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NUMERICAL APPROXIMATION OF MULTI-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS BY GALERKIN METHOD WITH CHEBYSHEV POLYNOMIAL BASIS

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ABSTRACT. The Galerkin method is a numerical technique used to approximate solutions to Partial Differential Equations (PDEs) or integral equations. To this end, this study employed the Galerkin method to address Multi-Order Fractional Differential Equations (MFDEs), utilizing Chebyshev Polynomials as the basis functions. The approach involved assuming an approximate solution using shifted Chebyshev polynomials, which was then substituted into the given problem. Subsequently, boundary conditions were applied. The residual equation, integrated over the interval of interest along with the weight function, resulted in a linear system of equations with unknown Chebyshev coefficient constants. Maple 18 was utilized to determine these unknown constants, which were then substituted back into the assumed solution to obtain the desired approximate solution. To assess the effectiveness of the proposed technique, numerical examples were solved, and the results were compared with existing literature. The comparison showcased that the suggested algorithm is not only accurate but also efficient in multi-order fractional differential equations. Tables and figures were employed to present and illustrate the obtained results.

1. INTRODUCTION

The exploration of differential equations stands as a fundamental pillar within the field of mathematics, playing a pivotal role in comprehending and resolving diverse real-world challenges. These mathematical constructs serve to depict the intricate connection between a function and its derivatives. Contemporary focus has been directed towards fractional differential equations (FDEs) as they hold the remarkable capacity to model complex systems encountered in physics, engineering,

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and various applied sciences. This surge of interest has been motivated by their potential to offer deeper insights and more accurate representations of the intricate dynamics present in such systems.

In recent years, researchers have shown significant interest in developing robust and efficient numerical methods for solving multi-order fractional differential equations. These equations involve multiple fractional derivatives of different orders, making their analytical solutions elusive, if not impossible, for all but a few special cases. Consequently, the quest for accurate and efficient numerical techniques has intensified to unlock the secrets hidden in these intricate mathematical models.

The Galerkin method, a highly influential and extensively employed numerical technique, has showcased its efficacy in tackling diverse differential equations. This approach involves employing a finite-dimensional subspace to approximate the sought-after solution, effectively transforming the original problem into a system of algebraic equations. By doing so, it becomes amenable to efficient resolution through standard numerical methods. The Galerkin method has garnered notable achievements in solving both ordinary and partial differential equations, arousing considerable interest among researchers keen to explore its potential in the realm of fractional calculus. Its adaptability and success in various differential equation scenarios make it a compelling option for investigating and solving complex mathematical models arising in practical applications and scientific investigations.

Accurate solutions using the Galerkin method for fractional differential equations necessitate careful selection of suitable basis functions. Among the plethora of options, Chebyshev polynomials emerge as a standout choice owing to their exceptional properties. As orthogonal polynomials, they contribute to efficient numerical computations while demonstrating remarkable approximation capabilities. Consequently, Chebyshev polynomials present themselves as an ideal candidate to address the intricacies posed by multi-order fractional differential equations, making them a promising avenue of exploration in this context.

Chebyshev polynomials, known for their excellent approximation properties, serve as the basis functions in our approach. Chebyshev polynomials are a set of orthogonal polynomials defined on a given interval, which possess several desirable properties for numerical approximation. They have been successfully applied in various areas of mathematics and engineering due to their ability to provide accurate and efficient solutions.

The main advantage of using the Galerkin method with Chebyshev polynomials lies in its ability to handle multi-order fractional differential equations. By considering different fractional orders in the equations, we can capture the dynamics of complex systems more effectively. This approach allows us to obtain numerical solutions that accurately represent the behavior of the systems under investigation. This work considers the numerical solution of multi-order fractional differential equation:

$$D^{\alpha}u(x) = \sum_{i=0}^{k} y_i D^{\beta i} u(x) + f(x), x \in [0,1] \tag{1}$$

with the initial conditions:

$$u^{(p)}(0) = d_p, \qquad p = 0, 1, \ldots, n-1, \tag{2}$$

where $n-1 < a \leq n$, the coefficient $y_i(i = 0, 1, \ldots, k)$ is constant, and $0 < \beta_0 < \cdots < \beta_k < a$, $f(x)$ is a known function.

Numerous researchers have contributed to the exploration of numerical solutions for Fractional Differential Equations(FDEs). Kilbas et al. [12] delved into the theory and application of FDEs. Chen et al. [6] focused on the numerical solution of nonlinear fractional integral Differential Equations(DE), employing the second kind Chebyshev wavelets. In another study, Chen et al. [7] proposed a numerical solution for a class of linear systems of FDEs using the Haar wavelet method, including a convergence analysis.

Pimenov and Hendy [19] conducted numerical studies on functional DEs with delay, utilizing backward differentiation Formula (BDF-type) shifted Chebyshev approximations. Babolian et al. [5] explored numerical solutions for MFDEs using Boubaker polynomials. Anastassiou et al. [2] established the monotone convergence of extended iterative methods and fractional calculus, showcasing various applications.

Saeedi [20] employed a fractional-order operational method to numerically treat multi-order fractional partial DEs with variable coefficients. Jafari et al. [11] provided an analysis of Riccati differential equations within a new fractional derivative without a singular kernel. Han et al. [10] proposed a numerical solution for a class of MFDEs, incorporating error correction and convergence analysis. Dabiri and Butche [8] obtained the numerical solution MFDEs with multiple delays through spectral collocation methods.

