon(\eth(\nu_{2nk}, \nu_{2mk+1})).|\eth(\nu_{2nk}, \nu_{2mk+1})| + |\eth(\nu_{2mk+2}, \nu_{2mk+1})|$$

$$+ \frac{|\eth(\nu_{2nk}, \nu_{2nk+1})|.|\eth(\nu_{2mk+2}, \nu_{2mk+1})|}{1 + |\eth(\nu_{2nk}, \nu_{2mk+1})|} + \frac{[\eth(\nu_{2nk+1}, \nu_{2mk+1})|.|\eth(\nu_{2mk+2}, \nu_{2mk+1})|}{1 + |\eth(\nu_{2nk}, \nu_{2mk+1})|}$$

$$\precsim |\eth(\nu_{2nk}, \nu_{2nk+1})| + |\eth(\nu_{2nk}, \nu_{2mk+1})| + |\eth(\nu_{2mk+2}, \nu_{2mk+1})| +$$
$$+ \frac{|\eth(\nu_{2nk}, \nu_{2nk+1})|.|\eth(\nu_{2mk+2}, \nu_{2mk+1})|}{1 + |\eth(\nu_{2nk}, \nu_{2mk+1})|} + \frac{[\eth(\nu_{2nk+1}, \nu_{2mk+1})|.|\eth(\nu_{2mk+2}, \nu_{2mk+1})|}{1 + |\eth(\nu_{2nk}, \nu_{2mk+1})|}$$

Letting k tends to ∞, by (13) and (19) we get

$$|c| \leq \lim_{k\to\infty} |\Upsilon(\eth(\nu_{2nk}, \nu_{2mk+1})|).|c| \leq |c|$$

Hence we get,

$$\lim_{k\to\infty} |\Upsilon(\eth(\nu_{2nk}, \nu_{2mk+1})|) = 1 \tag{23}$$

and we know that $\Upsilon \in \Im$ which means $|\eth(\nu_{2nk}, \nu_{2mk+1})| \to 0$ as k$\to \infty$, This is the contradict to fact that $\{\nu_{2n}\}$ is not a cauchy sequence. Hence $\{\nu_{2n}\}$ is a cauchy sequence this emplies $\{\nu_n\}$ is a cauchy sequence and we have Θ complete, then their exist a point $t \in \Theta$ such that $\nu_n \to t$ as $n \to \infty$,

$$\lim_{n\to\infty} \zeta\mu_{2n} = \lim_{n\to\infty} \psi\mu_{2n+1} = \lim_{n\to\infty} \xi\mu_{2n} = \lim_{n\to\infty} \varphi\mu_{2n+1} = t \tag{24}$$

Since, $\zeta(\Theta) \subseteq \varphi(\Theta)$ $\exists \vartheta \in \Theta$ such that $\varphi_\vartheta = t$ which gives (23) as,

$$\lim_{n\to\infty} \zeta\mu_{2n} = \lim_{n\to\infty} \psi\mu_{2n+1} = \lim_{n\to\infty} \xi\mu_{2n} = \lim_{n\to\infty} \varphi\mu_{2n+1} = t = \varphi_\vartheta \tag{25}$$

Next we have to show $\psi_\vartheta = \varphi_\vartheta$, then assume that

$$\eth(t, \psi_\vartheta) \precsim \eth(t, \zeta\mu_{2n}) + \eth(\zeta\mu_{2n}, \psi_\vartheta) \tag{26}$$

by using $\mu = \mu_{2n}$ and $\nu = \vartheta$ apply to $\eth(\zeta\mu_{2n}, \psi_\vartheta)$ and using condition (I),

$$\eth(t, \psi_\vartheta) \precsim \eth(t, \zeta\mu_{2n}) + \eth(\xi\mu_{2n}, \varphi_\vartheta) + \frac{\eth(\zeta\mu_{2n}, \xi\mu_{2n}).\eth(\psi_\vartheta, \varphi_\vartheta)}{1 + \eth(\xi\mu_{2n}, \varphi_\vartheta)} + \frac{\eth(\zeta\mu_{2n}, \varphi_\vartheta).\eth(\psi_\vartheta, \varphi_\vartheta)}{1 + \eth(\xi\mu_{2n}, \varphi_\vartheta)}$$

apply $\lim_{n\to\infty}$ and with the help of (24), $\eth(t, \psi_\vartheta) \precsim 0$ which gives $\eth(t, \psi_\vartheta) = 0$ and by (23) we get,

$$\psi_\vartheta = \varphi_\vartheta = t$$

and on the other side $\psi(\Theta) \subseteq \xi(\Theta)$, $\exists \varsigma$ such that $\xi_\varsigma = t$ by (23)

$$\lim_{n\to\infty} \zeta\mu_{2n} = \lim_{n\to\infty} \psi\mu_{2n+1} = \lim_{n\to\infty} \xi\mu_{2n} = \lim_{n\to\infty} \varphi\mu_{2n+1} = t = \xi_\varsigma \tag{27}$$

