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SUPERLATIVE TOTAL DOMINATION IN GRAPHS

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Abstract. Let $G = (V, E)$ be a simple graph with no isolated vertices and $p \geq 3$. A set $D \subseteq V$ is a dominating set, abbreviated as DS, of a graph G, if every vertex in $V - D$ is adjacent to some vertex in D, while a total dominating set, abbreviated as TDS, of G is a set $T \subseteq V$ such that every vertex in G is adjacent to a vertices in T. A set T is a superlative total dominating set, abbreviated as $STDS$, of G if $V - T$ is not contains a TDS but it contains a DS of G. The superlative total domination number $\gamma_{st}(G)$ is the minimum cardinality of a $STDS$ of G. In this paper, we initiate a study on $\gamma_{st}(G)$ and its exact values for some classes of graphs. Furthermore, bounds in terms of order, size, degree and other domination related parameters are investigated.

1. Introduction

All the graphs $G = (V, E)$ considered here are simple, finite, nontrivial and undirected, where $\mid V \mid= p$ denotes number of vertices and $\mid E \mid= q$ denotes number of edges of G. In general, we use $\langle A \rangle$ to denote the subgraph induced by the set of vertices A. The set of all vertices which are adjacent to a vertex v is called open neighborhood of v and denoted by $N(v)$. The closed neighborhood set of a vertex v is the set $N[v] = N(v) \cup \{v\}$. Let $deg(v)$ be the degree of vertex v and usual $\delta(G)$, the minimum degree and $\Delta(G)$, the maximum degree of G. If v has degree one, then the vertex v is known as end-vertex of G (i.e., $\delta(G) = 1$). The complement graph $\overline{G} = (V, \overline{E})$ is a graph with $uv \in \overline{E}(\overline{G})$ if and only if $uv \notin E(G)$ for all $\{u, v\} \subseteq V(G)$. For a real number $n > 0$, let $\lfloor n \rfloor$ (or, $\lceil n \rceil$) be the greatest (least) integer not greater (less) than or equal to n. For graph-theoretic terminology and notation not defined here, we follow [8].

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The domination in graphs has been an extremely researched branch of graph theory. For more and comprehensive details of domination-related parameters and their applications, the reader refereed to [6, 7, 9, 10, 11, 12].

A set $D \subseteq V$ is a dominating set, abbreviated as DS, of a graph G with no isolated vertex, if every vertex in $V - D$ is adjacent to some vertex in D, while a total dominating set T of G, abbreviated as TDS, is a subset of $V(G)$, such that every vertex in G is adjacent to a vertex in T, see [3].

Further, let D and T be a minimum DS and TDS of a graph G, respectively.

(i) If $V - D$ contains a DS, say D', then D' is called an inverse dominating set, abbreviated as IDS of G with respect to D, see [20].
(ii) If $V - D$ contains no a DS, then D is called a maximal dominating set, abbreviated as MDS, of G, see [19].
(iii) If $V - T$ contains a TDS, say T', then T' is called an inverse total dominating set, abbreviated as $ITDS$ of G with respect to T, see [5, 18].
(iv) If $V - T$ contains no a TDS, then T is called a maximal total dominating set, abbreviated as $MTDS$, of G, see [21].
(v) The domatic number of G, denoted $d(G)$, is the maximum number of disjoint DS of G, see [3, 10].
(vi) The total domatic number of G, denoted $d_t(G)$, is the maximum number of disjoint TDS of G, see [2, 4, 23].

Analogously, the domination, total, inverse, maximal, inverse total and maximal total domination number of G, denoted by $\gamma(G), \gamma_t(G), \gamma_i(G)$, $\gamma_m(G), \gamma_{it}(G)$ and $\gamma_{mt}(G)$, is the minimum cardinality of a DS, TDS, IDS, MDS, $ITDS$ and $MTDS$ of a graph G, respectively.

