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WEIGHTED SHARING OF ENTIRE FUNCTIONS CONCERNING LINEAR q - DIFFERENCE OPERATORS

HARINA P. WAGHAMORE, MEGHA M. MANAKAME

Abstract. In this research article, we investigates the value distribution of linear q-difference operators $L_k(f, \Delta_{q,c})$ and $L_k(g, \Delta_{q,c})$, for a transcendental entire functions of zero order. At the same time we also investigate the uniqueness problems when two linear q - difference operators of entire functions share one value with finite weight. Our results extends the previous theorems of existing studies [11], [12].

1. Introduction, Definitions and Main Results

In this paper, we assume that the reader is familiar with the fundamental results [6],[14],[15]. We adopt the standard notations of the Nevanlinna theory of meromorphic function $m(r, f)$, $N(r, f)$, $\overline{N}(r, 0; f)$ and $T(r, f)$ denote the proximity function, the counting function, the reduced counting function and the characteristic function of $f(z)$, respectively.

Let f and g be two non-constant meromorphic functions defined in the complex plane and $S(r, f)$ denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly exceptional set of finite linear measure. A meromorphic function $\alpha(\not\equiv 0, \infty)$ is called a small function with respect to f, if $T(r, \alpha) = S(r, f)$. If for some $a \in \mathbb{C} \cup \infty$, the zeros of $f - a$ and $g - a$ coincide in locations and multiplicity, we say that f and g share the value a CM(Counting Multiplicities). On the other hand, if the zeros of $f - a$ and $g - a$ coincide only in their locations, then we say that f and g share the value a IM(Ignoring Multiplicities).

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$ be a polynomial $(\neq 0)$, where $a_n(\neq 0), a_{n-1}, ..., a_0$ are complex non-variables. Denote Γ_1, Γ_2 by $\Gamma_1 = m_1 + m_2$, $\Gamma_2 = m_1 + 2m_2$ respectively, where m_1 is the number of first order zeros of $P(z)$, likewise m_2 gives number of higher order zeros of $P(z)$.

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Definition 1.1. [7],[8] *Let k be non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a, f)$ the set of all a-points of f where an a-points of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.*

Definition 1.2. [9] *Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.*

(1) *$N(r, a; f| \geq p)$, $\overline{N}(r, a; f| \geq p)$ denotes the counting function(reduced counting function) of those a-points of f whose multiplicities are not less than p.*
(2) *$N(r, a; f| \leq p)$. $\overline{N}(r, a; f| \leq p)$ denotes the counting function(reduced counting function) of those a-points of f whose multiplicities are not greater than p.*

Definition 1.3. [1] *Let f and g be two non-constant meromorphic functions such that f and g share the value a IM. Let z_0 be a a-point of f with multiplicity p, a a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the counting function of those a-points of f and g where $p > q$, by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f and g where $p = q = 1$ and $N_E^{(2}(r, a; f)$ the counting function of those a-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. Similarly, one can define $\overline{N}_L(r, a; g)$, $\overline{N}_E^{1)}(r, a; g)$, $\overline{N}_E^{(2}(r, a; g)$.*

Definition 1.4. [5] *For a meromorphic function f and c, $(q \neq 0) \in \mathbb{C}$, let us now denote its q- shift $E_{q,c}f$ and q-difference operators $\Delta_{q,c}f$ respectively by $E_{q,c}f(z) = f(qz+c)$ and $\Delta_{q,c}f(z) = f(qz+c) - f(z)$, $\Delta_{q,c}^k f(z) = \Delta_{q,c}^{k-1}(\Delta_{q,c}f(z))$, for all $k \in \mathbb{N} - \{1\}$.*

Definition 1.5. [5] *Let us define linear q-shift and q-difference operators, denoted by $L_k(f, E_{q,c})$ and $L_k(f, \Delta_{q,c})$ as follows,*

$$L_k(f, E_{q,c}) = a_k f(q_k z + c_k) + a_{k-1} f(q_{k-1} z + c_{k-1}) + ... + a_0 f(q_0 z + c_0)$$

and

$$L_k(f, \Delta_{q,c}) = a_k \Delta_{q_k,c_k} f(z) + a_{k-1} \Delta_{q_{k-1},c_{k-1}} f(z) + ... + a_0 \Delta_{q_0,c_0} f(z),$$

where $a_0, a_1, ..., a_k$; $q_0, q_1, ..., q_k$; $c_0, c_1, ..., c_k$ are complex constants. From the above definition we can easily observe that

$$L_k(f, \Delta_{q,c}) = L_k(f, E_{q,c}) - \sum_{j=0}^{k} a_j f(z).$$

If we choose $q_j = q^j$, $c_j = c$ and $a_j = (-1)^{k-j}\binom{k}{j}$ for $0 \leq j \leq k$, then $L_k(f, \Delta_{q,c})$ reduces $\Delta_{q,c}^k(f(z))$.