Our next claim is $\xi_\varsigma = \zeta_\varsigma$, assume that

$$\eth(t, \zeta_\varsigma) \precsim \eth(\psi\mu_{2n+1}, \zeta\varsigma) + \eth(\psi\mu_{2n+1}, t)$$

when $\mu = \varsigma, \nu = \mu_{2n+1}$ similar like above, we will get $\eth(t, \zeta_\varsigma) = 0$, hence $\xi_\varsigma = \zeta_\varsigma = t$

$$\psi_\vartheta = \varphi_\vartheta = \xi_\varsigma = \zeta_\varsigma = t \tag{28}$$

apply defination of weak compatibility of pair (ζ,ξ),(ψ,φ) and by (27),

$$\varphi_\vartheta = \psi_\vartheta \Rightarrow \psi\varphi_\vartheta = \varphi\psi_\vartheta \Rightarrow \psi_t = \varphi_t \tag{29}$$

$$\xi_\varsigma = \zeta_\varsigma \Rightarrow \zeta\xi_\varsigma = \xi\zeta_\varsigma \Rightarrow \zeta_t = \xi_t \tag{30}$$

Here t is co incident point of above pair in Θ, Next we cheak $\zeta, \psi, \xi, \varphi$ has fixed point t, $\zeta_t = t$ On contrary suppose $\zeta_t = t$ and $\zeta_\varsigma = \varsigma$ and $t \neq \varsigma$ lets put $\mu = t$ and $\nu = \vartheta$ in step (I),

$$\frac{\eth(\zeta_t, \psi_\vartheta)}{\eth(\xi_t, \varphi_\vartheta)} \leq \eta_1(\eth(\xi_t, \varphi_\vartheta)).[1+\eta_2(\eth(\xi_t, \varphi_\vartheta)).[1+\eth(\varphi_\vartheta, \psi_\vartheta)]] - \eta_3(\eth(\xi_t, \varphi_\vartheta))[\frac{\eth(\psi_\vartheta, \varphi_\vartheta)}{1 + \eth(\psi_\vartheta, \varphi_\vartheta)}]$$
$$- \eta_4(\eth(\xi_t, \varphi_\vartheta))[\frac{\eth(\psi_\vartheta, \varphi_\vartheta)}{\eth(\psi_\vartheta, \varphi_\vartheta) + max[1, [\frac{\eth(\varphi_\vartheta, \psi_t)}{\eth(\xi_t, \varphi_\vartheta).\eth(\xi_t, \psi_\vartheta)}], [\frac{\eth(\xi_t, \varphi_t)}{\eth(\xi_t, \varphi_\vartheta).\eth(\xi_t, \psi_\vartheta)}]}]$$

apply (27) and (28) we get,

$$\frac{\eth(\zeta_t, t)}{\eth(\xi_t, t)} \le \eta_1(\eth(\xi_t, t)).[1 + \eta_2(\eth(\xi_t, t)).[1 + \eth(t, t)]] - \eta_3(\eth(\xi_t, t))[\frac{\eth(t, t)}{1 + \eth(t, t)}] -$$

$$\eta_4(\eth(\xi_t, t))[\frac{\eth(t, t)}{\eth(t, t) + max[1, [\frac{\eth(t,\psi_t)}{\eth(\xi_t,t).\eth(\xi_t,t)}], [\frac{\eth(\xi_t,\varphi_t)}{\eth(\xi_t,t).\eth(\xi_t,t)}]}]$$

$$\frac{\eth(\zeta_t, t)}{\eth(\xi_t, t)} \le \eta_1(\eth(\xi_t, t)) + \eta_1(\eth(\xi_t, t)).\eta_2(\eth(\xi_t, t)) + \eta_1(\eth(\xi_t, t)).\eta_2(\eth(\xi_t, t)).\eth(t, t)]]$$

$$\frac{\eth(\zeta_t, t)}{\eth(\xi_t, t)} \le \eta_1(\eth(\xi_t, t)) + \eta_1(\eth(\xi_t, t)).\eta_2(\eth(\xi_t, t))$$

As we know $\zeta_t = \xi_t$,

$$1 \le \eta_1(\eth(\xi_t, t))(1 + \eta_2(\eth(\xi_t, t)))$$

Which is not possible, Our assumption is wrong. It gives $t = \varsigma \Rightarrow \zeta_t = t$ by (23), $\zeta_t = \xi_t = t$. Simillarly letting $\mu = \varsigma$ and $\nu = t$ in step-I 2 and using (27),(28), one can obtained $\psi_t = \varphi_t = t$.