2. **Superlative total domination**

In this paper, motivated by the work of Michael Henning and others [4, 13, 14, 15, 16, 17], towards the contributions of total domination and its related parameters, we introduce $STDS$ of graphs as follows:

A set $T \subseteq V$ is a superlative total dominating set, abbreviated as $STDS$, of a graph G with no isolated vertex if $V - T$ is not a TDS but it is a DS of G. The superlative total domination number $\gamma_{st}(G)$ is the minimum cardinality of a $STDS$ of G. A γ_{st}-set is a minimum $STDS$ of G. Similarly, other sets (i.e, domination related parameters) can be expected. We note that, if the connected graph G satisfying γ_{st}-set T with $p \geq 3$, then $\langle V - T \rangle$ is either disconnected or K_1.

Observation 2.1. *Let D and T be the γ-set and γ_t-set of a connected graph G with $p \geq 3$. Then, the vertex $v \in D'$, which is an IDS, as well as its support vertex (cut-vertex) $u \in D \subseteq T$ must be included in every DS, TDS, $STDS$ of G, where $e = uv$ represents an edge in G.*

To prove next couple of results towards $STDS$, we construct the graph G^* as follows (see, Figure 1):

Step 1. The graph G^* is obtained by taking a single copy of a non-trivial connected graph G_1 with vertices $\{u_1, u_2, \ldots, u_k\}$.

Step 2. Attach k copies of a subgraph H_i $(1 \leq i \leq k)$ with vertices $\{v_1, v_2, \ldots, v_t\}$ from graph G_2 to each vertex u_i in $V(G_1)$, such that $\cup_{i=1}^{k} V(H_i) = V(G_2)$, $| V(H_i) |= t$ and $| V(G_2) |= kt$. Here, the components H_i in G_2 may be connected or disconnected.

Step 3. Additionally, every vertex in G_1 is adjacent to k copies of totally disconnected subgraphs I_i $(1 \leq i \leq k)$ in a totally disconnected graph G_3. Each subgraph I_i has vertices $\{w_1, w_2, \ldots, w_s\}$, such that $\cup_{i=1}^{k} V(I_i) = V(G_3)$, $\mid V(I_i) \mid = s$ and $\mid V(G_3) \mid = ks$.

As a result, the total number of vertices in G^* is given by $|V(G^*)| = k(1+t+s)$.

In the graph G^*, we encounter the following cases:

Case 1. $G^* - G_3 \cong G_1 \circ H_i$ $(1 \leq i \leq k)$. This corresponds to a corona product graph, where G_1 is a non-trivial connected graph. In other words, the corona product of G_1 and H_i is defined as the graph obtianed by taking one copy of G_1 and $\mid V(G_1) \mid$ copies of H_i and joining the j^{th} vertex of G_1 to every vertex in the j^{th} copy of H_i.

Case 2. The double star graph $d(p_1, p_2)$ is formed by connecting two stars, each with p_1 and p_2 vertices, with an edge. This is characterized by $s = 0$, $k = 2$, and $\langle H_i \rangle \cong \overline{K_t}$ $(i = 1, 2)$, where $\overline{K_t}$ represents the complementary graph of the complete graph K_t with $t \geq 1$.

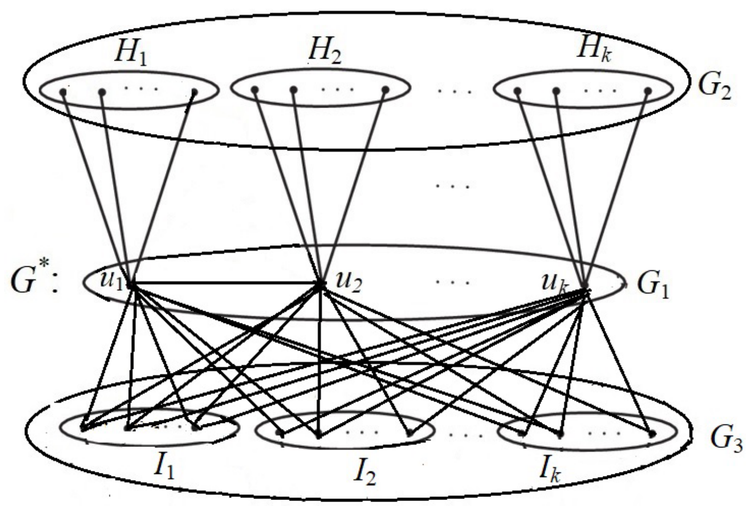

FIGURE 1. Graph of G^*.