Yusuf et al. [24] introduced soliton structures for some time-fractional nonlinear DEs with a conformable derivative. Bello et al. [4] utilized fourth kinds of Chebyshev polynomials as basis functions for the numerical treatment of MFDEs. El-Sayed et al. [9] established the Jacobi operational matrix for the numerical solution of multi-term variable-order FDEs. Shah and Khan [21] investigated a system of nonlinear fractional order hybrid DEs under usual boundary conditions for the existence of a solution.

Alshehri et al. [1] derived a Caputo (discretization) fractional-order model of glucoseinsulin interaction, presenting numerical solutions and comparisons with experimental data. Tajadodi et al. [22] presented exact solutions of conformable FDEs. Talib et al. [23] conducted a numerical study on multi-order fractional differential equations with constant and variable coefficients.[15]-[18] applied collocation method for solving integro-differential equations, while [22] applied least squares collocation for fractional integro-differential equations.

Mamadu and Njoseh [13] applied Galerkin method Volterra equations with certain orthogonal polynomials. Mamadu et al. [14] applied Galerkin Method with Mamadu-Njoseh Polynomials for the solution of Fractional Integro-Differential Equation. Motivated by the above work, this study propose Galakin Method with shifted Chebyshev polynomial polynomials for the solution of MFDEs.

2. Definition of Relevant Terms

2.1. **Fractional Differential Equations (FDEs).** A fractional differential equation is a type of differential equation that involves fractional derivatives of a function. It is a Differential Equation whose Order of Derivative is not an integer.

$$D^{\alpha}u(x) + au(x) = 0 \tag{3}$$

$x > 0$ and $\alpha > 0$ (α not necessarily an integer)

2.2. **Galerkin Method.** The approach, alternatively known as the weighted residual method, employs trial functions (or approximating functions) that adhere to the boundary conditions of the given problem. The trial function is inserted into the specified differential equation, and the outcome is termed the residual. Subsequently, the integral of the product between this residual and a weighted function across the domain is equated to zero, leading to a set of equations that determine the unknown parameters in the trial function.

2.3. **Chebyshev Polynomials.** The very known Chebyshev polynomial of degrees n denoted $T_n(x)$,defined on the interval $-1 \leq x \leq 1$ can be determined using the recurrence formular.

$$T_{n+1}(x) = 2T_1(x)T_n(x) - T_{n-1}(x) \qquad n = 0, 1, 2, \ldots \tag{4}$$

Therefore, we have only a few terms:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \end{aligned}$$

In order to derive these polynomials within the interval [0, 1], we introduce a variable transformation with respect to x.

$$\begin{aligned} x &= \frac{1}{2}X + \frac{1}{2} \\ \implies X &= 2x - 1 \end{aligned}$$

Now substitute X into $T_n(x) = \cos(n \cdot \cos^{-1}(X))$, $n = 0, 1, 2 \ldots$ The shifted Chebyshev polynomial is defined as follows:

$$\begin{aligned} T_0^*(x) &= 1 \\ T_1^*(x) &= 2x - 1 \end{aligned}$$

This leads to the recursive relation:

$$T_{n+1}^*(x) = 2(2x-1)T_n^*(x) - T_{n-1}^*(x), \qquad n \geq 1 \tag{5}$$

Utilizing this recurrence relation, we can generate the following terms:

$$\begin{aligned} T_0^*(x) &= 1 \\ T_1^*(x) &= 2x - 1 \\ T_2^*(x) &= 8x^2 - 8x + 1 \\ T_3^*(x) &= 32x^3 - 48x^2 + 18x - 1 \\ T_4^*(x) &= 128x^4 - 256x^3 + 160x^2 - 32x + 1 \\ T_5^*(x) &= 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1 \\ T_6^*(x) &= 2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1 \end{aligned}$$

2.4. **Multi-Order Fractional Differential Equations.** Multi-order Fractional Differential Equation (FDE) is a type of differential equation that involves multiple fractional derivatives of different orders. Fractional derivatives extend the concept of differentiation to non-integer orders, allowing for the description of systems with complex dynamics and memory effects. In a multi-order FDE, the fractional derivatives can have various orders, and the equation may include both fractional and integer-order derivatives.

Mathematically, a multi-order FDE can be represented as:

$$D^{\alpha_1}D^{\alpha_2}\dots D^{\alpha_k}y(t) = f(t, y(t), D^{\beta_1}y(t), \dots, D^{\beta_m}y(t))$$

where $y(t)$ is the unknown function, D^{α_i} denotes the fractional derivatives of different orders α_i, and D^{β_j} represents the integer-order derivatives of different orders β_j. The function $f(t, y(t), D^{\beta_1}y(t), \dots, D^{\beta_m}y(t))$ represents a given expression involving the variables t, $y(t)$, and the integer-order derivatives.

2.5. **Exact Solution.** An exact solution is a precise and complete analytical solution to a mathematical problem or equation that provides an explicit formula or expression satisfying all given conditions without any approximation or error.

2.6. **Approximate Solution.** An approximate solution refers to an estimation or an approach to solving a mathematical problem, such as a differential equation, that provides an approximation of the true solution with a certain level of accuracy.

2.7. **Absolute Error.** In the context of numerical computation and approximation, the absolute error is a measure of the difference between an approximate value or solution and the true or exact value. It quantifies the magnitude of the discrepancy between the approximation and the actual value, disregarding the direction of the error (whether it is positive or negative).