$$\zeta_t = \psi_t = \varphi_t = \xi_t = t \tag{31}$$

Hence, $\zeta, \psi, \varphi, \xi$ admits fixed point as t. Now we need to work for uniqeness of t, assume that $t^* = t$ be the fixed point of $\zeta, \psi, \varphi, \xi$ and put $\mu = t$ and $\nu = t^*$ then step(I),

$$\frac{\eth(\zeta_t, \psi_{t^*})}{\eth(\xi_t, \varphi_{t^*})} \le \eta_1(\eth(\xi_t, \varphi_{t^*})).[1+\eta_2(\eth(\xi_t, \varphi_{t^*})).[1+\eth(\varphi_{t^*}, \psi_{t^*})]] - \eta_3(\eth(\xi_t, \varphi_{t^*}))[\frac{\eth(\psi_{t^*}, \varphi_{t^*})}{1 + \eth(\psi_{t^*}, \varphi_{t^*})}]$$

$$- \eta_4(\eth(\xi_t, \varphi_{t^*}))[\frac{\eth(\psi_{t^*}, \varphi_{t^*})}{\eth(\psi_{t^*}, \varphi_{t^*}) + max[1, [\frac{\eth(\varphi_{t^*},\psi_t)}{\eth(\xi_t,\varphi_{t^*}).\eth(\xi_t,\psi_{t^*})}], [\frac{\eth(\xi_t,\varphi_t)}{\eth(\xi_t,\varphi_{t^*}).\eth(\xi_t,\psi_{t^*})}]}]$$

$$\frac{\eth(t, t^*)}{\eth(t, t^*)} \le \eta_1(\eth(t, t^*)).[1 + \eta_2(\eth(t, t^*)).[1 + \eth(t^*, t^*)]] - \eta_3(\eth(t, t^*))[\frac{\eth(t^*, t^*)}{1 + \eth(t^*, t^*)}] -$$

$$\eta_4(\eth(t, t^*))[\frac{\eth(t^*, t^*)}{\eth(t^*, t^*) + max[1, [\frac{\eth(t^*,t)}{\eth(t,t^*).\eth(t,t^*)}], [\frac{\eth(t,t)}{\eth(t,t^*).\eth(t,t^*)}]}]$$

$$\frac{\eth(t, t^*)}{\eth(t, t^*)} \le \eta_1(\eth(t, t^*)) + \eta_1(\eth(t, t^*)).\eta_2(\eth(t, t^*)).[1 + \eth(t^*, t^*)]]$$

$$1 \le \eta_1(\eth(t, t^*))(1 + \eta_2(\eth(t, t^*)))$$

Which is contradiction to our assumption that $t \neq t^*$. and $t = t^*$, Hence, $\zeta, \psi, \varphi, \xi$ admits unique fixed point as t.

□

By using the main result 3.1 we can derive the following corollaries and proof of every corollary similar to the above theorem so we omit it.

Corollary 3.0. *If* ζ,ψ *be two self mapping on a complex valued metric space* (Θ,$\eth$) *and* η_i:$\mathbb{C}_+ \to$*[0,1),* $\forall$ *i =1,2,3,4 be given mapping suppose we have following condition,*
Step I)

$$\frac{\eth(\zeta_\mu, \psi_\nu)}{\eth(\mu, \nu)} \le \eta_1(\eth(\mu, \nu)).[1 + \eta_2(\eth(\mu, \nu)).[1 + \eth(\nu, \psi_\nu)]] - [\frac{\eta_3(\eth(\mu, \nu)).\eth(\psi_\nu, \nu)}{1 + \eth(\psi_\nu, \nu)}] -$$

$$[\frac{\eta_4(\eth(\mu, \nu)).\eth(\psi_\nu, \nu)}{\eth(\psi_\nu, \nu) + max[1, [\frac{\eth(\nu,\psi_\mu)}{\eth(\mu,\nu).\eth(\mu,\psi_\nu)}], [\frac{\eth(\mu,\mu)}{\eth(\mu,\nu).\eth(\mu,\psi_\nu)}]]}] \tag{32}$$

Step II)

$$\eta_1(\mu)(1+\eta_2(\mu))+\eta_3(\mu)+\eta_4(\mu)<1, \eta_1(\mu)(1+\eta_2(\mu))<1 \tag{33}$$

Step III) The mapping Υ*:* $\mathbb{C}_+ \to$*[0,1) define as,*

$$\Upsilon(\mu)=[\frac{\eta_1(\mu).(1+\eta_2(\mu))}{1-[\mu.\eta_1(\mu).\eta_2(\mu)]}]-[\frac{\eta_3(\mu)+\eta_4(\mu)}{1-[\mu.\eta_1(\mu).\eta_2(\mu)]}], \forall\mu\in\mathbb{C}_+ \tag{34}$$

If (ζ,ξ) & (ψ,φ) *both weakly compatible then* ζ,ψ *admits unique fixed point in* Θ.