Many research works on entire and meromorphic functions whose differential polynomials share certain value or fixed point have been done by many mathematicians world wide. Recently, there has been an increasing interest in studying difference equations, the difference product and the q-difference in the complex plane $\mathbb{C}$, a number of papers have focused on the uniqueness of difference analouge of Nevanlinna Theory. The difference logarithmic derivative lemma, given by R. G. Halburd and R. J. Korhonen [4], in 2006 plays an important role in considering the difference analogues of Nevanlinna Theory. Barnett , Halburd, Korhonen and Morgan [3] also established an analogue of the logarithmic derivative lemma on q-difference operators.

In 2010, Zhang and Korhonen [16] studied the value distribution of q-difference polynomials of meromorphic functions and obtained the following result.

Theorem A. [16] *Let f be a transcendental meromorphic(resp. entire) function of zero order and q non-zero complex constant. Then for $n \geq 6$(resp $n \geq 2$), $f^n f(qz)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

In 2015, Xu, Liu and Cao [12] investigated value distributions for a q - shift of meromorphic functions and obtained the following results.

Theorem B. [12] *Let f be a zero-order transcendental meromorphic (resp.entire) function, $q \in \mathbb{C}\backslash\{0\}, \eta \in \mathbb{C}$ are complex constants. Then for $n > m+4$ (resp. $n > m$), $P(f)f(qz+\eta) = \alpha(z)$ has infinitely many solutions, where $\alpha(z) \in S(f)\backslash\{0\}$ and $S(f)$ denotes the family of all meromorphic functions α such that $T(r,\alpha) = S(r,f)$, where $r \to \infty$ outside a possible exceptional set of the finite logarithmic measure.*

Theorem C. [12] *Let f be a zero-order transcendental meromorphic (resp.entire) function, $q(\neq 0), \eta$ are complex constants. Then for $n > m+6$ (resp. $n > m+2$), $P(f)\{f(qz+\eta) - f(z)\} = \alpha(z)$ has infinitely many solutions, where $\alpha(z) \in S(f)\backslash\{0\}$ and $S(f)$ denotes the family of all meromorphic functions α such that $T(r,\alpha) = S(r,f)$, where $r \to \infty$ outside a possible exceptional set of the finite logarithmic measure.*

Theorem D. [12] *Let f and g be two transcendental entire functions of zero order, and let $q \in \mathbb{C}\backslash\{0\}, \eta \in \mathbb{C}$. If $P(f)f(qz+\eta)$ and $P(g)g(qz+\eta)$ share 1 CM and $n > 2\Gamma_2 + 1$ be an integer, then one of the following results holds:*

(1) *$f \equiv tg$ for a constant t such that $t^d = 1$ and $d =gcd(\lambda_0, \lambda_1, \lambda_2, ...\lambda_n)$;*
(2) *f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(w_1,w_2) = P(w_1)w_1(qz+\eta) - P(w_2)w_2(qz+\eta)$;*
(3) *$fg \equiv \mu$ where μ is a complex constant satisfying $a_n^2\mu^{n+1} \equiv 1$.*

In 2020, Waghamore and S. Anand [11] investigated the value distribution of linear difference polynomial in the form $(P(\phi)L(\phi))$,$(P(\psi)L(\psi))$ two difference products of entire functions share one value with finite weight and obtained the following results.