For example, let's consider the graph G^* constructed with a non-trivial connected graph G_1 with vertices $V(G_1) = \{u_1, u_2\}$. Consider a graph G_2 composed of two components: H_1 and H_2, where each component is isomorphic to a path P_2. Specifically, $H_1 \cong P_2 = \{v_1, v_2\}$ and $H_2 \cong P_2 = \{v_1', v_2'\}$. Each component of G_2 forms a path P_2. Now, introduce a totally disconnected graph G_3 composed of two components: I_1 and I_2, where each component is isomorphic to the complementary graph $\overline{K_2}$. Specifically, $I_1 \cong \overline{K_2} = \{w_1, w_2\}$ and $I_2 \cong \overline{K_2} = \{w_1', w_2'\}$.

Therefore, the vertices in the resulting graph G^* are arranged as $V(G^*) = \{u_1, u_2, v_1, v_2, v_1', v_2', w_1, w_2, w_1', w_2'\}$.

In this context, the following sets are $STDS$ of G^*: $\{u_1, u_2\}$, $\{u_1, u_2, v_1\}$ and $\{u_1, u_2, v_1, v_1'\}$. Among these sets, γ_{st}-set is $\{u_1, u_2\}$. Hence, we have $\gamma_{st}(G^*) = 2$ in this specific example, where $k = t = s = 2$ (see, Figure 2).

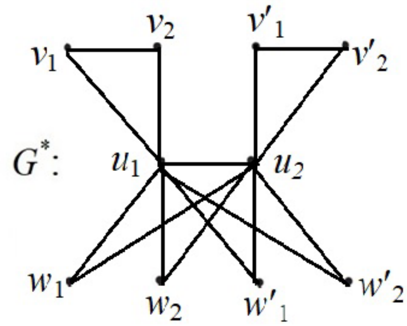

FIGURE 2. Graph with $STDS$ and $\gamma_{st}(G^*)$ for $s = t = k = 2$.

An application of $STDS$ is found in a computer network. Suppose $D \subseteq T \subseteq V(G)$ is a DS and TDS of a graph (or network) G, when the network fails in some vertices (or nodes) in D, the set $D' \subseteq V - T$ is an IDS will take care of the role of D. In this aspect, it is worthwhile to concentrate to protect the DS (or core group) D in such way that every node of network has a neighbor in T, whether or not it lies in T itself. This type of conceptional comparison is very essential to facilitate the communication between processors in parallel computers. So we require the minimum $STDS$ (or $\gamma_{st}(G)$) of a graph G.

The following computed values of $\gamma_{st}(G)$ for some specific families of graphs are stated without proof.

Proposition 2.1. *For any graph G^* with $p = k(t + 1) + s$ and $k \geq 2$,*

$$\gamma_{st}(G^*) = \begin{cases} k & \text{if } s, t \geq 1 \\ \frac{p}{2} & \text{if } s = 0, t = 1 \text{ or } s = 1, t = 0. \end{cases}$$

Proposition 2.2.

(i) *For any Path P_p with $p \geq 3$ vertices,*

$$\gamma_{st}(P_p) = \begin{cases} \frac{p}{2}; & p \equiv 0(mod\,4) \\ \frac{p+1}{2} + 1; & p \equiv 1(mod\,4) \\ \frac{p}{2} + 1; & p \equiv 2(mod\,4) \\ \frac{p+1}{2}; & p \equiv 3(mod\,4) \\ \text{does not exist} & p = 3, 5, 6, 9. \end{cases}$$

(ii) *For any Cycle C_p with $p \geq 3$ vertices,*

$$\gamma_{st}(C_p) = \begin{cases} \frac{p}{2} + 2; & p \equiv 0(mod4) \\ \frac{p+1}{2} + 1; & p \equiv 1(mod4) \\ \frac{p}{2} + 1; & p \equiv 2(mod4) \\ \frac{p+1}{2}; & p \equiv 3(mod4) \\ \text{does not exist} & p = 4, 5, 8. \end{cases}$$

