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CHEBYSHEV COMPUTATIONAL ALGORITHM FOR EIGHT ORDER BOUNDARY VALUE PROBLEMS

T. OYEDEPO, E.A. ADENIPEKUN, G. AJILEYE, A. M. AJILEYE

Abstract. In this research, we present a computational algorithm designed for solving eighth-order Boundary Value Problems(BVPs) using fourth-kind Chebyshev polynomials as basis functions. The method entails assuming an approximate solution employing fourth-kind shifted Chebyshev polynomials. Subsequently, this assumed solution is substituted into the general problem. The resulting equation is collocated at evenly spaced points within the interval, resulting in a linear system of equations with unknown Chebyshev coefficient constants. To solve this system, we employ a matrix inversion approach to determine the unknown constants, which are then substituted back into the assumed solution to obtain the desired approximate solution. To validate the effectiveness of the proposed technique, three numerical examples are selected from existing literature. The results obtained from our method are compared with those reported in the literature, demonstrating that the proposed algorithm is not only accurate but also efficient in solving BVPs. Tables and figures are employed to present and illustrate the results.

1. Introduction

In the realm of differential equations, a mathematical boundary value problem is constituted by a differential equation accompanied by additional constraints referred to as boundary conditions. A boundary value problem is deemed to be present when a differential equation possesses a solution that adheres to the specified boundary conditions. Given that virtually every physical differential equation inherently involves Boundary Value Problems(BVPs), these problems manifest across various domains of physics. BVPs serve as a prevalent method for framing challenges involving wave equations, such as the identification of normal modes. Sturm-Liouville problems encompass a substantial category of noteworthy

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BVPs, wherein the eigenfunctions of a differential operator play a crucial role in their examination. Numerous numerical techniques have been devised for solving BVPs. Several authors have contributed numerical solutions to BVPs, employing distinct methodologies. For instance, one approach involves solving special eighth-order BVPs using the modified decomposition method [30], while another employs non-polynomial spline methodology for eighth-order BVPs [26]. The homotopy perturbation method is applied to address eighth-order BVPs [10], and non-polynomial splines are employed for sixth-order BVPs [27]. Various other methods, such as the variational iteration technique, cubic B-spline method, collocation method, reproducing kernel space, Daftardar Jafari method, and tau-collocation approximation method, have been employed for solving BVPs with varying orders [[1],[3],[8],[11],[12],[13],[14],[16], [17], [18],[19],[24],[25], [29]]. Additionally, Vieta-Lucas polynomials, Haar wavelet method, and modified variation iteration method using Chebyshev polynomials have been utilized for specific cases [[2],[28],[15]]. Also, [[4],[20],[21],[22]] applied collocation method for solving integro-differential equations, while [23] applied least squares collocation for fractional integro-differential equations. Building upon the aforementioned studies, this work focuses on the numerical solution of an eighth-order BVP, expressed as:

$$w^{viii}(u) + k_1(u)w^{vii}(u) + k_2(u)w^{vi}(u) + k_3(u)w^{v}(u) + k_4(u)w^{iv}(u)+$$
$$k_5(u)w^{iii}(u) + k_6(u)w^{ii}(u) + k_7(u)w(u) = g(u), u \in [a, b], \tag{1}$$

subject to the boundary conditions:

$$w^i(a) = \alpha_i, w^i(b) = \beta_i,\ i = 0, 1, 2, \tag{2}$$

Here, α_0,α_1, α_1 and β_0, β_1,β_2 are predetermined real constants, $k_i(u), i = 0, 1, 2, \ldots n$ and $g(u)$ are known functions on the an interval $\in [a, b]$ and $w(u)$ is the unknown function to be determined.