Theorem E. [11] *Consider a zero order transcendental meromorphic functions(resp. entire) $(P(\phi)L(\phi))$ and $(P(\psi)L(\psi))$ and $A(z) \in S(\phi)\backslash\{0\}$. Let $q_j \in \mathbb{C}\backslash\{0\}(j = 1, 2..., s)$, b_j and $c_j \in \mathbb{C}$ be constants such that $L(\phi) = \sum_{j=1}^{s} b_j\phi(q_jz + c_j) \not\equiv 0$. Suppose n and $k \in \mathbb{Z}^+$. Again for $n > \Gamma_1 + km_2 + 2s + 1$(resp. $n > \Gamma_1 + km_2$), $(P(\phi)L(\phi))^{(k)} - \alpha(z) = 0$ has solutions which are infinite in number.*

Theorem F. [11] *Let $q_j \in \mathbb{C}\backslash\{0\}$ $(j = 1, 2...s)$, b_j and $c_j \in \mathbb{C}$. Consider a zero order transcendental meromorphic functions(resp. entire) $(P(\phi)L(\phi))$ and $(P(\psi)L(\psi))$, if $E_l(1; P(\phi)L(\phi))^{(k)} = E_l(1; P(\psi)L(\psi))^{(k)}$ and l,m,n and s are selected as integers with one of the subsequent options:*

(1) *$l \geq 2$, $n > 2\Gamma_2 + 2km_2 + s$;*
(2) *$l = 1$, $n > \frac{1}{2}(\Gamma_1 + 4\Gamma_2 + 5km_2 + 3s)$;*
(3) *$l = 0$, $n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4s$,*

then one of the following results hold:
(i) $\phi = t\psi$ such that $t^d = 1$ and t is constant and $d =gcd(\lambda_0, \lambda_1, \lambda_2, ..., n)$;
(ii) ϕ and ψ satisfy $R(\phi,\psi) = 0$, where $R(x_1,x_2) = P(x_1)L(x_1) - P(x_2)L(x_2)$;
(iii) $\phi\psi \equiv \mu$, where complex constant μ satisfies $a_n^2 b^2 \mu^{n+1} = 1$.

In this paper, we study distribution of values in a more general form of linear $q-$difference operator $L_k(f, \Delta_{q,c})$ as defined in Definition 1.5, and we obtain Theorems 1.1, 1.2 which extends the existing results of Theorem C respectively.

Theorem 1.1. *Let $f(z)$ be a transcendental entire function of zero order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that c is a non-zero complex constant, n and p are positive integers. Then for $n > \Gamma_1 + pm_2 + 2k + 3$, $(P(f)L_k(f, \Delta_{q,c}))^{(p)} - \alpha(z) = 0$ has infinitely many solutions.*

Theorem 1.2. *Let f and g be two transcendental entire functions of zero order, and let $q \in \mathbb{C}\backslash\{0\}, c \in \mathbb{C}$. If $E_l(1; (P(f)L_k(f, \Delta_{q,c}))^{(p)}) = E_l(1; (P(g)L_k(g, \Delta_{q,c}))^{(p)})$ and l, m, n are integers satisfying one of the following conditions:*
(i) $l \geq 2$, $n > 2\Gamma_2 + 2pm_2 + 3k + 5$;
(ii) $l = 1$, $n > \frac{1}{2}[\Gamma_1 + 4\Gamma_2 + 5pm_2 + 7k + 12]$;

(iii) $l=0$, $n>3\Gamma_1+2\Gamma_2+5pm_2+6k+11$.
Then either $(P(f)L_k(f,\Delta_{q,c}))^{(p)}.(P(g)L_k(g,\Delta_{q,c}))^{(p)}\equiv a^2$ *or one of the following results holds:*

1. $f\equiv tg$ *for a constant* t *such that* $t^d=1$ *and* $d=gcd(\lambda_0,\lambda_1,\lambda_2,...,\lambda_n)$*;*
2. f *and* g *satisfy algebraic equation* $R(f,g)=0$ *where* $R(w_1,w_2)=P(w_1)L_k(w_1,\Delta_{q,c})-P(w_2)L_k(w_2,\Delta_{q,c})$.

Remark 1. The zero order growth restriction in Theorem 1.1 cannot be extended to finite order. By taking $P(z)=z^n, f(z)=e^z, c=0$ and $q=-n$. Then $P(f(z))[f(qz)-f(z)]-1$ have no zeros.

2. Preliminary Lemmas

In this section, we state some Lemmas which will play key roles in proving the main results of the paper.

$$H=\Big(\frac{F''}{F'}-\frac{2F'}{F-1}\Big)-\Big(\frac{G''}{G'}-\frac{2G'}{G-1}\Big).$$

where F and G are non-constant meromorphic functions defined in the complex plane $\mathbb{C}$.