Corollary 3.0. *If* ζ *be self mapping on a complex valued metric space* $(\Theta,\eth)$ *and* η_i*:*$\mathbb{C}_+\to$*[0,1),* $\forall$ *i =1,2,3,4 be given mapping suppose we have following condition,*
Step I)

$$\frac{\eth(\zeta_\mu,\zeta_\nu)}{\eth(\mu,\nu)}\le\eta_1(\eth(\mu,\nu)).[1+\eta_2(\eth(\mu,\nu)).[1+\eth(\nu,\zeta_\nu)]]-[\frac{\eta_3(\eth(\mu,\nu)).\eth(\zeta_\nu,\nu)}{1+\eth(\zeta_\nu,\nu)}]-$$

$$[\frac{\eta_4(\eth(\mu,\nu)).\eth(\zeta_\nu,\nu)}{\eth(\zeta_\nu,\nu)+max[1,[\frac{\eth(\nu,\zeta_\mu)}{\eth(\mu,\nu).\eth(\mu,\zeta_\nu)}],[\frac{\eth(\mu,\mu)}{\eth(\mu,\nu).\eth(\mu,\zeta_\nu)}]]}] \tag{35}$$

Step II)

$$\eta_1(\mu)(1+\eta_2(\mu))+\eta_3(\mu)+\eta_4(\mu)<1, \eta_1(\mu)(1+\eta_2(\mu))<1 \tag{36}$$

Step III) The mapping Υ*:* $\mathbb{C}_+ \to$*[0,1) define as,*

$$\Upsilon(\mu)=[\frac{\eta_1(\mu).(1+\eta_2(\mu))}{1-[\mu.\eta_1(\mu).\eta_2(\mu)]}]-[\frac{\eta_3(\mu)+\eta_4(\mu)}{1-[\mu.\eta_1(\mu).\eta_2(\mu)]}], \forall\mu\in\mathbb{C}_+ \tag{37}$$

then ζ *admits unique fixed point in* Θ.

Theorem 3.2. *If* ρ *be two self mapping on a complex valued metric space* $(\Theta,\eth)$ *and* η_i*:*$\mathbb{C}_+\to$*[0,1),* $\forall$ *i =1,2,3,4 be given mapping suppose we have following condition,*
Step I)

$$\frac{\eth(\rho^n_\mu,\rho^n_\nu)}{\eth(\mu,\nu)}\le\eta_1(\eth(\mu,\nu)).[1+\eta_2(\eth(\mu,\nu)).[1+\eth(\nu,\rho^n_\nu)]]-\eta_3(\eth(\mu,\nu))[\frac{\eth(\rho^n_\nu,\nu)}{1+\eth(\rho^n_\nu,\nu)}]-$$

$$\eta_4(\eth(\mu,\nu))[\frac{\eth(\rho^n_\nu,\nu)}{\eth(\rho^n_\nu,\nu)+max[1,[\frac{\eth(\nu,\rho^n_\mu)}{\eth(\mu,\nu).\eth(\mu,\rho^n_\nu)}],[\frac{\eth(\mu,\mu)}{\eth(\mu,\nu).\eth(\mu,\rho^n_\nu)}]]}] \tag{38}$$

Step II)

$$\eta_1(\mu)(1+\eta_2(\mu))+\eta_3(\mu)+\eta_4(\mu)<1, \eta_1(\mu)(1+\eta_2(\mu))<1 \tag{39}$$

Step III) The mapping Υ*:* $\mathbb{C}_+ \to$*[0,1) define as,*

$$\Upsilon(\mu)=[\frac{\eta_1(\mu).(1+\eta_2(\mu))}{1-[\mu.\eta_1(\mu).\eta_2(\mu)]}]-[\frac{\eta_3(\mu)+\eta_4(\mu)}{1-[\mu.\eta_1(\mu).\eta_2(\mu)]}], \forall\mu\in\mathbb{C}_+ \tag{40}$$

then ρ *admits unique fixed point in* Θ.

