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CENTRAL INDEX ORIENTED GROWTH ANALYSIS OF COMPOSITE ENTIRE FUNCTIONS FROM THE VIEW POINT OF (α, β, γ)-ORDER

TANMAY BISWAS, CHINMAY BISWAS, SUDIPTA KUMAR PAL

Abstract. Complex analysis is a very important branch of research in Pure Mathematics and so many works in different directions have been explored in this field. Moreover, in this regard, the study of growth analysis of composite entire functions in terms of maximum modulus or maximum terms is one of the most important part of research. Order and lower order are the classical growth indicators which are the main tools to determine the growth rate of composite entire functions. Continuing the research work and proceed ahead, the idea of order has been modified and extended to iterated p-order, (p, q)-th order, generalized order, (p, q)-φ order etc. by different researchers. Recently, Belaïdi et al. [1] have introduced the concept of (α, β, γ)-order of entire function which is considerably extended and generalized all the previous ideas of different growth indicators. However, study of growth properties of composite entire functions in terms of their central index is another active side of research and in this paper, we have discussed some central index based growth properties of composite entire functions on the basis of their (α, β, γ)-order and (α, β, γ)-lower order.

1. Introduction

Let $f = \sum_{n=0}^{+\infty} a_n z^n$ be an entire function defined on $\mathbb{C}$, the set of all finite complex numbers. The maximum modulus function $M_f(r)$ and the maximum term function $\mu_f(r)$ of f, are respectively defined as $M_f = \max_{|z|=r} |f(z)|$ and $\mu_f = \max_{n \geq 0} (|a_n| r^n)$. The central index $\nu_f(r)$ of an entire function f is the greatest exponent n for which $|a_n| r^n = \mu_f(r)$. Clearly, like $M_f(r)$ and $\mu_f(r)$, $\nu_f(r)$ is also real and increasing function of r. Though $\nu_f(r)$ is much weaker than $M_f(r)$ and $\mu_f(r)$ in some sense, from another angle of view $\frac{\nu_f(r)}{\nu_g(r)}$ is also called the growth of f with respect to g in terms of the central index. Order and lower order are classical growth indicators of entire and meromorphic functions in complex analysis. Several authors have made the close investigations on the growth properties of entire and meromorphic in different directions using the concepts of order, iterated

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p-order [9, 10], (p,q)-th order [7, 8], (p,q)-φ order [11] and achieved many valuable results. The standard notations and definitions of the theory of entire functions are available in [12, 13] and therefore we do not explain those in details. To start our paper, we just recall the following definition:

Definition 1.1. *The order ρ_f and the lower order λ_f of an entire function f are defined as:*

$$\rho_f = \limsup_{r\to+\infty} \frac{\log\log M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to+\infty} \frac{\log\log M_f(r)}{\log r}.$$

He and Xiao [5] gave the alternative definitions of order and lower order of entire function f in terms of its central index which are as follows:

$$\rho_f = \limsup_{r\to+\infty} \frac{\log \nu_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to+\infty} \frac{\log \nu_f(r)}{\log r}.$$

First of all, let L be a class of continuous non-negative on $(-\infty,+\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a)+\alpha(b)+c$ for all $a, b \geq R_0$ and fixed $c \in (0,+\infty)$. Further we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x+O(1)) = (1+o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_3$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a)+\alpha(b)$ for all $a, b \geq R_0$, i.e., α is subadditive. Clearly $L_3 \subset L_1$.

Particularly, when $\alpha \in L_3$, then one can easily verify that $\alpha(mr) \leq m\alpha(r)$, $m \geq 2$ is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0,+\infty)$ and satisfies $\alpha(0) \geq 0$, then for $t \in [0,1]$,

$$\begin{aligned} \alpha(tx) &= \alpha(tx + (1-t)\cdot 0) \\ &\geq t\alpha(x) + (1-t)\alpha(0) \geq t\alpha(x), \end{aligned}$$

so that by choosing $t = \frac{a}{a+b}$ or $t = \frac{b}{a+b}$,

$$\begin{aligned} \alpha(a+b) &= \frac{a}{a+b}\alpha(a+b) + \frac{b}{a+b}\alpha(a+b) \\ &\leq \alpha\left(\frac{a}{a+b}(a+b)\right) + \alpha\left(\frac{b}{a+b}(a+b)\right) \\ &= \alpha(a) + \alpha(b),\ a, b \geq 0. \end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r) \leq \alpha(r+R_0) \leq \alpha(r) + \alpha(R_0)$$

for any $R_0 \geq 0$. This yields that $\alpha(r) \sim \alpha(r+R_0)$ as $r \to +\infty$. Throughout the present paper we take $\alpha \in L_1$, $\beta \in L_2$, $\gamma \in L_3$.