Lemma 2.1. [14] *Let* f *be a non-constant meromorphic function and* $P(f)=a_nf^n+a_{n-1}f^{n-1}+...+a_0$*, where* $a_n(\neq 0),a_{n-1},...,a_0$ *are complex constants. Then*

$$T(r,P(f))=nT(r,f)+S(r,f).$$

Lemma 2.2. [12] *Let* f *be a non-constant meromorphic function of zero order and* q *and* c *two non-zero complex constants. Then*

$$T(r,f(qz+c))=T(r,f)+S(r,f),$$

$$N(r,\infty;f(qz+c))\leq N(r,\infty;f(z))+S(r,f);\ N(r,0;f(qz+c))\leq N(r,0;f)+S(r,f),$$

$$\overline{N}(r,\infty;f(qz+c))\leq\overline{N}(r,\infty;f)+S(r,f);\ \overline{N}(r,0;f(qz+c))\leq\overline{N}(r,0;f)+S(r,f).$$

Lemma 2.3. [17] *Let* f *be a non-constant meromorphic function, and* p, k *be two positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right)\leq T(r,f^{(k)})-T(r,f)+N_{p+k}(r,0;f)+S(r,f), \tag{1}$$

$$N_p\left(r,0;f^{(k)}\right)\leq k\overline{N}(r,\infty;f)+N_{p+k}(r,0;f)+S(r,f). \tag{2}$$

Lemma 2.4. [10] *Let* f *be a zero-order meromorphic function, and* $q(\neq 0)$*,* $c\in\mathbb{C}$*. Then*

$$m\left(r,\frac{f(qz+c)}{f(z)}\right)=S(r,f).$$

Lemma 2.5. [5] *Let* f *be a non-constant meromorphic function of zero order, then* $S(r,L_k(f,\Delta_{q,c}))$ *can be replaced by S(r,f).*

In view of Lemma 2.2, we get
$T(r,L_k(f,\Delta_{q,c}))\leq\sum_{j=0}^{k}T(r,f(q_jz))+T(r,f(z))+S(r,f)\leq(k+2)T(r,f)+S(r,f)$.

Lemma 2.6. [7] *Let* f *and* g *be two non-constant meromorphic functions. If* $E_2(1;f)=E_2(1;g)$ *then one of the following cases holds:*
(i) $T(r)\leq N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)+S(r)$,
(ii) $f=g$,
(iii) $fg=1$,
where $T(r)=max\{T(r,f),T(r,g)\}$ *and* $S(r)=o\{T(r)\}$.

Lemma 2.7. [2] *Let F and G be two non-constant meromorphic functions. If $E_1(1;F) = E_1(1;G)$ and $H \not\equiv 0$ then*
$T(r,F) \leq N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+\frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,\infty;F)+S(r,F)+S(r,G)$;
the same inequality holds for $T(r,G)$.

Lemma 2.8. [2] *Let F and G be two non-constant meromorphic functions sharing 1 IM and $H \not\equiv 0$ then*
$T(r,F) \leq N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,F)+S(r,G)$;
the same inequality holds for $T(r,G)$.

Lemma 2.9. *Let $f(z)$ be a transcendental meromorphic function of zero order and $P(f) = a_n f^n + a_{n-1}f^{n-1} + \ldots + a_0$, Let $F_1 = P(f)L_k(f,\Delta_{q,c})$, where n is a positive integer. Then $(n-k-1)T(r,f) + S(r,f) \leq T(r,F_1)$.*
Proof. *From First Fundamental Theorem, Lemmas 2.1 and 2.4, we obtain*

$$\begin{aligned}
(n+1)T(r,f) &= T(r,f(z)P(f(z))) + S(r,f)\\
&\leq T\left(r, \frac{f(z).F_1}{L_k(f,\Delta_{q,c})}\right) + S(r,f)\\
&\leq T(r,F_1) + T\left(r, \frac{L_k(f,\Delta_{q,c})}{f(z)}\right) + S(r,f)\\
&\leq T(r,F_1) + T\left(r, \frac{\sum_{j=0}^{k} a_j f(q_j z + c_j) - \sum_{j=0}^{k} a_j f(z)}{f(z)}\right) + S(r,f)\\
&\leq T(r,F_1) + \sum_{j=0}^{k} T\left(r, \frac{a_j f(q_j z + c_j)}{f(z)}\right) + S(r,f)\\
&\leq T(r,F_1) + \sum_{j=0}^{k} m\left(r, \frac{a_j f(q_j z + c_j)}{f(z)}\right) + \sum_{j=0}^{k} N\left(r, \frac{a_j f(q_j z + c_j)}{f(z)}\right) + S(r,f)
\end{aligned}$$

$(n+1)T(r,f) \leq T(r,F_1) + (k+2)T(r,f) + S(r,f)$.