Mathematically, for a true or exact value x_{true} and an approximate value x_{approx}, the absolute error (E_{abs}) is calculated as:

$$E_{\text{abs}} = |x_{\text{true}} - x_{\text{approx}}|$$

3. Method of solution

In this section, we shall consider the general Multi-Order Fractional differential equation of the form Equ. (1) and Equ. (2) as:

$$D^{\alpha}u(x) = \sum_{i=0}^{k} y_i D^{\beta_i}u(x) + f(x), \quad x \in [0,1] \tag{6}$$

with initial conditions

$$u^0(0) = d_0 \qquad u^1(0) = d_1 \tag{7}$$

We assumed an approximate solution of the form:

$$u_m(x) = \sum_{n=0}^{m} a_n T_n^*(x), \tag{8}$$

where $T_n^*(x)$ is the shifted Chebyshev Polynomial and a_n are the constants to be determined. Equ. (7) is imposed on (8) and manipulated to give:

$$u_m^*(x) = \sum_{n=2}^{m} a_n T_n^*(x). \tag{9}$$

Substituting (9) in (6) gives:

$$\begin{aligned} D^\alpha(u_m^*) = y_0 D^{\beta_0} u_m^*(x) + y_1 D^{\beta_1} u_m^*(x) + y_2 D^{\beta_2} u_m^*(x) \\ + y_3 D^{\beta_3} u_m^*(x) + f(x). \end{aligned} \tag{10}$$

Hence, the residual equation is obtained as:

$$\begin{aligned} R(x) = D^\alpha(u_m^*) - y_0 D^{\beta_0} u^*(x) - y_1 D^{\beta_1} u^*(x) \\ - y_2 D^{\beta_2} u^*(x) - y_3 D^{\beta_3} u^*(x) - f(x) \end{aligned} \tag{11}$$

Let

$$I_n = \int_a^b T_n^*(x) R(x) dx = 0, \tag{12}$$

considering that $T_n^*(x))$ is the weight function defined in the interval [0,1], setting Equation (1) to zero results in a system of algebraic equations with unknown constants. These constants, denoted as $a_i (i = 2, 3, 4, 5, \dots n)$, are then determined by solving the system using Maple 18. Subsequently, the unknowns the unknowns a_i are substituted into (9) to derive the desired approximate solution.

4. Numerical Examples

4.1. **Example 4.1.** Consider the following multi-order fractional differential equation

$$\begin{aligned} D^\alpha u(x) = y_0 D^{\beta_0} u(x) + y_1 D^{\beta_1} u(x) + y_2 D^{\beta_2} u(x) \\ + y_3 D^{\beta_3} u(x) + f(x); \quad x \in [0, 1], \end{aligned} \tag{13}$$

with initial conditions

$$u^0(0) = d_0 \qquad u^1(0) = d_1$$

where $\alpha = 2$, $d_0 = d_1 = 0$, the coefficient is $y_0 = y_2 = -1$, $y_1 = 2$, $y_3 = 0$ and $\beta_0 = 0$, $\beta_1 = 1$, $\beta_2 = \frac{1}{2} \in (0, 1)$,

$$f(x) = x^7 + \frac{2048}{429\sqrt{\pi}} x^{6.5} - 14x^6 + 42x^5 - x^2 - \frac{8}{3\sqrt{\pi}} x^{1.5} + 4x - 2$$

The exact solution is $u(x) = x^7 - x^2$.

Solution: Here, Equ. (13) assumed an approximate solution of the form:

$$u(x) = u_7(x) = \sum_{n=0}^{7} a_n T_n(x) \tag{14}$$

Expanding Equation (14) yields:

$$\begin{aligned} u(x) = a_0 + a_1(2x-1) + a_2(8x^2-8x+1) + a_3(32x^3-48x^2+18x-1) \\ + a_4(128x^4-256x^3+160x^2-32x+1) + a_5(512x^5 \\ -1280x^4+1120x^3-400x^2+50x-1) + a_6(2048x^6 \\ -6144x^5+6912x^4-3584x^3+840x^2-72x+1) \\ + a_7(8192x^7-28672x^6+39424x^5-26880x^4 \\ +9408x^3-1568x^2+98x-1) \end{aligned} \tag{15}$$

Applying the initial conditions to Equ. (15) as outlined in Equations (9) and (10), Equation (15) transforms into:

$$\begin{aligned} u^*(x) = 8a_2x^2 + a_3(32x^3-48x^2) + a_4(128x^4-256x^3+160x^7) \\ + a_5(512x^5-1280x^4+1120x^3-400x^7) + a_6(2048x^6 \\ -6144x^5+6912x^4-3584x^3+840x^2) + a_7(8192x^7 \\ -28672x^6+39424x^5-26880x^4+9408x^3-1568x^2) \end{aligned} \tag{16}$$

Inserting Equation (16) into Equation (13) results in the expression for $R(x)$.