Heittokangas et al. [6] introduced a new concept of φ-order of entire function considering φ as subadditive function. For details, one may see [6]. Recently, Belaïdi et al. [1] have extended this idea and have introduced the definitions of (α,β,γ)-order and (α,β,γ)-lower order of an entire function f in terms of maximum moduli in the following way:

Definition 1.2. [1] *The (α, β, γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$ and (α, β, γ)-lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$, of an entire function f, are defined as:*

$$\rho_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]}(M(r,f)))}{\beta\left(\log(\gamma(r))\right)}$$

$$\text{and } \lambda_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]}(M(r,f)))}{\beta\left(\log(\gamma(r))\right)}.$$

Further Belaïdi et al. [1] have also introduced the equivalent definitions of (α, β, γ)-order and (α, β, γ)-lower order of an entire function f in terms of its central index which are as follows:

Definition 1.3. [1] *The (α, β, γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$ and (α, β, γ)-lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$ of an entire function f are defined as:*

$$\rho_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log(\nu_f(r)))}{\beta\left(\log(\gamma(r))\right)}$$

$$\text{and } \lambda_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\log(\nu_f(r)))}{\beta\left(\log(\gamma(r))\right)}.$$

Remark 1. *Let $\alpha(r) = \beta(r) = r$, the Definition 1.3 coincides with the definition of order and lower order given by He et al. [5].*

In this paper, we have studied some growth properties relating to the composition of entire functions on the basis of (α, β, γ)-order and (α, β, γ)-lower order in terms of central index. In fact some works in this area have also been explored in [2, 3].

2. Main results

In this section, the main results of the paper are presented.

Theorem 2.1. *Let f and g are entire functions such that $0 < \lambda_{(\alpha,\beta,\gamma)}[f] \leq \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$ and $\lambda_{(\alpha,\beta,\gamma)}[f \circ g] = +\infty$. Then*

$$\lim_{r\to+\infty} \frac{\alpha(\log(\nu_{f\circ g}(r)))}{\alpha(\log(\nu_f(r)))} = +\infty.$$

Proof. If possible, let the conclusion of the theorem does not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of r tending to infinity

$$\alpha(\log(\nu_{f\circ g}(r))) \leq \Delta \cdot \alpha(\log(\nu_f(r))). \tag{1}$$

Again from the definition of $\rho_{(\alpha,\beta,\gamma)}[f]$, it follows for all sufficiently large values of r that

$$\alpha(\log(\nu_f(r))) \leq (\rho_{(\alpha,\beta,\gamma)}[f] + \epsilon)\beta(\log(\gamma(r))). \tag{2}$$

From (1) and (2), for a sequence of values of r tending to $+\infty$,we have

$$\alpha(\log(\nu_{f\circ g}(r))) \leq \Delta(\rho_{(\alpha,\beta,\gamma)}[f] + \epsilon)\beta(\log(\gamma(r))),$$

$$i.e., \ \frac{\alpha(\log(\nu_{f\circ g}(r)))}{\beta(\log(\gamma(r)))} \leq \Delta(\rho_{(\alpha,\beta,\gamma)}[f] + \epsilon),$$

$$i.e., \ \liminf_{r\to+\infty} \frac{\alpha(\log(\nu_{f\circ g}(r)))}{\beta(\log(\gamma(r)))} < +\infty,$$

$$i.e., \ \lambda_{(\alpha,\beta,\gamma)}[f \circ g] < +\infty.$$

This is a contradiction.