(iii) *For any Complete graph K_p with $p \geq 3$ vertices,*

$$\gamma_{st}(K_p) = p - 1.$$

(iv) *For any Wheel graph $W_p = C_{p-1} + K_1$ with $p \geq 4$ vertices,*

$$\gamma_{st}(W_p) = 3.$$

(v) *For any Fan graph $F_p = P_{p-1} + K_1$ with $p \geq 3$ vertices,*

$$\gamma_{st}(F_p) = 2.$$

(vi) *For any Complete bipartite graph $K_{m,n}$ with $1 \leq m \leq n$ vertices, $\gamma_{st}(K_{m,n})$ does not exist.*

3. Bounds and Characterizations

3.1. In terms of domination number.

Theorem 3.1. *For any connected graph G with $p \geq 3$ vertices,*

$$\gamma(G) \leq \gamma_{st}(G) \leq p - \gamma(G).$$

Furthermore, the lower bound is achieved on the graph G ($p \geq 3$) contains at least one end-vertex or the graph $G \cong G^*$.

Similarly, the upper bound is achieved when $G \cong K_p$ ($p \geq 3$), $G_1 \circ K_1$, or the path P_4, where G_1 is any non-trivial connected graph.

Proof. We will prove the inequalities separately for the lower and upper bounds.

Let D be a γ-set and T be a γ_t-set of a connected graph G. Consider the case where G contains at least one end-vertex. In this case, any end-vertex v must be included in either D or T since it has degree one. This means that $\gamma_{st}(G) \geq \gamma(G)$, achieving the lower bound.

Proof of the Upper Bound: Consider a connected graph G with p vertices. Let D be a γ-set of G, and let $T = V - D$ be the corresponding γ_t-set. We want to show that $\gamma_{st}(G) \leq p - \gamma(G)$.

Notice that, the set D is a DS of G, which means that every vertex in $V - D$ must be adjacent to at least one vertex in D. Therefore, $V - T$ is a DS of G, as it includes all the vertices not in T that are adjacent to at least one vertex in T.

Now, consider the induced graph $\langle V - T \rangle$, which consists of the vertices in $V - T$ and the edges between them that are present in G. Since $\langle V - T \rangle$ is a DS of G, every vertex in $\langle V - T \rangle$ is adjacent to at least one other vertex in $\langle V - T \rangle$, forming a subgraph with no isolated vertices.

By definition, a set T is a $STDS$, which must be a satisfy the following two properties: $V - T$ is not a DS, and $V - T$ is a DS of the induced graph $\langle V - T \rangle$. Since $V - T$ is a DS of G, it follows that $V - T$ is a DS of $\langle V - T \rangle$. However, this means that $V - T$ cannot satisfy the first property of an $STDS$, which requires that $V - T$ is not a DS. Therefore, T must be an $STDS$ of G, and by definition, $\gamma_{st}(G) \leq |T| = p - |D| = p - \gamma(G)$.

Thus, we have shown that $\gamma_{st}(G) \leq p - \gamma(G)$, completing the proof of the upper bound. □

3.2. In terms of order, size and degrees.

Theorem 3.2. *For any connected graph G with $p \geq 3$,*

$$2 \leq \gamma_{st}(G) \leq p - 1.$$

Furthermore, the lower bound is achieved one of the following condition:

(i) *$G \cong H^*$, where H^* is a graph forming an exactly one vertex u in a complete graph K_n; $n \geq 3$ or a cycle C_4, which u is adjacent to at least one end-vertex.*
(ii) *$G \cong H^{**}$, where H^{**} is a graph forming an exactly two adjacent vertices u and v in a complete graph K_n; $n \geq 2$ or a cycle C_4, which each u and v are adjacent to at least one end-vertex.*
(iii) *$G \cong K_2 + \overline{K_t}$; $t \geq 2$ or each adjacent vertices of K_2 in G is adjacent to at least one end-vertex.*

Similarly, the upper bound is achieved on $G \cong K_p$; $p \geq 3$.