Thus $(n-k-1)T(r,f) + S(r,f) \leq T(r,F_1)$
This completes the proof of lemma 2.9.

Lemma 2.10. *Let f and g be two entire functions, n,k be two positive integers, $q \neq 0, c$ complex constants and let*

$$F = (P(f)L_k(f,\Delta_{q,c}))^{(p)}, G = (P(g)L_k(g,\Delta_{q,c}))^{(p)}$$

If there exists two non-zero constants c_1 and c_2 such that $\overline{N}(r,c_1;F) = \overline{N}(r,0;G)$ and $\overline{N}(r,c_2;G) = \overline{N}(r,0;F)$ then $n \leq 2\Gamma_1 + 2pm_2 + 3k + 5$.
Proof. *We put $F_1 = (P(f)L_k(f,\Delta_{q,c}))$ and $G_1 = (P(g)L_k(g,\Delta_{q,c}))$, by the Second fundamental theorem of Nevanlinna we have*

$$\begin{aligned}
T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,c_1;F) + S(r,F)\\
&\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F). \qquad (3)
\end{aligned}$$

Using equations (1), (2) and (3), Lemmas 2.1, 2.2 and 2.9 we get

$$\begin{aligned}
(n-k-1)T(r,f) &\leq T(r,F) - \overline{N}(r,0;F) + N_{p+1}(r,0;F_1) + S(r,f)\\
&\leq \overline{N}(r,0;G) + N_{p+1}(r,0;F_1) + S(r,f)\\
&\leq N_{p+1}(r,0;F_1) + N_{p+1}(r,0;G_1) + S(r,f) + S(r,g)\\
&\leq [m_1 + (p+1)m_2 + k + 2][T(r,f) + T(r,g)] + S(r,f) + S(r,g) \qquad (4)
\end{aligned}$$

Similarly,

$$(n-k-1)T(r,g) \leq [m_1+(p+1)m_2+k+2][T(r,f)+T(r,g)]+S(r,f)+S(r,g) \quad (5)$$

In view of (4) and (5) we have
$[n-2m_1-2m_2-2pm_2-3k-5][T(r,f)+T(r,g)] \leq S(r,f)+S(r,g)$
which gives $n \leq 2\Gamma_1+2pm_2+3k+5$.
This completes the proof of lemma 2.10.

3. Proof of the Theorems

Proof of Theorem 1.1. Let $F_1 = P(f)L_k(f,\Delta_{q,c})$. Then F_1 is a transcendental entire function. If possible, we may assume that $F_1^{(p)} - \alpha(z)$ has only finitely many zeros. Then we have

$$N(r,\alpha;F_1^{(p)}) = O\{logr\} = S(r,f). \quad (6)$$

Using (1), (6) and Nevanlinna's theorem for three small functions we deduce

$$\begin{aligned} T(r,F_1^{(p)}) &\leq \overline{N}(r,0;F_1^{(p)})+\overline{N}(r,\alpha;F_1^{(p)})+S(r,f) \\ &\leq T(r,F_1^{(p)})-T(r,F_1)+N_{p+1}(r,0;F_1)+S(r,f). \end{aligned} \quad (7)$$

By Lemma 2.9 we obtain from (7)

$$\begin{aligned} (n-k-1)T(r,f) &\leq N_{p+1}(r,0;F_1)+S(r,f) \\ &\leq N_{p+1}(r,0;P(f))+N(r,0;L_k(f,\Delta))+S(r,f) \\ &\leq (m_1+(p+1)m_2+k+2)T(r,f)+S(r,f). \end{aligned}$$

This gives
$(n-k-1-m_1-pm_2-m_2-k-2)T(r,f) \leq S(r,f)$,
a contradiction to the assumption that $n > \Gamma_1+pm_2+2k+3$.
This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let $F_1 = P(f)L_k(f,\Delta_{q,c})$, $G_1 = P(g)L_k(g,\Delta_{q,c})$, $F = F_1^{(p)}$ and $G = G_1^{(p)}$. Then F and G are transcendental entire functions satisfying $E_l(1;F) = E_l(1;G)$. Using (1) and Lemma 2.9 we get