$$\begin{aligned} R(x) = \Big[16a_2 + a_3(-96+192x) + a_4(320-1536x+1536x^2) \\ +a_5(-800+6720x-15360x^2+10240x^3) + a_6(1680-21504x \\ +82944x^2-122880x^3+61440x^4) + a_7(-3136+56448x \\ -322560x^2+788480x^3-860160x^4+344064x^5)\Big] + \Big[8a_2x^2 \\ +a_3(32x^3-42x^2) + a_4(128x^4-256x^3+160x^2) + a_5(512x^5 \\ -1280x^4+1120x^3-400x^2) + a_6(2048x^6-6144x^5+6912x^4 \\ -3584x^3+840x^2) + a_7(8192x^7-28672x^6+39424x^5-26880x^4 \\ +9408x^3-1568x^2)\Big] - 32a_2x + a_3(192x-192x^2) + a_4(-640x \\ +1536x^2x-1024x^3) + a_5(1600x-6720x^2+10240x^3-5120x^4) \\ +a_6(-3360x+21504x^2-55290x^3+61440x^4-24570x^5) + a_7(6272x \\ -56448x^2+215040x^3-394240x^4+344064x^5-114688x^6) \\ +\Big[a_2(\frac{64x^{\frac{3}{2}}}{3\sqrt{\pi}}) + a_3(512\frac{x^{\frac{5}{2}}}{5\sqrt{\pi}} - \frac{128x^{\frac{3}{2}}}{\sqrt{\pi}}) + a_4(16384\frac{x^{\frac{7}{2}}}{35\sqrt{\pi}} - 4096\frac{x^{\frac{5}{2}}}{5\sqrt{\pi}} + \\ \frac{1280x^{\frac{3}{2}}}{3\sqrt{\pi}}) + a_5(131072\frac{x^{\frac{9}{2}}}{63\sqrt{\pi}} - 32768\frac{x^{\frac{7}{2}}}{7\sqrt{\pi}} + 3584\frac{x^{\frac{5}{2}}}{\sqrt{\pi}} - 3200\frac{x^{\frac{3}{2}}}{3\sqrt{\pi}}) \\ +a_6(2097152\frac{x^{\frac{11}{2}}}{231\sqrt{\pi}} - 3524288\frac{x^{\frac{9}{2}}}{21\sqrt{\pi}} + \frac{884736x^{\frac{7}{2}}}{35\sqrt{\pi}} - 57344\frac{x^{\frac{5}{2}}}{5\sqrt{\pi}} \\ +2240\frac{x^{\frac{3}{2}}}{\sqrt{\pi}}) + a_7(16777216\frac{x^{\frac{13}{2}}}{429\sqrt{\pi}} - 4194304\frac{x^{\frac{11}{2}}}{33\sqrt{\pi}} \\ +1441792\frac{x^{\frac{9}{2}}}{9\sqrt{\pi}} - 98304\frac{x^{\frac{7}{2}}}{\sqrt{\pi}} + 150528\frac{x^{\frac{5}{2}}}{5\sqrt{\pi}} - \frac{112544x^{\frac{3}{2}}}{3\sqrt{\pi}}) - f(x)\Big] \end{aligned} \tag{17}$$

Let

$$I_n = \int_0^1 T_n^*(x)R(x)dx = 0 \tag{18}$$

Here, $T^*n(x)$ (where $n = 2, 3, 4, 5, \ldots m$) represents the weight functions, selected from the coefficients of ai (where $n = 2, 3, 4, 5, \ldots m$) in Equation (16). Substituting Equation (17) and the weight functions into Equation (18) yields:

$$I_1 = \int_0^1 x^2[R(x)]\,dx = 0$$

$$I_2 = \int_0^1 [32x^3 - 48x^2][R(x)]\,dx = 0$$

$$I_3 = \int_0^1 [128x^4 - 256x^3 + 160x^2][R(x)]\,dx = 0$$

$$I_4 = \int_0^1 [512x^5 - 1280x^4 + 1120x^3 - 400x^2][R(x)]\,dx = 0$$

$$I_5 = \int_0^1 [2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2][R(x)]\,dx = 0$$

$$I_6 = \int_0^1 [8192x^7 - 28672x^6 + 39424x^5 - 26880x^4 + 9408x^3$$

$$-1568x^2][R(x)]dx = 0$$

Solving I_1 to I_6,we have the unknown constants as:

$$\left.\begin{array}{c} a_2 = 0.1193847656 \\ a_3 = 0.1221923828 \\ a_4 = 0.04443359375 \\ a_5 = 0.011108398437 \\ a_6 = 0.001708984375 \\ a_7 = 0.000122070312305 \end{array}\right\} \tag{19}$$

Substituting a_i's in equation (16) gives:

$$\begin{aligned} u^*(x) = 8(0.119385)x^2 + 0.122192(32x^3 - 48x^2) + 0.0444336(128x^4 \\ - 256x^3 + 160x^2) + 0.0111084(512x^5 - 1280x^4 + 1120x^3 \\ - 400x^2) + 0.00170898(2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 \\ + 840x^2) + 0.00012207(8192x^7 - 28672x^6 + 39424x^5 \\ - 26880x^4 + 9408x^3 - 1568x^2) \end{aligned} \tag{20}$$

Thus, the required approximation solution is given as:

$$\begin{aligned} u^*(x) = -x^2 - 4.1682 \times 10^{-9}x^3 + 9.2183 \times 10^{-9}x^4 \\ + 1.09061 \times 10^{-8}x^5 + 6.58295 \times 10^{-9}x^6 + x^7 \end{aligned} \tag{21}$$

4.2. **Example 4.2.** Consider the following Multi-Order fractional differential equation

$$\begin{aligned} D^\alpha u(x) = y_0 D^{\beta_0} u(x) + y_1 D^{\beta_1} u(x) + y_2 D^{\beta_2} u(x) \\ + y_3 D^{\beta_3} u(x) + f(x) \quad x \in [0, 1] \end{aligned} \tag{22}$$

with initial conditions

$$u^0(0) = d_0 \qquad u^1(0) = d_1$$

Where $\alpha = 2$, $d_0 = d_1 = 0$, the coefficient is $y_0 = y_2 = -1$, $y_1 = 0$, $y_3 = 2$ and $\beta_0 = 0$, $\beta_1 = 1$, $\beta_2 = \frac{2}{3} \in (0, 1)$, $\beta_3 = \frac{5}{3} \in (1, 2)$,

$$f(x) = x^3 + 6x + \frac{12}{\sqrt{\frac{7}{3}}} x^{\frac{4}{3}} + \frac{6}{\sqrt{\frac{10}{3}}} x^{\frac{7}{3}}$$

The exact solution is $u(x) = x^3$.