Proof. Let T be a γ_t-set of a connected graph G with $p \geq 3$ vertices. Then every vertex of T in G are adjacent to each other. So that $\langle T \rangle$ of G has no isolates. Therefore $2 \leq \gamma_t(G) \leq \gamma_{st}(G)$ and hence the lower bound follows.
For every vertex in T of G dominates at least one vertex in $V - T$. Therefore $\mid V - T \mid \geq 1$. Since $V - T$ is not contains TDS and contain a DS of a graph G. Hence T is a $STDS$ of a graph G. This implies that $\mid T \mid \leq \mid V \mid -1 = p - 1$.
Thus the upper bound follows.
Further, the lower and upper bounds are achieved some specific families of graphs, hence we omit the proof. □

Theorem 3.3. *For any connected graph G with $p \geq 3$ vertices,*

$$Max.\Big\{p - q + 1, \Big\lceil \frac{p}{q} \Big\rceil \Big\} \leq \gamma_{st}(G) \leq Min.\Big\{q, 2q - p + 1\Big\}.$$

Proof. Let G be a connected graph with $p \geq 3$ vertices. If D is a DS and $|V - D| = r$, then there are at least q edges from $V - D$ to D, and $|D| = p - r$. Since $r \leq q$, we have $p - q \leq |D| = \gamma(G)$ and $(D \cup \{v\}) \subseteq T$ for $v \in V - D$. Also, we know that $\gamma_t(G) \leq \gamma_{st}(G)$. This implies that $p - q + 1 \leq \gamma_{st}(G)$. Since $\lceil \frac{p}{q} \rceil \leq 2 \leq \gamma_t(G)$, we have $\lceil \frac{p}{q} \rceil \leq \gamma_{st}(G)$. Thus, the lower bound follows.

By Theorem 3.2, we have $\gamma_{st}(G) \leq p-1 \leq q$ and $\gamma_{st}(G) \leq p-1 = 2(p-1)-p+1 \leq 2q - p + 1$. Thus the upper bound follows. □

To prove our next result, we make use of the following theorems.

Theorem 3.4. [1, 22] *For any non-trivial graph G with no isolates,*

$$\Big\lceil \frac{p}{\Delta(G) + 1} \Big\rceil \leq \gamma(G) \leq p - \Delta(G).$$

Theorem 3.5. [15] *For any non-trivial graph G with no isolated vertex,*

$$\gamma_t(G) \geq \frac{p}{\Delta(G)}.$$

Theorem 3.6. *For any connected graph G with $p \geq 3$,*

$$\Big\lceil \frac{p}{\Delta(G)} \Big\rceil \leq \gamma_{st}(G) \leq \Big\lfloor \frac{p\Delta(G)}{\Delta(G) + 1} \Big\rfloor.$$

Proof. By Theorem 3.5 and every STD-set is a TD-set of a connected graph G (i.e., $\gamma_t(G) \leq \gamma_{st}(G)$), the lower bound follows.
By Theorem 3.4 and $\gamma_{st}(G) \leq p - \gamma(G)$, we have the desired upper bound in terms of order and maximum degree of a graph G. □

3.3. In terms of Total domination number.

Observation 3.1. *The difference between $\gamma_{st}(G)$ and $\gamma_t(G)$ is arbitrarily large. For example, $\gamma_{st}(K_p) - \gamma_t(K_p) = p - 3$ with $p \geq 3$.*

Theorem 3.7. *Let G be a connected graph with $p \geq 3$ vertices. Then every $STDS$ is a TDS of G is achieved one of the following conditions:*

(i) *$G \cong H^*$, where H^* is a graph forming an exactly one vertex u in a complete graph K_n; $n \geq 3$ vertices or cycle C_4, which u is adjacent to at least one end-vertex.*

(ii) *$G \cong H^{**}$, where H^{**} is a graph forming an exactly two adjacent vertices u and v in a complete graph K_n; $n \geq 2$ vertices or cycle C_4, which each u and v are adjacent to at least one end-vertex.*