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;(F_1)^{(p)})+S(r,f) \\ &\leq T(r,(F_1)^{(p)})-T(r,F_1)+N_{p+2}(r,0;F_1)+S(r,f) \\ &\leq T(r,F)-(n-k-1)T(r,f)+N_{p+2}(r,0;F_1)+S(r,f). \end{aligned}$$

From this we get

$$(n-k-1)T(r,f) \leq T(r,F)-N_2(r,0;F)+N_{p+2}(r,0;F_1)+S(r,f). \quad (8)$$

Again from (2) we have

$$N_2(r,0;F) \leq N_2(r,0;(F_1^{(p)})+S(r,f) \leq N_{p+2}(r,0;F_1)+S(r,f). \quad (9)$$

We now discuss the following three cases sepeartely.

Case 1. Let $l \geq 2$. Suppose, if possible , that (i) of Lemma 2.6 holds. Then using (9) we obtain (8)

$$\begin{aligned} (n-k-1)T(r,f) &\leq N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+N_{p+2}(r,0;F_1) \\ &\quad +S(r,f)+S(r,g) \\ &\leq N_{p+2}(r,0;F_1)+N_{p+2}(r,0;G_1)+S(r,f)+S(r,g) \end{aligned}$$

$$(n-k-1)T(r,f) \leq (m_1+(p+2)m_2+k+2)[T(r,f)+T(r,g)]+S(r,f)+S(r,g) \quad (10)$$

Similarly,

$$(n-k-1)T(r,g) \leq (m_1+(p+2)m_2+k+2)[T(r,f)+T(r,g)]+S(r,f)+S(r,g) \quad (11)$$

Combining (10) and (11) we obtain

$$(n-k-1-2m_1-2pm_2-4m_2-2k-4)[T(r,f)+T(r,g)] \leq S(r,f)+S(r,g),$$

contradicting the fact that $n > 2\Gamma_2+2pm_2+3k+5$. Therefore , by Lemma 2.6, we have either $FG=1$ or $F=G$. We assume that $F=G$ then

$$(P(f)L_k(f,\Delta_{q,c}))^{(p)} = (P(g)L_k(g,\Delta_{q,c}))^{(p)}$$

Integrating once we obtain

$$(P(f)L_k(f,\Delta_{q,c}))^{(p-1)} = (P(g)L_k(g,\Delta_{q,c}))^{(p-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, from Lemma 2.10, it follows that $n \leq 2\Gamma_1 + 2(p-1)m_2+3k+5$, contrary to the fact that $n > 2\Gamma_2+2pm_2+3k+5$ and $\Gamma_2 \geq \Gamma_1$. Hence we must have $c_{k-1}=0$. Repeating the process k-times we deduce that

$$(P(f)L_k(f,\Delta_{q,c}) = (P(g)L_k(g,\Delta_{q,c}) \quad (12)$$

i.e $[a_nf^n+a_{n-1}f^{n-1}+...+a_1f+a_0]\{a_k[(f(q_kz+c_k)-f(z)]+a_{k-1}[f(q_{k-1}z+c_{k-1}-f(z)]+ ... + a_1[f(q_1z+c_1)-f(z)]+a_0[f(q_0z+c_0)-f(z)]\} = [a_ng^n+a_{n-1}g^{n-1}+...+a_1g+a_0]\{a_k[(g(q_kz+c_k)-g(z)]+a_{k-1}[g(q_{k-1}z+c_{k-1})-g(z)]+...+a_1[g(q_1z+c_1)-g(z)]+a_0[g(q_0z+c_0)-g(z)]\}$

Set $h = \frac{f}{g}$, we consider the following two subcases.

Subcase 1.1. Suppose h is non-constant then using (12), f and g will be a solution of the algebraic equation $R(f,g)=0$, with $R(w_1,w_2) = P(w_1)L_k(w_1,\Delta_{q,c}) - P(w_2)L_k(w_2,\Delta_{q,c})$.