2. Basic definitions

Definition 2.1. *Chebyshev Polynomials of the Fourth Kind (CPFK) [[6], [7],[9]]:The CPFK are orthogonal polynomials with respect to the weight function* $\sqrt{\frac{1-u}{1+u}}$ $\forall u \in [-1, 1]$*, is defined by* $m_n(u) = \frac{\sin{(n+\frac{1}{2})\theta}}{\sin{(\frac{\theta}{2})}}$*, where* $u = \cos\theta$ *and the recurrence relation*

$$m_{n+1}(u) = 2um_n(u) - m_{n-1}(u); n \geq 1,$$

starting with

$$m_0(u) = 1, m_1(u) = 2u + 1$$

Definition 2.2. *Shifted Chebyshev Polynomials of the Fourth Kind (SCPFK):The SCPFK is orthogonal polynomials with respect to the weight function* $\sqrt{\frac{1-u}{u}}$ $\forall u \in [0, 1]$*,is defined by* $m^*_n(u) = m_n(2u - 1)$ *where* $\omega_n(\zeta)$ *is CPFK .*

$$m^*_{n+1}(u) = 2(2u - 1)m^*_n(u) - m^*_{n-1}(u); n \geq 1,$$

starting with

$$m^*_0(u) = 1, m^*_1(u) = 4u - 1$$

3. Demonstration of the method

The work assumed an approximate solution by means of the SCPFK in the form:

$$w(u) = \sum_{i=0}^{n} m^{*}(u)a_i \tag{3}$$

where a_i, $i = 0, 1, 2 \ldots n$ are the unknown shifted Chebyshev coefficients constants to be determined.

Thus, substituting Equ. (3) in Equ. (1) gives

$$\sum_{i=0}^{n} m^{*viii}(u)a_i + k_1(u)\sum_{i=0}^{n} m^{*vii}(u)a_i + k_2(u)\sum_{i=0}^{n} m^{*vi}(u)a_i(u)+$$

$$k_3(u)\sum_{i=0}^{n} m^{*v}(u)a_i + k_4(u)\sum_{i=0}^{n} m^{*iv}(u)a_i + k_5(u)\sum_{i=0}^{n} m^{*iii}(u)a_i+$$

$$k_6(u)\sum_{i=0}^{n} m^{*ii}(u)a_i + k_7(u)\sum_{i=0}^{n} m(u)a_i = g(u) \tag{4}$$

$$Let\ z(u) = \sum_{i=0}^{n} m^{*viii}(u)a_i,\ z^{*}(u) = \sum_{i=0}^{n} m^{*vii}(u)a_i$$

$$y(u) = \sum_{i=0}^{n} m^{*vi}(u)a_i(u),\ y^{*}(u) = \sum_{i=0}^{n} m^{*v}(u)a_i,$$

$$r(u) = \sum_{i=0}^{n} m^{*iv}(u)a_i,\ r^{*}(u) = \sum_{i=0}^{n} m^{*iii}(u)a_i,$$

$$s(u) = \sum_{i=0}^{n} m^{*ii}(u)a_i,\ ,s^{*}(u) = \sum_{i=0}^{n} m(u)a_i$$

Thus, Equ. (4) becomes

$$z(u) + k_1(u)z^{*}(u) + k_2(u)y(u) + k_3(u)y^{*}(z) + k_4(u)r(u) + k_5(u)r^{*}(u)+$$

$$k_6(u)s(u) + k_7(u)s^{*}(u) = g(u) \tag{5}$$

The system of linear algebraic equations involving $(n + 1)$ unknown constants, represented as constants $a_i's$ is obtained by positioning Eq.(5) at evenly distributed locations determined by points $u_i = a + \frac{(b-a)i}{n}$, where $(i = 0(1)(n))$. Additional equations are derived from Eq. (2) and expressed using matrix representation:

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & \cdots & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & \cdots & \cdots & P_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ P_{m1} & P_{m2} & P_{m3} & \cdots & \cdots & \cdots & P_{mn} \\ P^0_{11} & P^0_{12} & P^0_{13} & \cdots & \cdots & \cdots & P^0_{1n} \\ P^1_{21} & P^1_{22} & P^1_{23} & \cdots & \cdots & \cdots & P^1_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ P^n_{m1} & P^n_{m2} & P^n_{m3} & \cdots & \cdots & \cdots & P^n_{mn} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} Q_{11} \\ Q_{22} \\ \vdots \\ \vdots \\ Q_{mn} \\ Q^0_{11} \\ Q^1_{22} \\ \vdots \\ \vdots \\ Q^{n-1}_{mn} \end{pmatrix} \tag{6}$$

Here,$P_i's$ and P_i^n's represent the coefficients of $a_i's$,provided as, while $Q_i's$ denotes are values of $g(u_i)$. Following this, the matrix inversion method is utilized to resolve the system of equations and ascertain the unknown constants.