Proof. With the help of technique given in the proof of theorem (3.6) in [6] and our main result 3.1 we get the proof. □

Remark 1. *Using map $\zeta, \xi, \varphi, \psi$ in Theorem 3.1 subsequently results 3.0, 3.0, 3.2 & point dependent control function $\{\eta_i, 1 \leq i \leq 4\}$ one can deduce multitude of Result from literature in CVMS including Banach contraction Result.*

4. Application of main result to the Urysohn integral equations.

In this section, we use our main theoretical result to prove the existence and uniqueness of a common fixed point solution to the Urysohn integral equation. Many mathematicians successfully studied the nonlinear integral equation and its common and unique fixed point solution. In this section we are dealing with Urysohn integral equation of type,

$$\omega(t) = \gamma_i(t) + \int_\alpha^\beta \chi_i(t, s, \omega(s))ds. \tag{41}$$

Where $\omega, \gamma_i \in C([\alpha, \beta], \mathbb{R}^n)$, $[\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}^n \to \mathbb{R}^n$, i=1,2,3,4 and $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$, $t \in [\alpha, \beta]$ and in following proof we use

$$\triangle_i(\omega(t)) = \int_\alpha^\beta \chi_i(t, s, \omega(s))ds. \tag{42}$$

Theorem 4.3. *Consider Urysohn Integral equation of type 41 and we have following assumption for each $t \in [\alpha, \beta]$*

$$(P_1): \gamma_1(t)+3\gamma_3(t) = 4\omega(t)-2\triangle_3(\omega(t))-\triangle_1[\triangle_1\omega(t)+\gamma_1(t)]-\triangle_3(2\omega(t)-\triangle_3\omega(t)-\gamma_3(t))$$

$$\gamma_2(t)+3\gamma_4(t) = 4\omega(t)-2\triangle_4(\omega(t))-\triangle_2[\triangle_2\omega(t)+\gamma_2(t)]-\triangle_4(2\omega(t)-\triangle_4\omega(t)-\gamma_4(t))$$

$$(P_2): \gamma_1(t) + \gamma_4(t) = \triangle_4(\triangle_1\omega(t) + \gamma_1(t) + \gamma_4(t)) - \triangle_1\omega(t)$$

$$\gamma_2(t) + \gamma_3(t) = \triangle_3(\triangle_2\omega(t) + \gamma_2(t) + \gamma_3(t)) - \triangle_2\omega(t)$$

there exist η_i:$\mathbb{C}_+ \to [0,1)$, $\forall$ i =1,2,3,4 be given mapping suppose we have following condition,
Step I) The mapping Υ: $\mathbb{C}_+ \to [0,1)$ define as,

$$\Upsilon(\mu) = [\frac{\eta_1(\mu).(1 + \eta_2(\mu))}{1 - [\mu.\eta_1(\mu).\eta_2(\mu)]}] - [\frac{\eta_3(\mu) + \eta_4(\mu)}{1 - [\mu.\eta_1(\mu).\eta_2(\mu)]}], \forall \mu \in \mathbb{C}_+ \tag{43}$$

Which belongs to $\Im$;
Step II)

$$\eta_1(\mu)(1 + \eta_2(\mu)) + \eta_3(\mu) + \eta_4(\mu) < 1, \eta_1(\mu)(1 + \eta_2(\mu)) < 1$$

Step III) for every $\mu, \nu \in \Theta$ and $t \in [\alpha, \beta]$,

$$\frac{\mathbb{E}_{\mu\nu(t)}.\sqrt{1 + (\alpha)^2}e^{icot\alpha}}{\mathbb{F}_{\mu\nu(t)}.\sqrt{1 + (\alpha)^2}e^{icot\alpha}} \precsim \eta_1(\max_{t \in [\alpha, \beta]} \mathbb{F}_{\mu\nu(t)}.\sqrt{1 + (\alpha)^2}e^{icot\alpha}).[1+\eta_2(\max_{t \in [\alpha, \beta]} \mathbb{F}_{\mu\nu(t)}.\sqrt{1 + (\alpha)^2}e^{icot\alpha})$$

$$.[1+(\mathbb{G}_{\mu\nu(t)}.\sqrt{1 + (\alpha)^2}e^{icot\alpha})]-\eta_3(\max_{t \in [\alpha, \beta]} \mathbb{F}_{\mu\nu(t)}.\sqrt{1 + (\alpha)^2}e^{icot\alpha})[(\mathbb{H}_{\mu\nu(t)}.\sqrt{1 + (\alpha)^2}e^{icot\alpha})]$$

$$- \eta_4(\max_{t \in [\alpha, \beta]} \mathbb{F}_{\mu\nu(t)}.\sqrt{1 + (\alpha)^2}e^{icot\alpha}).[\mathbb{I}_{\mu\nu(t)}.\sqrt{1 + (\alpha)^2}e^{icot\alpha}] \tag{44}$$