Subcase 1.2. If h is constant, then substituting $f=gh$ in (12), we get

$[a_ng^nh^n + a_{n-1}g^{n-1}h^{n-1} + ... + a_1gh + a_0]\{a_kh[(g(q_kz+c_k)-g(z)] + a_{k-1}h[g(q_{k-1}z + c_{k-1}-g(z)] + ... + a_1h[g(q_1z+c_1)-g(z)] + a_0h[g(q_0z+c_0)-g(z)]\} =$
$[a_ng^n+a_{n-1}g^{n-1}+...+a_1g+a_0]\{a_k[(g(q_kz+c_k)-g(z)]+a_{k-1}[g(q_{k-1}z+c_{k-1})-g(z)]+ ... + a_1[g(q_1z+c_1)-g(z)]+a_0[g(q_0z+c_0)-g(z)]\}$

which implies,

$$a_ng_n[L_k(g,\Delta_{q,c})](h^{n+1}-1) + a_{n-1}g^{n-1}[L_k(g,\Delta_{q,c})](h^n-1)+...+$$

$$a_1g[L_k(g,\Delta_{q,c})](h^2-1) + a_0[L_k(g,\Delta_{q,c})](h-1) = 0 \quad (13)$$

If $a_n \neq 0, a_{n-1} = a_{n-2} = ... = a_0 = 0$, then we get $h^{n+1}=1$.
Let $a_n \neq 0$ and suppose there is some $a_i \neq 0 (i \in 0,1,...,n-1)$. Let $h^{n+1} \neq 1$ from (13), we have $T(r,g) = S(r,g)$ which contadicts the transcendental function g. So $h^{n+1}=1$. Likewise, we have $h^{j+1}=1$ provided $a_j \neq 0 (j=0,1,2...n)$, which implies $f=tg$ where t is a constant such that $t^d=1$.

Case 2. Let $l = 1$ and $H \not\equiv 0$. Using Lemma 2.7 and (9), we obtain from (8)

$$\begin{aligned}(n-k-1)T(r,f) \leq\ & N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) \\ & + \frac{1}{2}\overline{N}(r,\infty;F) + N_{p+2}(r,0;F_1) + S(r,f) + S(r,g) \\ \leq\ & N_{p+2}(r,0;F_1) + N_{p+2}(r,0;G_1) + \frac{1}{2}N_{p+1}(r,0;F_1) + S(r,f) + S(r,g) \\ \leq\ & [m_1 + (p+2)m_2 + k + 2]T(r,f) + \frac{1}{2}[m_1 + (p+1)m_2 + k + 2]T(r,f) \\ & + [m_1 + (p+2)m_2 + k + 2]T(r,g) \\ \leq\ & \frac{1}{2}[5m_1 + (5p+9)m_2 + 5k + 10]T(r) + S(r), \end{aligned} \tag{14}$$

where $T(r)$ and $S(r)$ are same as in Lemma 2.6. Similarly, we obtain

$$(n-k-1)T(r,g) \leq \frac{1}{2}[5m_1 + (5p+9)m_2 + 5k + 10]T(r) + S(r) \tag{15}$$

From the above inequalities (14) and (15) we have

$$\left[n - \frac{5m_1 + (5p+9)m_2 + 7k + 12}{2}\right] T(r) \leq S(r),$$

contradicting the fact that $n > \frac{1}{2}[\Gamma_1 + 4\Gamma_2 + 5pm_2 + 7k + 12]$.
We now assume that $H \equiv 0$. Then

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get,

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{16}$$

where $A(\neq 0)$ and B are constants. From (16) it is obvious that F, G share the value 1 CM and so they share (1, 2). Hence we have $n > 2\Gamma_2 + 2pm_2 + 3k + 5$.
Now we discuss the following three subcases.
Subcase 2.1. Let $B \neq 0$ and $A = B$ then from (16), we get

$$\frac{1}{F-1} = \frac{BG}{G-1}, \tag{17}$$

If $B = -1$, then from (17) we obtain $FG = 1$ i.e

$$(P(f)L_k(f,\Delta_{q,c}))^{(p)}.(P(g)L_k(g,\Delta_{q,c}))^{(p)} = a^2$$

which is one of the conclusion of Theorem 1.2.
If $B \neq -1$, from (17), we have

$$\frac{1}{F} = \left(\frac{BG}{(1+B)G - 1}\right)$$

and so $\overline{N}\left(r, \frac{1}{1+B}; G\right) = \overline{N}(r,0;F)$.
Now from the second fundamental theorem and Lemma 2.3, we have

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq N_{p+1}(r,0;F_1) + T(r,G) + N_{p+1}(r,0;G_1) - (n-k-1)T(r,g) + S(r,g). \end{aligned}$$

This gives

$$(n-k-1)T(r,g) \leq [(m_1 + (p+1)m_2 + k + 2)]\{T(r,f) + T(r,g)\} + S(r,g). \tag{18}$$

Similary,

$$(n-k-1)T(r,f) \leq [(m_1+(p+1)m_2+k+2)]\{T(r,f)+T(r,g)\}+S(r,f) \quad (19)$$

By combining (18) and (19) we obtain

$$(n-k-1-2m_1-2(p+1)m_2-2k-4)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g), \quad (20)$$

a contradiction that $n > 2\Gamma_2 + 2pm_2 + 3k + 5$.