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & \cdots & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & \cdots & \cdots & P_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ P_{m1} & P_{m2} & P_{m3} & \cdots & \cdots & \cdots & P_{mn} \\ P^0_{11} & P^0_{12} & P^0_{13} & \cdots & \cdots & \cdots & P^0_{1n} \\ P^1_{21} & P^1_{22} & P^1_{23} & \cdots & \cdots & \cdots & P^1_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ P^n_{m1} & P^n_{m2} & P^n_{m3} & \cdots & \cdots & \cdots & P^n_{mn} \end{pmatrix}^{-1} \begin{pmatrix} Q_{11} \\ Q_{22} \\ \vdots \\ \vdots \\ Q_{mn} \\ Q^0_{11} \\ Q^1_{22} \\ \vdots \\ \vdots \\ Q^{n}_{mn} \end{pmatrix} \tag{7}$$

Solving equation (7) yields the unknown constants, which are then substituted into the assumed approximate solution in equation (5) to obtain the required approximate solution.

4. Numerical examples

Example 1 [18]: Consider the eight Order boundary value problem.

$$w^8(u) = -uw(u) - (48 + 15u + u^3)e^u, 0 \leq u \leq 1, \tag{8}$$

subject to the boundary conditions $w(0) = 0$,$w^{'}(0) = 0$,$w^{''}(0) = 0$, $w^{'''}(0) = -3$, $w(1) = 0$,$w^{'}(1) = -e$,$w^{''}(1) = -4e$, $w^{'''}(1) = -9e$. The exact solution is $w(u) = u(1-u)e^u$. We obtained the following unknown constants via the above-described method:
$a_0 = 0.281718155870810$, $a_1 = 0.0839128319848488$,
$a_2 = -0.269776938652414$,$a_3 = -0.0828153503388682$,

$a_4 = -0.0118660202349515$,$a_5 = -0.00109887552220600$,
$a_6 = -0.0000748839394258634$,$a_7 = -0.00000396409454284097$
$a_8 = -1.79014618310089 \times 10^{-7}$, $a_9 = -6.76616483126564 \times 10^{-9}$,
$a_{10} = -2.22475939204455 \times 10^{-10}$,$a_{11} = -6.47174478183181 \times 10^{-12}$,
$a_{12} = -1.68773483902462 \times 10^{-13}$, $a_{13} = -3.98781361334468 \times 10^{-15}$
$a_{14} = -8.61430236746842 \times 10^{-17}$,$a_{15} = -1.72344429321310 \times 10^{-18}$,
$a_{16} = -3.22781901642560 \times 10^{-20}$.
Thus, the approximate solution is given as:

$$w(u) = 0.9999926704u - 0.001190476319u^8 - 0.006718200148u^7$$
$$-0.03412353920u^6 - 0.1239179239u^5 - 0.3340662441u^4$$
$$-0.4997423268u^3 - 0.00004627786939u^2 - 0.0001736110049u^9$$
$$-0.00002204596836u^{10} - 0.000002480032208u^{11} - 2.506602467 \times 10^{-7}u^{12}$$
$$-2.283419914 \times 10^{-8}u^{13} - 2.014019880 \times 10^{-9}u^{14} - 1.121221076 \times 10^{-10}u^{15}$$
$$-1.940178713 \times 10^{-11}u^{16} + 0.000005498522491$$

Example 2 [5]: Consider the Boundary Value Problem

$$w^8(u) + w^7(u) + 2w^6(u) + 2w^5(u) + 2w^4(u) + 2w^{'''}(u) + 2w^{''}(u) +$$
$$w^{'}(u) + w(u) = 14\cos(u) - 16\sin(u) - 4u\sin(u), 0 \le u \le 1, \tag{9}$$