Solution
Here, equation (22) assumed an approximate solution of the form:

$$u(x) = u_7(x) = \sum_{n=0}^{7} a_n T_n(x). \tag{23}$$

Expanding Equ. (23) gives:

$$\begin{aligned} u(x) = a_0 &+ a_1(2x - 1) + a_2(8x^2 - 8x + 1) + a_3(32x^3 - 48x^2 + 18x - 1) \\ &+ a_4(128x^4 - 256x^3 + 160x^2 - 32x + 1) + a_5(512x^5 \\ &- 1280x^4 + 1120x^3 - 400x^2 + 50x - 1) + a_6(2048x^6 \\ &- 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1) \\ &+ a_7(8192x^7 - 28672x^6 + 39424x^5 - 26880x^4 \\ &+ 9408x^3 - 1568x^2 + 98x - 1) \end{aligned} \tag{24}$$

Applying the initial conditions to Equation (24) as outlined in Equations (9) and (10), Equation (24) transforms into:

$$\begin{aligned} u^*(x) = 8a_2x^2 &+ a_3(32x^3 - 48x^2) + a_4(128x^4 - 256x^3 + 160x^7) \\ &+ a_5(512x^5 - 1280x^4 + 1120x^3 - 400x^7) + a_6(2048x^6 \\ &- 6144x^5 + 6912x^4 - 3584x^3 + 840x^2) + a_7(8192x^7 \\ &- 28672x^6 + 39424x^5 - 26880x^4 + 9408x^3 - 1568x^2) \end{aligned} \tag{25}$$

Substituting (25) in (22) gives:

$$
\begin{aligned}
R(x) = \Big[&16a_2 + a_3(-96+192x) + a_4(320-1536x+1536x^2) \\
&+a_5(-800+6720x-15360x^2+10240x^3) + a_6(1680-21504x \\
&+82944x^2-122880x^3+61440x^4) + a_7(-3136+56448x \\
&-322560x^2+788480x^3-860160x^4+344064x^5)\Big] + \Big[8a_2x^2 + a_3(32x^3 \\
&-42x^2) + a_4(128x^4-256x^3+160x^2) + a_5(512x^5-1280x^4+1120x^3 \\
&-400x^2) + a_6(2048x^6-6144x^5+6912x^4-3584x^3+840x^2)+ \\
&a_7(8192x^7-28672x^6+39424x^5-26880x^4+9408x^3-1568x^2)\Big] \\
&+\Big[16a_2\frac{x^{\frac{4}{3}}}{\Gamma\frac{7}{3}} + a_3(192\frac{x^{\frac{7}{3}}}{\Gamma\frac{10}{3}} - \frac{96x^{\frac{4}{3}}}{\Gamma\frac{7}{3}}) + a_4(3072\frac{x^{\frac{10}{3}}}{\Gamma\frac{13}{3}} - 1536\frac{x^{\frac{7}{3}}}{\Gamma\frac{10}{3}} + 320\frac{x^{\frac{4}{3}}}{\Gamma\frac{7}{3}}) \\
&+a_5(61440\frac{x^{\frac{13}{3}}}{\Gamma\frac{16}{3}} - 30720\frac{x^{\frac{10}{3}}}{\Gamma\frac{13}{3}} - 800\frac{x^{\frac{4}{3}}}{\Gamma\frac{7}{3}}) + a_6(1474560\frac{x^{\frac{16}{3}}}{\Gamma\frac{19}{3}} - 737280\frac{x^{\frac{13}{3}}}{\Gamma\frac{16}{3}} \\
&+165888\frac{x^{\frac{10}{3}}}{\Gamma\frac{13}{3}} - 21504\frac{x^{\frac{7}{3}}}{\Gamma\frac{10}{3}} + 1680\frac{x^{\frac{4}{3}}}{\Gamma\frac{7}{3}}) + a_7(41287680\frac{x^{\frac{19}{3}}}{\Gamma\frac{22}{3}} \\
&-20643840\frac{x^{\frac{16}{3}}}{\Gamma\frac{19}{3}} + 4730880\frac{x^{\frac{13}{3}}}{\Gamma\frac{16}{3}} - 645120\frac{x^{\frac{10}{3}}}{\Gamma\frac{13}{3}} + 56448\frac{x^{\frac{7}{3}}}{\Gamma\frac{10}{3}} - \frac{3136x^{\frac{4}{3}}}{\Gamma\frac{7}{3}})\Big] \\
&-32a_2\frac{x^{\frac{1}{3}}}{\Gamma\frac{4}{3}} - 2a_3(\frac{192x^{\frac{4}{3}}}{\Gamma\frac{7}{3}} - \frac{96x^{\frac{1}{3}}}{\Gamma\frac{4}{3}}) + 2a_4(3072\frac{x^{\frac{7}{3}}}{\Gamma\frac{10}{3}} - 1536\frac{x^{\frac{4}{3}}}{\Gamma\frac{7}{3}} + 320\frac{x^{\frac{1}{3}}}{\Gamma\frac{4}{3}}) \\
&-2a_5(61440\frac{x^{\frac{10}{3}}}{\Gamma\frac{13}{3}} - 30720\frac{x^{\frac{7}{3}}}{\Gamma\frac{10}{3}} + 6720\frac{x^{\frac{4}{3}}}{\Gamma\frac{7}{3}} - 800\frac{x^{\frac{1}{3}}}{\frac{4}{3}}) \\
&-2a_6(1474560\frac{x^{\frac{13}{3}}}{\Gamma\frac{16}{3}} - 737280\frac{x^{\frac{10}{3}}}{\Gamma\frac{13}{3}} + 165880\frac{x^{\frac{7}{3}}}{\Gamma\frac{10}{3}} - 21504\frac{x^{\frac{4}{3}}}{\Gamma\frac{7}{3}} \\
&+1680\frac{x^{\frac{1}{3}}}{\Gamma\frac{4}{3}}) + 2a_7(41287680\frac{x^{\frac{16}{3}}}{\Gamma\frac{19}{3}} - \frac{206431640x^{\frac{13}{3}}}{\Gamma\frac{16}{3}} + 4730880\frac{x^{\frac{10}{3}}}{\Gamma\frac{13}{3}} \\
&-645120\frac{x^{\frac{7}{3}}}{\Gamma\frac{10}{3}} + 56448\frac{x^{\frac{4}{3}}}{\Gamma\frac{7}{3}} - 3136\frac{x^{\frac{1}{3}}}{\Gamma\frac{4}{3}}) - x^3 - 6x + \frac{12x^{\frac{4}{3}}}{\Gamma\frac{7}{3}} + \frac{6x^{\frac{7}{3}}}{\Gamma\frac{10}{3}}
\end{aligned}
\tag{26}
$$