Subcase 2.2. Let $B \neq 0$ and $A \neq B$. Then from (16) we obtain $F = \left(\frac{(B+1)G-(B-A+1)}{BG+(A-B)}\right)$ and therefore $\overline{N}\left(r, \frac{B-A+1}{B+1}; G\right) = \overline{N}(r,0;F)$. Proceeding similarly as in subcase 2.1, we can get a contradiction.

Subcase 2.3. Let $B = 0$ and $A \neq 0$. Then from (16) we get $F = \left(\frac{G+A-1}{A}\right)$ and $G = AF-(A-1)$. If $A \neq 1$, we have $\overline{N}\left(r, \frac{A-1}{A}; F\right) = \overline{N}(r,0;G)$ and $\overline{N}(r,1-A;G) = \overline{N}(r,0;F)$ Then by Lemma 2.10, it follows that $n \leq 2\Gamma_1 + 2pm_2 + 3k + 5$, a contradiction. Thus $A = 1$ and then $F = G$. Now the result follows from proof of case 1.

Case 3. Let $l = 0$ and $H \not\equiv 0$. Using Lemma 2.8 and (9), we obtain from (8)

$$\begin{aligned}
(n-k-1)T(r,f) \leq\ & N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) \\
& + \overline{N}(r,0;G) + N_{p+2}(r,0;F_1) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\
& + S(r,f) + S(r,g) \\
\leq\ & N_{p+2}(r,0;F_1) + N_{p+2}(r,0;G_1) + 2N_{p+1}(r,0;F_1) + N_{p+1}(r,0;G_1) \\
& + S(r,f) + S(r,g) \\
\leq\ & [3m_1 + (3p+4)m_2 + 3k + 6]T(r,f) + [2m_1 + (3+2p)m_2 + 2k + 4]T(r,g) \\
& + S(r,f) + S(r,g) \\
\leq\ & [5m_1 + (5p+7)m_2 + 5k + 10]T(r) + S(r)
\end{aligned}$$

$$(n-k-1)T(r,f) \leq [5m_1 + (5p+7)m_2 + 5k + 10]T(r) + S(r). \quad (21)$$

Similarly,

$$(n-k-1)T(r,g) \leq [5m_1 + (5p+7)m_2 + 5k + 10]T(r) + S(r). \quad (22)$$

From (21) and (22) we obtain $(n - 5m_1 - (5p-7)m_2 + 6k + 11)T(r) \leq S(r)$,
Contradicting the assumption that $n > 3\Gamma_1 + 2\Gamma_2 + 5pm_2 + 6k + 11$.
Therefore $H \equiv 0$ and then proceeding in a manner, similar to case 2, the result follows.
This completes the proof of Theorem 1.2.

Applications. Value distribution of zero order meromorphic functions is a powerful tool and their weighted value sharing is a fascinating area of research within complex analysis. Through the analysis of various research articles, we have gained valuable insights into the properties of these functions and their applications in mathematics, engineering and physics. By explore the idea of q-shift difference-differential polynomials, and finding a useful tool to comprehend the behaviour of meromorphic (entire) functions.

Also, we can pose the following open questions.

Open questions:

1. What happens to Theorems 1.1 and 1.2, if we replace the linear q - difference operator $L_k(f, \Delta_{q,c})$ by $L(z,f)$ where $L(z,f) = b_1 f(qz+c) + b_0 f(z)$, $b_1(\neq 0)$ and b_0 are complex constants.
2. Can the condition for the lower bound n in Theorems 1.1 and 1.2 be reduced any further?

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Harina P. Waghamore

Department of Mathematics, Jnanabharathi Campus, Bangalore University, Bangalore, India- 560056

Email address: harinapw@gmail.com, harina@bub.ernet.in

Megha M. Manakame

Department of Mathematics, Jnanabharathi Campus, Bangalore University, Bangalore, India- 560056

Email address: megha.manakame80@gmail.com, meghamm@bub.ernet.in