Thus the theorem follows. □

Remark 2. *If we take "*$0 < \lambda_{(\alpha,\beta,\gamma)}[g] \leq \rho_{(\alpha,\beta,\gamma)}[g] < +\infty$*" instead of "*$0 < \lambda_{(\alpha,\beta,\gamma)}[f] \leq \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$*" and other conditions remain same, the conclusion of Theorem 2.1 remains true with "*$\alpha(\log(\nu_g(r)))$*" in place of "*$\alpha(\log(\nu_f(r)))$*" in the denominator.*

Remark 3. *Theorem 2.1 and Remark 2 are also valid with "limit superior" instead of "limit" if "*$\lambda_{(\alpha,\beta,\gamma)}[f \circ g] = +\infty$*" is replaced by "*$\rho_{(\alpha,\beta,\gamma)}[f \circ g] = +\infty$*" and the other conditions remain the same.*

Theorem 2.2. *Let f and g are entire functions such that $0 < \lambda_{(\alpha,\beta,\gamma)}[f \circ g] \leq \rho_{(\alpha,\beta,\gamma)}[f \circ g] < +\infty$ and $0 < \lambda_{(\alpha,\beta,\gamma)}[f] \leq \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$. Then*

$$\frac{\lambda_{(\alpha,\beta,\gamma)}[f \circ g]}{\rho_{(\alpha,\beta,\gamma)}[f]} \leq \liminf_{r \to +\infty} \frac{\alpha\left(\log(\nu_{f \circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)}$$
$$\leq \min\left\{\frac{\lambda_{(\alpha,\beta,\gamma)}[f \circ g]}{\lambda_{(\alpha,\beta,\gamma)}[f]}, \frac{\rho_{(\alpha,\beta,\gamma)}[f \circ g]}{\rho_{(\alpha,\beta,\gamma)}[f]}\right\}$$
$$\leq \max\left\{\frac{\lambda_{(\alpha,\beta,\gamma)}[f \circ g]}{\lambda_{(\alpha,\beta,\gamma)}[f]}, \frac{\rho_{(\alpha,\beta,\gamma)}[f \circ g]}{\rho_{(\alpha,\beta,\gamma)}[f]}\right\}$$
$$\leq \limsup_{r \to +\infty} \frac{\alpha\left(\log(\nu_{f \circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)} \leq \frac{\rho_{(\alpha,\beta,\gamma)}[f \circ g]}{\lambda_{(\alpha,\beta,\gamma)}[f]}.$$

Proof. From the definitions of $\lambda_{(\alpha,\beta,\gamma)}[f \circ g]$, $\rho_{(\alpha,\beta,\gamma)}[f \circ g]$, $\lambda_{(\alpha,\beta,\gamma)}[f]$, $\rho_{(\alpha,\beta,\gamma)}[f]$ and we have for arbitrary positive ε and for all sufficiently large values of r such that

$$\alpha\left(\log(\nu_{f \circ g}(r))\right) \geqslant \left(\lambda_{(\alpha,\beta,\gamma)}[f \circ g] - \varepsilon\right) \beta(\log(\gamma(r))), \tag{3}$$
$$\alpha\left(\log(\nu_{f \circ g}(r))\right) \leq \left(\rho_{(\alpha,\beta,\gamma)}[f \circ g] + \varepsilon\right) \beta(\log(\gamma(r))), \tag{4}$$
$$\alpha\left(\log(\nu_f(r))\right) \geqslant \left(\lambda_{(\alpha,\beta,\gamma)}[f] - \varepsilon\right) \beta(\log(\gamma(r))) \tag{5}$$
$$\text{and } \alpha\left(\log(\nu_f(r))\right) \leq \left(\rho_{(\alpha,\beta,\gamma)}[f] + \varepsilon\right) \beta(\log(\gamma(r))). \tag{6}$$

Again for a sequence of values of r tending to infinity,

$$\alpha\left(\log(\nu_{f \circ g}(r))\right) \leq \left(\lambda_{(\alpha,\beta,\gamma)}[f \circ g] + \varepsilon\right) \beta(\log(\gamma(r))), \tag{7}$$
$$\alpha\left(\log(\nu_{f \circ g}(r))\right) \geqslant \left(\rho_{(\alpha,\beta,\gamma)}[f \circ g] - \varepsilon\right) \beta(\log(\gamma(r))), \tag{8}$$
$$\alpha\left(\log(\nu_f(r))\right) \leq \left(\lambda_{(\alpha,\beta,\gamma)}[f] + \varepsilon\right) \beta(\log(\gamma(r))) \tag{9}$$
$$\text{and } \alpha\left(\log(\nu_f(r))\right) \geqslant \left(\rho_{(\alpha,\beta,\gamma)}[f] - \varepsilon\right) \beta(\log(\gamma(r))). \tag{10}$$