subject to the boundary conditions $w(0) = 0$,$w^{'}(0) = 0$,$w^{''}(0) = 0$, $w^{'''}(0) = -3$, $w(1) = 0$,$w^{'}(1) = 2\sin(1)$,$w^{''}(1) = 4\cos(1) + 2\sin(1)$, $w^{'''}(1) = 6\cos(1) - 6\sin(1)$. The exact solution is $w(u) = (u^2 - 1)\sin(u)$. We obtained the following unknown constants via the above-described method:
$a_0 = 0 - .236453418579600$, $a_1 = -0.0350603754105561$, $a_2 = 0.242017589579213$,
$a_3 = 0.0355705478924233$,$a_4 = -0.00559375374286199$,$a_5 = -0.000512124356115776$,
$a_6 = 0.0000296479773765308$,$a_7 = 0.00000194979875187218$
$a_8 = -6.530136009768909 \times 10^{-8}$, $a_9 = -3.34327069741497 \times 10^{-9}$,
$a_{10} = 7.73870492666299 \times 10^{-11}$, $a_{11} = 3.24108224717461 \times 10^{-12}$,
$a_{12} = -5.69095186049598 \times 10^{-14}$, $a_{13} = -2.01601279935148 \times 10^{-15}$
$a_{14} = 2.84240173218021 \times 10^{-17}$,$a_{15} = 8.72355728249490 \times 10^{-19}$,
$a_{16} = -1.19580783642360 \times 10^{-20}$,$a_{17} = -2.32648466341516 \times 10^{-22}$,
$a_{18} = 1.48503961553844 \times 10^{-22}$

Thus, the approximate solution is given as:
$w(u) = -1.000000004u + 3.711555375 \times 10^{-8}u^2 + 1.166666542u^3 - 2.015805540 \times 10^{-7}u^4 - 0.1749991918u^5$
$-7.437779736 \times 10^{-7}u^6 + 0.008531958092u^7 + 1.349063911 \times 10^{-8}u^8$
$-0.0002011754998u^9 + 4.032032446 \times 10^{-10}u^{10} + 0.0000002780738215u^{11} + 1.296127930 \times 10^{-10}u^{12} - 2.539435635 \times 10^{-8}u^{13} + 1.728983111 \times 10^{-10}u^{14}$
$+4.92381638^{-11}u^{15} + 4.750340936 \times 10^{-11}u^{16} - 1.267215181 \times 10^{-11}u^{17}$
$5.417729697 \times 10^{-9}$

Example 3 [1]: Consider the Boundary Value Problem

$$w^8(u) - w(u) = -8(2u\cos(u) + 7\sin(u)), -1 \le u \le 1, \tag{10}$$

subject to the boundary conditions $w(-1) = 0$,$w^{'}(-1) = 2\sin(1)$,$w^{''}(-1) = -4\cos(1) - 2\sin(1)$, $w^{'''}(-1) = 6\cos(1) - 6\sin(1)$, $w(1) = 0$,$w^{'}(1) = 2\sin(1)$,$w^{''}(-1) =$

$-4\cos(1) + 2\sin(1)$, $w'''(1) = 6\cos(1) - 6\sin(1)$. The exact solution is $w(u) = (u^2 - 1)\sin(u)$

$a_0 = 5.01911910271564\times10^{-15}$, $a_1 = -.372210312147683$, $a_2 = -5.23414184073522\times 10^{-15}$,
$a_3 = 0.392430628322382$,$a_4 = 4.88076041811298\times10^{-15}$,$a_5 = -0.205209074968138$,
$a_6 = -1.00404615697957 \times 10^{-17}$,$a_7 = 0.000302626404121425$
$a_8 = 1.31664068892446\times10^{-20}$, $a_9 = -0.204290734608008e{-5}$,$a_{10} = -1.74772634793934\times 10^{-21}$
$a_{11} = 7.84757782804694\times10^{-9}$,$a_{12} = 6.50222905153518\times10^{-22}$, $a_{13} = -1.94032989928556\times 10^{-11}$
$a_{14} = -2.89209606088613\times10^{-22}$,$a_{15} = 3.34189256202840\times10^{-14}$, $a_{16} = 9.01110406364609\times 10^{-23}$
$a_{17} = -4.17541481985765 \times 10^{-17}$, $a_{18} = -1.28508730979663 \times 10^{-23}$.