Let

$$I_n = \int_0^1 T_n^*(x)R(x)dx = 0 \tag{27}$$

Here, $T^*n(x)$ (where $i = 2, 3, 4, 5, \ldots n$) represents the weight functions, selected from the coefficients of ai (where $i = 2, 3, 4, 5, \ldots n$) in Equation (25). Substituting

Equation (26) and the weight functions into Equation (27) yields:

$$I_1 = \int_0^1 x^2[R(x)]\,dx = 0$$
$$I_2 = \int_0^1 [32x^3 - 48x^2][R(x)]\,dx = 0$$
$$I_3 = \int_0^1 [128x^4 - 256x^3 + 160x^2][R(x)]\,dx = 0$$
$$I_4 = \int_0^1 [512x^5 - 1280x^4 + 1120x^3 - 400x^2][R(x)]\,dx = 0$$
$$I_5 = \int_0^1 [2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2][R(x)]\,dx = 0$$
$$I_6 = \int_0^1 [8192x^7 - 28672x^6 + 39424x^5 - 26880x^4 + 9408x^3$$

$-1568x^2][R(x)]dx$

Solving I_1 to I_6, we have,

$$\left.\begin{array}{c} a_2 = 0.1875 \\ a_3 = 0.03125 \\ a_4 = 2.88769 \times 10^{-11} \\ a_5 = 7.16425 \times 10^{-12} \\ a_6 = 1.20112 \times 10^{-12} \\ a_7 = 1.58142 \times 10^{-12} \end{array}\right\} \tag{28}$$

substituting a_i's in equation (25), gives:

$$\begin{aligned} u^*(x) = -1.46412 \times 10^{-10}x^2 + x^3 - 6.12385 \times 10^{-9}x^4 + 1.09369 \times 10^{-8}x^5 \\ - 9.11392 \times 10^{-9}x^6 + 3.52933 \times 10^{-9}x^7 \end{aligned} \tag{29}$$

4.3. **Example 4.3.** Consider the following Multi-Order fractional differential equation

$$D^{\beta_1}u(x) + x^{0.3}D^{\beta_2}u(x) = F(x) \quad x \in [0,1] \tag{30}$$

with initial conditions

$$u^0(0) = 0 \qquad u^1(0) = 0$$

Where $d_0 = d_1 = 0$, $\beta_1 = 1.5$ and $\beta_2 = 0.8$, $f(x) = \frac{2}{\sqrt{1.5}}x^{0.5} + \frac{2}{\sqrt{2.2}}x^{1.5}$.
The exact solution is $u(x) = x^2$

Solution: Likewise, by applying the same methodology as in Examples 4.1 and 4.2 to solve Example 3.3, we obtained the following unknown constants:

$$\left.\begin{array}{c} a_2 = 0.125 \\ a_3 = 1.8751 \times 10^{-11} \\ a_4 = 5.61708 \times 10^{-12} \\ a_5 = 1.02262 \times 10^{-12} \\ a_6 = 9.47084 \times 10^{-14} \\ a_7 = 7.46314 \times 10^{-15} \end{array}\right\}, \tag{31}$$

and the required approximate solution as:

$$u^*(x) = x^2 - 1.01487 \times 10^{-10}x^3 + 2.65671 \times 10^{-10}x^4 - 3.52535 \times 10^{-10}x^5 + 4.07946 \times 10^{-10}x^6 + 6.1138 \times 10^{-11}x^7 \quad (32)$$

5. Numerical Results

Table 1. Numerical Results for Example 4.1

x	**Exact Solution**	**Approximate Solution**	**Absolute Error**
0.0	0.0000000000	0.0000000000	0
0.1	-0.0099999000	-0.0099999000	0
0.2	-0.0399872000	-0.0399872000	0
0.3	-0.0897813000	-0.0897813000	0
0.4	-0.1583616000	-0.1583616001	1×10^{-10}
0.5	-0.2421875000	-0.2421875002	2×10^{-10}
0.6	-0.3320064000	-0.3320064002	2×10^{-10}
0.7	-0.4076457000	-0.4076457003	3×10^{-10}
0.8	-0.4302848000	-0.4302848002	2×10^{-10}
0.9	-0.3317031000	-0.3317030999	1×10^{-10}
1.0	0.0000000000	-0.0000000000	0