Now from (3) and (6) it follows for all sufficiently large values of r that

$$\frac{\alpha\left(\log(\nu_{f \circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)} \geqslant \frac{\lambda_{(\alpha,\beta,\gamma)}[f \circ g] - \varepsilon}{\rho_{(\alpha,\beta,\gamma)}[f] + \varepsilon}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r \to +\infty} \frac{\alpha\left(\log(\nu_{f \circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)} \geqslant \frac{\lambda_{(\alpha,\beta,\gamma)}[f \circ g]}{\rho_{(\alpha,\beta,\gamma)}[f]}. \tag{11}$$

Combining (5) and (7) , we have for a sequence of values of r tending to infinity that

$$\frac{\alpha\left(\log(\nu_{f \circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)} \leq \frac{\lambda_{(\alpha,\beta,\gamma)}[f \circ g] + \varepsilon}{\lambda_{(\alpha,\beta,\gamma)}[f] - \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\liminf_{r\to+\infty}\frac{\alpha\left(\log(\nu_{f\circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)}\leq\frac{\lambda_{(\alpha,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha,\beta,\gamma)}[f]}.\tag{12}$$

Again from (3) and (9), for a sequence of values of r tending to infinity, we get

$$\frac{\alpha\left(\log(\nu_{f\circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)}\geq\frac{\lambda_{(\alpha,\beta,\gamma)}[f\circ g]-\varepsilon}{\lambda_{(\alpha,\beta,\gamma)}[f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\alpha\left(\log(\nu_{f\circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)}\geq\frac{\lambda_{(\alpha,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha,\beta,\gamma)}[f]}.\tag{13}$$

Now, it follows from (4) and (5) , for all sufficiently large values of r that

$$\frac{\alpha\left(\log(\nu_{f\circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)}\leq\frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]+\varepsilon}{\lambda_{(\alpha,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha\left(\log(\nu_{f\circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)}\leq\frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha,\beta,\gamma)}[f]}.\tag{14}$$

Now from (4) and (10) , it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha\left(\log(\nu_{f\circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)}\leq\frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]+\varepsilon}{\rho_{(\alpha,\beta,\gamma)}[f]-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha\left(\log(\nu_{f\circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)}\leq\frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha,\beta,\gamma)}[f]}.\tag{15}$$

So combining (6) and (8) , we get for a sequence of values of r tending to infinity that

$$\frac{\alpha\left(\log(\nu_{f\circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)}\geqslant\frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]-\varepsilon}{\rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\alpha\left(\log(\nu_{f\circ g}(r))\right)}{\alpha\left(\log(\nu_f(r))\right)}\geqslant\frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha,\beta,\gamma)}[f]}.\tag{16}$$

Thus the theorem follows from (11) , (12) , (13), (14) , (15) and (16) . □

Remark 4. *If we take "$0<\lambda_{(\alpha,\beta,\gamma)}[g]\leq\rho_{(\alpha,\beta,\gamma)}[g]<+\infty$" instead of "$0<\lambda_{(\alpha,\beta,\gamma)}[f]\leq\rho_{(\alpha,\beta,\gamma)}[f]<+\infty$" and other conditions remain same, the conclusion of Theorem 2.2 remains true with "$\lambda_{(\alpha,\beta,\gamma)}[g]$", "$\rho_{(\alpha,\beta,\gamma)}[g]$" and "$\alpha\left(\log(\nu_g(r))\right)$" in place of "$\lambda_{(\alpha,\beta,\gamma)}[f]$", "$\rho_{(\alpha,\beta,\gamma)}[f]$" and "$\alpha\left(\log(\nu_f(r))\right)$" respectively in the denominators.*

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T. Biswas
Rajbari, Rabindrapally, R. N. Tagore Road
P.O.-Krishnagar, P.S.-Katwali, Dist.-Nadia, PIN- 741101, West Bengal, India.
E-mail address: tanmaybiswas_math@rediffmail.com

C. Biswas
Department of Mathematics, Nabadwip Vidyasagar College
Nabadwip, Dist.- Nadia, PIN-741302, West Bengal, India.
E-mail address: chinmay.shib@gmail.com

S. K. Pal
Department of Mathematics, Jangipur College
P.O.-Jangipur, Dist.-Murshidabad, PIN-742213, West Bengal, India.
E-mail address: palsudipto2017@gmail.com