$w(u) = -1.000000000u - 4.448830113 \times 10^{-19}u^{18} - 7.433909603 \times 10^{-13}u^{17}$
$+2.771251043 \times 10^{-18}u^{16} + 1.612625507 \times 10^{-10}u^{15} - 7.443354431 * 10 \times 10^{-18}u^{14}$
$-2.521249308810\times10^{-8}u^{13} + 7.822360441\times10^{-15} + 1.129540269\times10^{-17}u^{12} + 0.2780783755\times 10^{-5}u^{11} - 1.075614572 \times 10^{-17}u^{10} - 0.2011684302 \times 10^{-3}u^9$
$+7.279890681 \times 10^{-18}u^8 + 0.008531746030u^7 - 1.487136127 \times 10^{-16}u^6$
$-0.1750000000u^5 + 2.334238035\times10^{-15}u^4 + 1.166666666u^3 - 9.747562328\times10^{-15}u^2$

5. Numerical Results

Approximate Solution=AS,Absolute Error=AE

Table 1. Numerical Results for Example 1

u_i	Exact	Our Method AS	AE of our method	AE [18]
0.0	0.00000000000000	0.00000549852249	$5.499E-06$	–
0.1	0.09946538262000	0.09946987985000	$4.497E-06$	$3.576279E-07$
0.2	0.19542444130000	0.19542781020000	$3.369E-06$	$6.318092E-06$
0.3	0.28347034970000	0.28347260760000	$2.258E-06$	$1.895428E-05$
0.4	0.35803792750000	0.35803903280000	$1.105E-06$	$3.099442E-05$
0.5	0.41218031780000	0.41218021950000	$9.820E-08$	$3.641844E-05$
0.6	0.43730851200000	0.43730723290000	$1.279E-06$	$3.170967E-05$
0.7	0.42288806850000	0.42288570000000	$2.369E-06$	$1.925230E-05$
0.8	0.35608654850000	0.35608316770000	$3.381E-06$	$7.182360E-06$
0.9	0.22136428000000	0.22135989690000	$4.383E-06$	$1.460314E-06$
1.0	0.00000000000000	−0.00000523199237	$5.232E-06$	–

Table 2. Numerical Results for Example 2

u_i	Exact	Our Method AS	AE of our Result	AE [30]
0.0	0.00000000000000	0.00000000541773	$5.418E-09$	–
0.1	−0.09883508248000	−0.09883507720000	$5.251E-09$	$4.239380E-06$
0.2	−0.19072255760000	−0.19072255250000	$5.033E-09$	$9.983778E-06$
0.3	−0.26892338810000	−0.26892338410000	$4.062E-09$	$5.096197E-06$
0.4	−0.32711140750000	−0.32711140530000	$2.166E-09$	$7.629395E-06$
0.5	−0.35956915400000	−0.35956915390000	$3.729E-11$	$1.493096E-05$
0.6	−0.36137118300000	−0.36137118540000	$2.333E-09$	$2.288818E-05$
0.7	−0.32855102050000	−0.32855102460000	$4.047E-09$	$2.276897E-05$
0.8	−0.25824819270000	−0.25824819750000	$4.726E-09$	$1.943111E-05$
0.9	−0.14883211280000	−0.14883211780000	$4.710E-09$	$1.323223E-05$
1.0	0.00000000000000	−0.00000000444248	$4.442E-09$	–

Table 3. Numerical Results for Example 3

u_i	Exact	Our Method AS	AE of our Result
-1.0	0.00000000000000	0.00000000099999	$1.000E-09$
-0.8	0.25824819270000	0.25824819310000	$4.000E-10$
-0.6	0.36137118300000	0.36137118300000	$3.509E-15$
-0.4	0.32711140750000	0.32711140760000	$9.000E-11$
-0.2	0.19072255760000	0.19072255760000	$3.100E-11$
0.0	0.00000000000000	0.00000000000001	$7.822E-15$
0.2	-0.19072255760000	-0.19072255760000	$3.100E-11$
0.4	-0.32711140750000	-0.32711140760000	$9.000E-11$
0.6	-0.36137118300000	-0.36137118300000	$3.509E-15$
0.8	-0.25824819270000	-0.25824819310000	$4.000E-10$
1.0	0.00000000000000	-0.00000000100001	$1.000E-09$