where,

$$\mathbb{E}_{\mu\nu(t)} = \|\gamma_1(t)-\gamma_2(t)+\triangle_1(\mu(t))-\triangle_2(\nu(t))\|_\infty, \mathbb{F}_{\mu\nu(t)} = \|2\mu(t)-2\nu(t)-\gamma_3(t)+\gamma_4(t)-\triangle_3(\mu(t))+\triangle_4(\nu(t))\|_\infty,$$

$$\mathbb{G}_{\mu\nu(t)} = \|\gamma_4(t) + \triangle_4(\nu(t)) - 2\nu(t) - \triangle_2(\nu(t)) - \gamma_2(t)\|_\infty,$$

$$\mathbb{H}_{\mu\nu(t)} = \| \frac{\gamma_4(t) + \triangle_4(\nu(t)) - 2\nu(t) - \triangle_2(\nu(t)) - \gamma_2(t)}{1 + [\gamma_4(t) + \triangle_4(\nu(t)) - 2\nu(t) - \triangle_2(\nu(t)) - \gamma_2(t)]} \|_\infty,$$

$$\mathbb{I}_{\mu\nu(t)} = \| \frac{\gamma_4(t) + \triangle_4(\nu(t)) - 2\nu(t) - \triangle_2(\nu(t)) - \gamma_2(t)}{\gamma_4(t) + \triangle_4(\nu(t)) - 2\nu(t) - \triangle_2(\nu(t)) - \gamma_2(t) + max\{1, [\mathbb{A}_{\mu\nu(t)}], [\mathbb{B}_{\mu\nu(t)}]\}} \|_\infty$$

where, $\mathbb{A}_{\mu\nu(t)}$ *and* $\mathbb{B}_{\mu\nu(t)}$ *are resp.,*

$$\frac{\triangle_2(\mu(t)) + \gamma_2(t) - \gamma_4(t) + 2\nu(t) - \triangle_4(\nu(t))}{[2\mu(t) - 2\nu(t) - \gamma_3(t) + \gamma_4(t) - \triangle_3(\mu(t)) + \triangle_4(\nu(t))].[\triangle_2(\nu(t)) + \gamma_2(t) - \gamma_3(t) - \triangle_3(\mu(t)) + 2\mu(t)]} \&$$

$$\frac{2\mu(t) - \triangle_3(\mu(t)) - \gamma_3(t) + \gamma_4(t) - 2\mu(t) + \triangle_4(\mu(t))}{[2\mu(t) - 2\nu(t) - \gamma_3(t) + \gamma_4(t) - \triangle_3(\mu(t)) + \triangle_4(\nu(t))].[\triangle_2(\nu(t)) + \gamma_2(t) - \gamma_3(t) - \triangle_3(\mu(t)) + 2\mu(t)]}$$

Then Urysohn integral equation admits unique common solution.

Proof. Let's define a map $\eth : \Theta \times \Theta \to \mathbb{C}$ and

$$\eth(\mu, \nu) = \max_{t\in[\alpha,\beta]} \|\mu(t) - \nu(t)\|_\infty . \sqrt{1 + (\alpha)^2} e^{icot\alpha}$$

and $C([\alpha, \beta], \mathbb{R}^n)$, then $(\Theta, \eth)$ is complete complex valued metric space. Now we take four self mapping,

$$\zeta_{\mu(t)} = \triangle_1(\mu(t)) + \gamma_1(t) = \gamma_1(t) + \int_\alpha^\beta \chi_1(t, s, \omega(s))ds,$$

$$\psi_{\mu(t)} = \triangle_2(\mu(t)) + \gamma_2(t) = \gamma_2(t) + \int_\alpha^\beta \chi_2(t, s, \omega(s))ds,$$

$$\xi_{\mu(t)} = 2(\mu(t)) - \triangle_3(\mu(t)) + \gamma_3(t) = 2\mu(t) - \gamma_3(t) - \int_\alpha^\beta \chi_3(t, s, \omega(s))ds,$$

$$\varphi_{\mu(t)} = 2(\mu(t)) - \triangle_4(\mu(t)) - \gamma_4(t) = 2\mu(t) - \gamma_4(t) - \int_\alpha^\beta \chi_4(t, s, \omega(s))ds, \quad (45)$$

take condition of 44 in above theorem,which gives

$$\frac{\mathbb{E}_{\mu\nu(t)} . \sqrt{1 + (\alpha)^2} e^{icot\alpha}}{\mathbb{F}_{\mu\nu(t)} . \sqrt{1 + (\alpha)^2} e^{icot\alpha}} \precsim \eta_1(\max_{t\in[\alpha,\beta]} \mathbb{F}_{\mu\nu(t)} . \sqrt{1 + (\alpha)^2} e^{icot\alpha}).[1+\eta_2(\max_{t\in[\alpha,\beta]} \mathbb{F}_{\mu\nu(t)} . \sqrt{1 + (\alpha)^2} e^{icot\alpha})$$