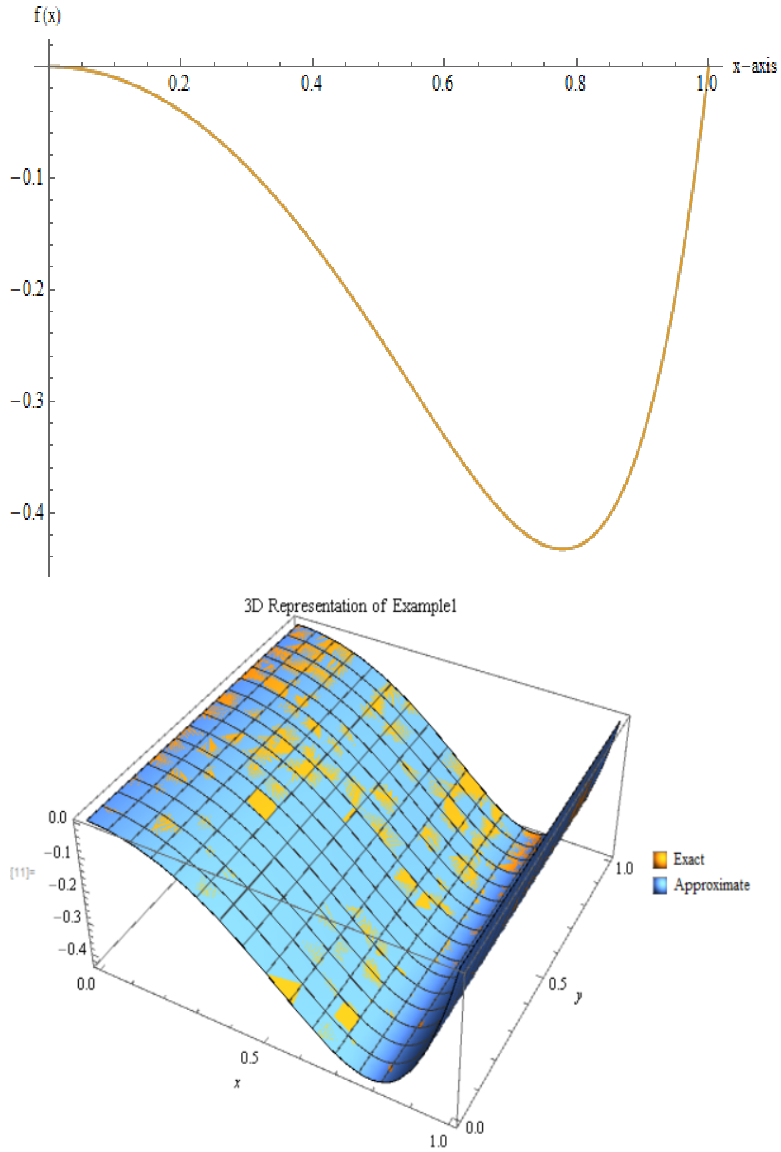

FIGURE 1. depicts error between exact and approximate solution for example 4.1

Table 2. Numerical Results for Example 4.2

x	Exact Solution	Approximate Solution	Absolute Error
0.0	0.0000000000	0.0000000000	0
0.1	0.0010000000	0.0009999991	9×10^{-10}
0.2	0.0080000000	0.0007999991	1×10^{-10}
0.3	0.0270000000	0.0269999999	1×10^{-10}
0.4	0.0640000000	0.0639999999	1×10^{-10}
0.5	0.1250000000	0.1249999999	2×10^{-10}
0.6	0.2160000000	0.2159999997	3×10^{-10}
0.7	0.3430000000	0.3429999995	5×10^{-10}
0.8	0.5120000000	0.5119999993	7×10^{-10}
0.9	0.7290000000	0.7289999992	8×10^{-10}
1.0	1.0000000000	0.9999999999	1×10^{-10}