(iii) *$G \cong K_2 + \overline{K_t}$; $t \geq 2$ or each adjacent vertices of K_2 in G is adjacent to at least one end-vertex.*

(iv) *$G \cong G^*$; $s, t \geq 1$ and $k \geq 2$.*

Proof. Since every $STDS$ is a TDS of a connected graph G with $p \geq 3$ vertices. Clearly, $\gamma_t(G) \leq \gamma_{st}(G)$. Further, let T be a γ_{st}-set of G. Then an induced subgraph $\langle V - T \rangle$ contains at least one isolated vertex. Thus the conditions (i)-(iv) holds. □

3.4. In terms of Maximal total domination number.

Theorem 3.8. *For any connected graph G with $p \geq 3$,*

$$\gamma_t(G) \leq \gamma_{mt}(G) \leq \gamma_{st}(G).$$

Proof. Since every $STDS$ is a $MTDS$ and every $MTDS$ is a TDS of a connected graph G with $p \geq 3$. Hence the desired bounds follow. Further, let v be an end-vertex of a connected graph G and T be a γ_t-set of G. $v \notin T$ and is adjacent to some vertex in $u \in T$. This implies that v is an isolate in $\langle V - T \rangle$ and hence T is a γ_{mt}-set as well as γ_{st}-set of G.

Further, the bounds are attained if G contains an end-vertex or $G \cong C_p$ with $p = 4n + 2$ and $p = 4n + 3$ for $n \geq 2$. Also, $\gamma_{st}(C_p) = \gamma_{mt}(C_p) + 1 = \gamma_t(C_p) + 2$ with $p = 4n$ for $n \geq 3$. □

3.5. In terms of Inverse domination number.

Observation 3.2. *The difference between $\gamma_{st}(G)$ and $\gamma_i(G)$ is arbitrarily large. For example, $\gamma_{st}(K_p) - \gamma_i(K_p) = p - 2$ for $p \geq 3$.*

Observation 3.3. *There is no good relation between IDS and $STDS$ of a connected graph G.*
For example,

(i) *$\gamma_i(G) = \gamma_{st}(G)$ if $G \cong G_1 \circ K_1$.*

(ii) *$\gamma_i(G) > \gamma_{st}(G)$ if $G \cong G_1 \circ \overline{K_t}$; $t \geq 2$.*

(iii) *$\gamma_i(G) < \gamma_{st}(G)$ if $G \cong K_p$; $p \geq 3$,*

where G_1 is a non-trivial connected graph.

By the definitions of DS, IDS, TDS, $ITDS$ and $STDS$ of a connected graph with $p \geq 4$ vertices, we have the following schematic representation as in Figure-3 to obtain some results in terms of two disjoint IDS (say, D_1' and D_2') of a connected graph G without proof.

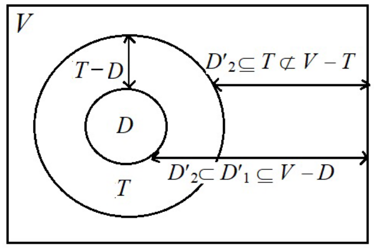

FIGURE 3. Schematic Representation of domination related parameters.

Theorem 3.9. *Let D and T are the γ-set and γ_t-set of a connected graph G with $p \geq 3$, respectively. Then*

(i) $\mid D_1' \mid = \mid D_2' \mid$ *if and only if* $T - D = \phi$.
(ii) $\mid D_1' \mid > \mid D_2' \mid$ *if and only if* $T - D \neq \phi$,

Theorem 3.10. *Let G be a connected graph with $p \geq 3$ vertices. Then*

(i) $\gamma(G) \leq \gamma_i(G) \leq p - \gamma(G)$.
(ii) $\gamma_t(G) \leq \gamma_{it}(G) \leq p - \gamma_t(G)$.
(iii) $\gamma_i(G) \leq \gamma_{it}(G)$.
(iv) $\gamma_{st}(G) \leq p - \gamma_i(G)$.