$$.[1+(\mathbb{G}_{\mu\nu(t)} . \sqrt{1 + (\alpha)^2} e^{icot\alpha})] - \eta_3(\max_{t\in[\alpha,\beta]} \mathbb{F}_{\mu\nu(t)} . \sqrt{1 + (\alpha)^2} e^{icot\alpha})[(\mathbb{H}_{\mu\nu(t)} . \sqrt{1 + (\alpha)^2} e^{icot\alpha})]$$

$$- \eta_4(\max_{t\in[\alpha,\beta]} \mathbb{F}_{\mu\nu(t)} . \sqrt{1 + (\alpha)^2} e^{icot\alpha}).[\mathbb{I}_{\mu\nu(t)} . \sqrt{1 + (\alpha)^2} e^{icot\alpha}]$$

$$\frac{\max_{t\in[\alpha,\beta]} \mathbb{E}_{\mu\nu(t)} \sqrt{1 + (\alpha)^2} e^{icot\alpha}}{\max_{t\in[\alpha,\beta]} \mathbb{F}_{\mu\nu(t)} \sqrt{1 + (\alpha)^2} e^{icot\alpha}} \precsim \eta_1(\max_{t\in[\alpha,\beta]} \mathbb{F}_{\mu\nu(t)} \sqrt{1 + (\alpha)^2} e^{icot\alpha})[1+\eta_2(\max_{t\in[\alpha,\beta]} \mathbb{F}_{\mu\nu(t)} \sqrt{1 + (\alpha)^2} e^{icot\alpha})$$

$$.[1+\max_{t\in[\alpha,\beta]}(\mathbb{G}_{\mu\nu(t)} \sqrt{1 + (\alpha)^2} e^{icot\alpha})] - \eta_3(\max_{t\in[\alpha,\beta]} \mathbb{F}_{\mu\nu(t)} \sqrt{1 + (\alpha)^2} e^{icot\alpha})[\max_{t\in[\alpha,\beta]}(\mathbb{H}_{\mu\nu(t)} \sqrt{1 + (\alpha)^2} e^{icot\alpha})]$$

$$- \eta_4(\max_{t\in[\alpha,\beta]} \mathbb{F}_{\mu\nu(t)} \sqrt{1 + (\alpha)^2} e^{icot\alpha}) \max_{t\in[\alpha,\beta]}[\mathbb{I}_{\mu\nu(t)} \sqrt{1 + (\alpha)^2} e^{icot\alpha}]$$

when $\mu, \nu \in \Theta$ one obtained

$$\eth(\xi_\mu, \psi_\nu) = \max_{t\in[\alpha,\beta]} \|2\mu(t) - \gamma_3(t) + \gamma_2(t) - \triangle_3(\mu(t)) + \triangle_2(\nu(t))\|_\infty,$$

$$\eth(\zeta_\mu, \psi_\nu) = \max_{t\in[\alpha,\beta]} \|\gamma_1(t) - \gamma_2(t) + \triangle_1(\mu(t)) - \triangle_2(\nu(t))\|_\infty,$$

$$\eth(\xi_\mu, \varphi_\nu) = \max_{t\in[\alpha,\beta]} \|2\mu(t) - 2\nu(t) - \gamma_3(t) + \gamma_4(t) - \triangle_3(\mu(t)) + \triangle_4(\nu(t))\|_\infty,$$

$$\eth(\xi_\mu, \varphi_\mu) = \max_{t\in[\alpha,\beta]} \|\gamma_3(t) - \gamma_4(t) + \triangle_3(\mu(t)) - \triangle_4(\mu(t))\|_\infty,$$

$$\eth(\varphi_\nu, \psi_\mu) = \max_{t\in[\alpha,\beta]} \|\triangle_4(\nu(t)) - 2\nu(t) + \gamma_4(t) - \gamma_2(t) - \triangle_2(\mu(t))\|_\infty,$$

$$\eth(\varphi_\nu, \psi_\nu) = \max_{t\in[\alpha,\beta]} \|\gamma_2(t) - \gamma_4(t) + \triangle_2(\nu(t)) - 2\nu(t) - \triangle_4(\nu(t))\| \tag{46}$$