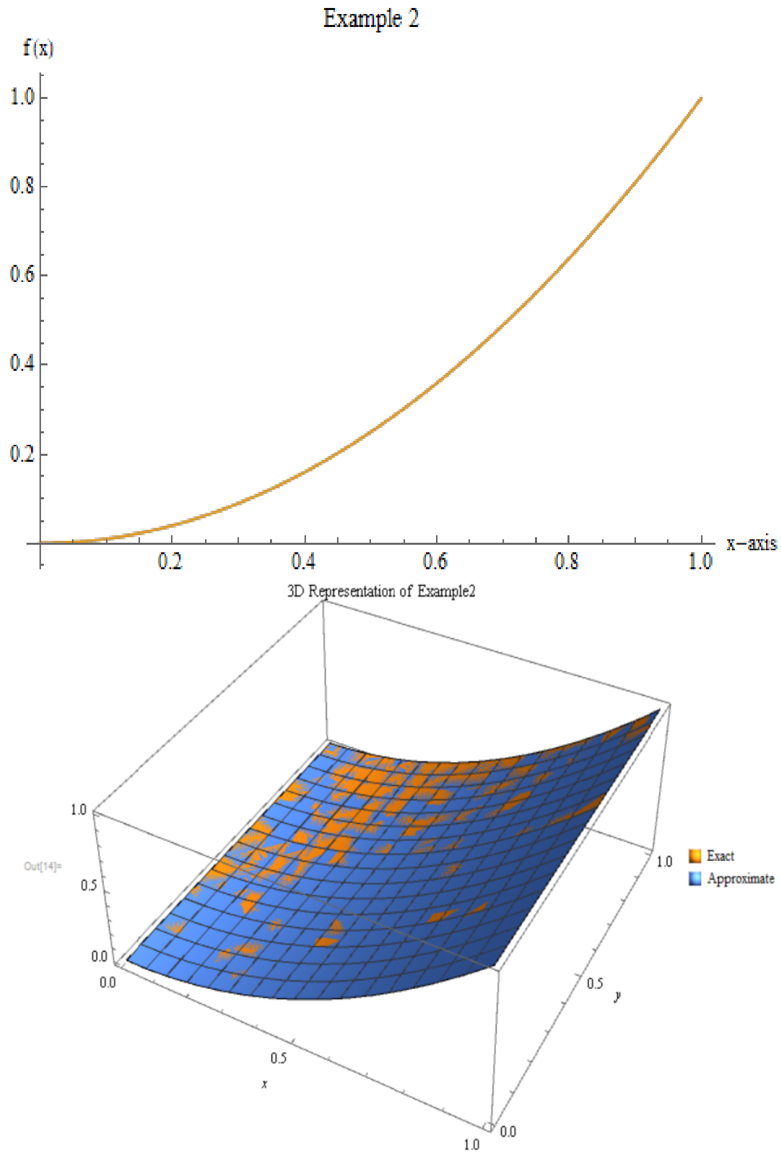

FIGURE 2. depicts error between exact and approximate solution for example 4.2

TABLE 3. Numerical Results for Example 4.3

x	**Exact Solution**	**Approximate Solution**	**Absolute Error**
0.0	0.0000000000	0.0000000000	0
0.1	0.0100000000	0.0100000000	0
0.2	0.0400000000	0.0400000000	0
0.3	0.0900000000	0.0900000000	0
0.4	0.1600000000	0.1600000000	0
0.5	0.2500000000	0.2500000000	0
0.6	0.3600000000	0.3600000000	0
0.7	0.4900000000	0.4900000000	0
0.8	0.6400000000	0.6400000000	0
0.9	0.8100000000	00.8100000000	0
1.0	1.0000000000	1.0000000001	1×10^{-10}

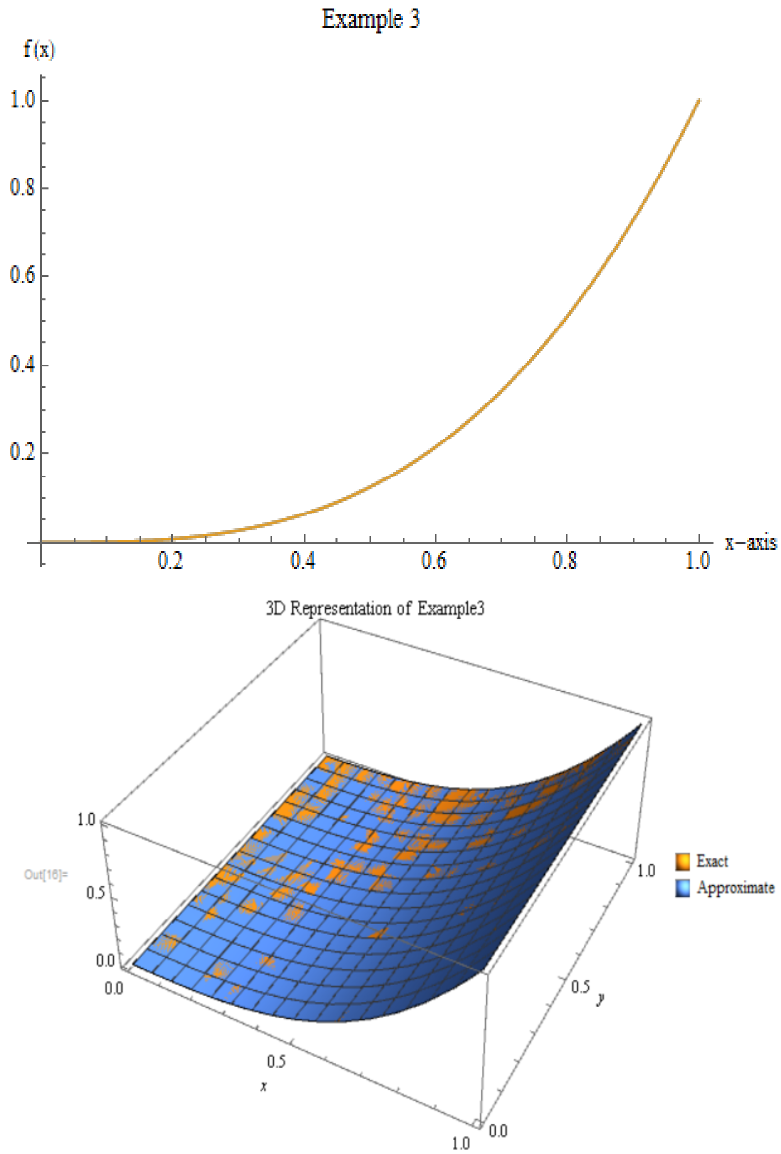

FIGURE 3. depicts error between exact and approximate solution for example 4.3

6. Conclusion

This study effectively applies the Galerkin method to address MFDEs, obtaining numerical solutions through the use of shifted Chebyshev polynomials. The integration of Chebyshev polynomials within the Galerkin method is a pivotal aspect of this approach. The methodology results in rapidly converging series solutions, particularly well-suited for addressing physical problems. The efficacy of this approach is evident from the results presented in Tables 1, 2, and 3, demonstrating its convergence when compared to exact solutions. Additionally, figures 1-3 illustrate an excellent agreement with the exact solution. Moreover, the numerical findings emphasize the robustness of the proposed technique as a potent mathematical tool for tackling the specific class of MFDEs.

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