Further, we have

(i) $\gamma(G) = \gamma_i(G) = \gamma_t(G) = \gamma_{st}(G) = \frac{p}{2}$ if $G \cong G_1 \circ K_1$, where G_1 is a non-trivial connected graph.
(ii) $\gamma_t(G) = \gamma_{it}(G) = \frac{p}{2}$ if $G \cong C_p$; $p \equiv 0(mod\, 4)$ or K_4 or $K_4 - e$; $e \in E(K_4)$.
(iii) $\gamma_t(G) = \gamma_{it}(G) = 2$ if $G \cong K_p$ or $K_{m,n}$; $2 \leq m \leq n$
(iv) $\gamma_t(G) = \gamma_{it}(G) = 4$ if $G \cong \overline{K_{m,n}}$; $2 \leq m \leq n$.

3.6. In terms of domatic and total domatic number.

We known that, $d(G) \leq \delta(G) + 1$ and $d_t(G) \leq \delta(G)$ for any simple graph G. If these bounds are attains, then we call as domatically full and total domatically full, respectively. Similarly, we can define the superlative total domatically full (i.e., $d_{st}(G) = \delta(G)$) of a graph G.

By Theorem 3.9, we have

Theorem 3.11. *Let D and T are the γ-set and γ_t-set of a connected graph G with $p \geq 3$, respectively. Then*

(i) $d(G) = 1$ *if and only if* γ_{st}*-set of* G *does not exist.*
(ii) $d(G) = 2$ *if and only if* γ_{st}*-set of* G *exist for* $D_1' \subseteq V - D$; $D_1' \subseteq V - T$ *and* $D_1' \subseteq T - D$.
(iii) $d(G) = 3$ *if and only if* γ_{st}*-set of* G *exist for* $D_1' \subseteq T - D$ *and* $D_1' \subseteq V - T$,

where D_1' and D_2' are two disjoint IDS of G.

Theorem 3.12. *Let G be a connected graph $p \geq 3$. Then $d_t(G) = 1$ if and only if γ_{st}-set of G exists and $\langle V - D_t \rangle$ has at least one isolated vertex.*

Proof. On the contrary, suppose $d_t(G) = 1$ holds. Then there exist at least four vertices u_1, u_2, u_3 and u_4 such that $u_1 - u_2 - u_3 - u_4 - u_1$ form a cycle. This implies that $\{u_1, u_2\}$ and $\{u_3, u_4\}$ form a disjoint TDS of G (i.e., $d_t(G) = 2$). But γ_{st}-set does not exist for this G, which is a contradiction. Further, if any two non-adjacent vertices are adjacent, say u_1, u_3 such that $e = u_1u_3$ in $G + e$, then again the sets $\{u_1, u_2\}$ and $\{u_3, u_4\}$ form a disjoint TDS of $G + e$ (i.e., $d_t(G) = 2$) and the set $\{u_1, u_3\}$ form a γ_{st}-set of $G + e$, which is again a contradiction.
The converse is obvious. □

Observation 3.4. *Let G be a connected graph $p \geq 4$. If $d_t(G) = 2$ with $3 \leq \Delta(G) \leq p - 1$, then the γ_{st}-set of G exists.*
Converse need not be true. For example, the complete bipartite graph $K_{m,n}$ with $3 \leq m \leq n$ does not exist.

4. Conclusion and Open Problems

In this paper, we initiated a new total domination-related parameter as a superlative total domination in graphs, which connects to both dominating set (or, total dominating set) and the inverse dominating set of graphs. Being a new parameter for the comparative advantages, applications and mathematical properties point of view, we pose the following open problems:

1. Find the complexity issues of $\gamma_{st}(G)$.
2. Obtain some bounds and characterizations of $\gamma_{st}(G)$ in terms of $\gamma(G)$, $\gamma_t(G)$, $\gamma_i(G)$, $\gamma_m(G)$, $\gamma_{mt}(G)$ and other domination related parameters.
3. Find some results towards the superlative total domatically full.
4. Characterize when $D'_1 \cap D'_2 = \phi$, where $D'_1 \subseteq V - D$ and $D'_2 \subseteq V - T$.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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