Table 4. Numerical Results for Example 3

Our Method	[1]	[26]
$3.509E-15$	$4.90E-09$	$1.02E-08$

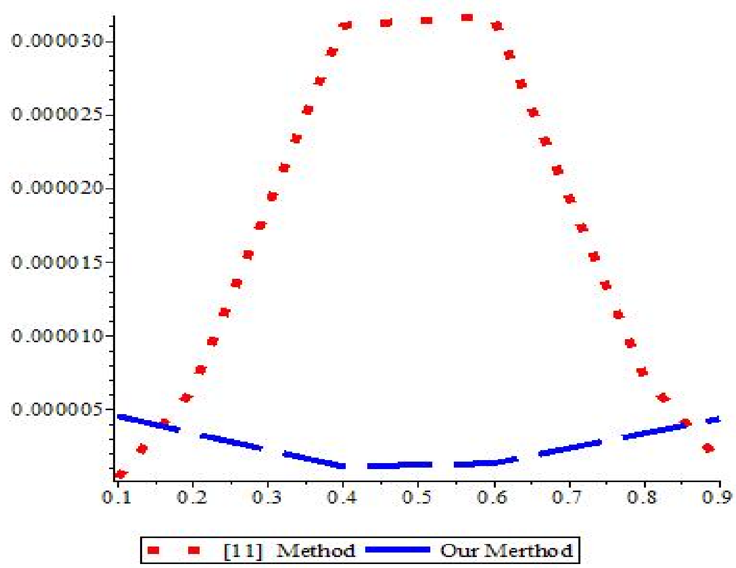

Figure 1. depicts comparison of the absolute errors of example 1

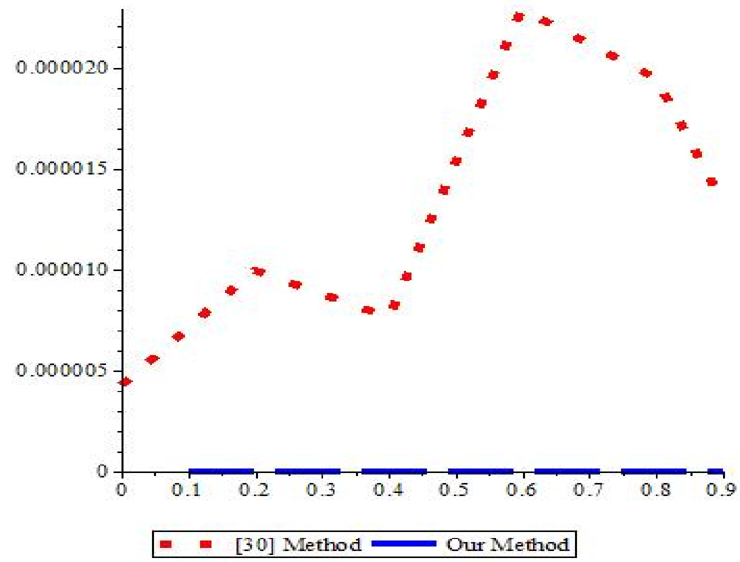

FIGURE 2. depicts comparison of the absolute errors of example 2

6. Conclusion

This study investigates the Chebyshev Computational Algorithm for solving eighth-order BVPs. All the examples presented in this research were addressed using Maple 18. In example 1, the fem-based collocation method employed by [18] was compared, revealing that the proposed method exhibited greater accuracy. Similarly, in example 2, the Galerkin Method with Septic B-splines used by [5] was surpassed in accuracy by the proposed method, as indicated in the result table. Example 3, previously addressed by [26] and [1], also demonstrated the superior accuracy of the proposed method. Additionally, figures 1 and 2 illustrate that the absolute errors obtained with the proposed method are smaller than those using the methods of [18] and [5]. Our findings suggest that the proposed method can effectively tackle further boundary value problems.

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T. Oyedepo
Department of Applied Sciences, Federal University of Allied Health Sciences, Enugu, Nigeria
E-mail address: oyedepotaiye12@gmail.com

E.A. Adenipekun
Department of Statistics,Federal Polytechnic Ede, Osun, Nigeria
E-mail address: waltolxy@yahoo.com

G. Ajileye
Department of Mathematics and Statistics, Federal University Wukari Taraba,670101 Nigeria
E-mail address: ajileye@fuwukari.edu.ng

A. M. Ajileye
Department of Mathematics, University Ilesa, Osun, Nigeria
E-mail address: adewale_ajileye@unilesa.ng