with the help of above 45 we get,

$$\frac{\eth(\zeta_\mu, \psi_\nu)}{\eth(\xi_\mu, \varphi_\nu)} \leq \eta_1(\eth(\xi_\mu, \varphi_\nu)).[1+\eta_2(\eth(\xi_\mu, \varphi_\nu)).[1+\eth(\varphi_\nu, \psi_\nu)]]-\eta_3(\eth(\xi_\mu, \varphi_\nu))[\frac{\eth(\psi_\nu, \varphi_\nu)}{1+\eth(\psi_\nu, \varphi_\nu)}]-$$

$$\eta_4(\eth(\xi_\mu, \varphi_\nu))[\frac{\eth(\psi_\nu, \varphi_\nu)}{\eth(\psi_\nu, \varphi_\nu) + max[1, [\frac{\eth(\varphi_\nu, \psi_\mu)}{\eth(\xi_\mu, \varphi_\nu).\eth(\xi_\mu, \psi_\nu)}], [\frac{\eth(\xi_\mu, \varphi_\mu)}{\eth(\xi_\mu, \varphi_\nu).\eth(\xi_\mu, \psi_\nu)}]}]$$

we have to prove, $\zeta(\Theta) \subseteq \varphi(\Theta)$ with the help of equation (45),

$$\varphi(\mu(t)) = 2(\mu(t)) - \triangle_4(\mu(t)) - \gamma_4(t)\&\mu(t) = \zeta\mu(t) + \gamma_4(t)$$

$$\varphi(\zeta\mu(t)+\gamma_4(t)) = 2(\zeta\mu(t)+\gamma_4(t))-\triangle_4(\zeta\mu(t)+\gamma_4(t))-\gamma_4(t) = \zeta\mu(t)+\zeta\mu(t)+\gamma_4(t)-\triangle_4(\zeta\mu(t)+\gamma_4(t))$$
$$= \zeta\mu(t) + \gamma_4(t) - \triangle_4(\gamma_1(t) + \triangle_1\mu(t) + \gamma_4(t)) + \triangle_1(\mu(t)) + \gamma_1(t)$$

with the help of (P_2), we write

$$\varphi(\zeta\mu(t) + \gamma_4(t)) = \zeta\mu(t)$$

which gives $\zeta(\Theta) \subseteq \varphi(\Theta)$ similarly as above we get, $\psi(\Theta) \subseteq \xi(\Theta)$, our next claim for each $t \in [\alpha, \beta]$, $(\zeta,\ \xi)$ and $(\psi,\ \varphi)$ are weakly compatible.

$$\|\xi\zeta\mu(t) - \zeta\xi\mu(t)\| = \|\xi(\triangle_1(\mu(t)) + \gamma_1(t)) - \zeta(2\mu(t) - \triangle_3(\mu(t)) - \gamma_3(t))\|$$
$$= \|2(\triangle_1(\mu(t)+\gamma_1(t))-\triangle_3(\triangle_1\mu(t)+\gamma_1(t))-\gamma_3(t)-\triangle_1(2\mu(t)-\triangle_3\mu(t)-\gamma_3(t))-\gamma_1(t)\| \tag{47}$$

for every $\mu \in \Theta$ when $\zeta\mu = \xi\mu$,

$$\triangle_1\mu(t) + \gamma_1(t) = 2\mu(t) - \triangle_3\mu(t) - \gamma_3(t)$$

By equation 47 we write, $\|\xi\zeta\mu(t) - \zeta\xi\mu(t)\|$ as

$$= \|2(2\mu(t)-\triangle_3\mu(t)-\gamma_3(t))-\triangle_3(2\mu(t)-\triangle_3\mu(t)-\gamma_3(t))-\gamma_3(t)-\triangle_1(\triangle_1\mu(t)+\gamma_1(t))-\gamma_1(t)\|$$

$$= \|4\mu(t)-2\triangle_3\mu(t)-3\gamma_3(t)-\triangle_3(2\mu(t)-\triangle_3\mu(t)-\gamma_3(t))-\triangle_1(\triangle_1\mu(t)+\gamma_1(t))-\gamma_1(t)\|$$

and finally with the help of (P_2) we get

$$\|\xi\zeta\mu(t) - \zeta\xi\mu(t)\| = 0 \Rightarrow \xi\zeta\mu(t) = \zeta\xi\mu(t)$$

whenever, $\xi\zeta\mu = \zeta\xi\mu$, Hence for every $t \in [\alpha, \beta]$, $(\zeta,\ \xi)$ is weakly compatible and similarly we get $(\psi,\ \varphi)$ is also weakly compatible. Hence there exist unique common fixed point for ζ,ψ,ξ,φ in Θ and Hence 41 has unique common solution. □

5. Conclusion

In our paper we proved CFP theorem under more generalized rational type contraction condition using class function and four self mapping. Our main results extend and improved many results given in the literature. Moreover we presented an application part to the integral equation 41.

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