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Effect of Rotation in a Semiconductor Medium Under Photothermal Theory with Ramp Type Heating

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Abstract: The goal of this work is to study the rotation of two-dimensional deformation in a semi-infinite semiconducting medium by ramp-type heating. The interaction of thermal-elastic mechanical-plasma waves is utilized in the photo-thermoelasticity theory model. The governing equations are presented in 2D and are of the thermal ramp type. The normal mode and approximation eigenvalue approaches have been applied to solve the given problem. The different physical quantities, such as displacement components, stress components, and the temperature distribution have been presented graphically using MATLAB software. These results were compared with previous results in the same direction, and it was found that the treatment method for the aforementioned problem may form a basis for examining the effects of rotation, angular frequency, time, and ramp-type heating parameter on a thermally elastic body.

Keywords: Ramp-type heating, Photo-thermoelasticity, Rotation, Semiconductor, Normal mode, Eigenvalue approach.

1. Introduction

In recent times, extensive research has focused on materials science, exploring their physical properties and wave propagation. Semiconductors, crucial in modern sectors like electronics, electrical circuits, solar cells, and transistors, occupy a unique position. Unlike insulators (such as glass) or natural conductors (like aluminum), semiconductors exhibit intermediate behavior. Early studies considered materials like silicon and carbon as elastic substances, analyzed through thermoelastic theories. However, after a thorough examination of these materials, it became clear that their resistance to electrical conduction is affected by temperature changes, as the thermal effect on these materials causes the excitation of free electrons on the surface, resulting in wide transformations known as electronic deformation (ED). During the process of electronic distortion, what is known as the carrier density is produced, which causes plasma waves, and in this case, the photothermal theory (PT) can be applied. Mechanical loads are generated within the material during thermal excitation processes, causing so-called thermoelastic deformation (TE). As a result of all of this, the interaction between thermal elasticity theory and PT theory can be studied in what is known as the photothermal elasticity theory of semiconductors. A good number of researchers have shown their dedication to this direction. Hobiny et al. [1] studied the photo-thermo-elastic distributed waves in a semiconductor medium due to the ramp-type heating. In the above studies, the coupling between the electrons and holes free charges under the impact of magnetic field is neglected when thermo-diffusive processes occur in semiconductors. However, the distribution of electrons and holes on the free surface of semiconductors is important and cannot be ignored when semiconductors are studied [2–6].

Researchers have previously been interested in deformation in a material caused by ramp type heating. Authors have found solutions to a variety of ramp-type loading and heating problems. Generalized thermoelasticity was developed by Youssef et al. [7–11] for infinite materials heated by ramp type in many external fields, where the thermal conductivity is varied. The physical characteristics of semiconductor materials that are dependent on temperature fluctuations under the influence of Hall current have not been taken into account in previous studies due to the overlap between photothermal and thermoelasticity theories. One generalized theory of thermoelasticity is developed by Lord and Shulman (L–S) [12].

A second theory with two thermal relaxation time parameters is produced by Green and Lindsay (G–L) [13]. The discussion of such extended theories was then covered in a few pieces that published as Chandrasekharaiah [14] and Hetnarski and Ignaczak [15]. Yahya and Abd Alla have investigated the radial vibrations of a spinning elastic hollow cylinder using the elasticity theory [16]. Numerous applications of the thermoelectricity theory may be found in the fields of geophysics, building constructions, and the behavior of sensitive biological tissues. Rotation's impact on a generalized thermoelastic medium with hydrostatic starting stress under ramp-type heating and loading was examined by Ailawalia and Narah [17]. The effects of variable thermal conductivity and hyperbolic two-temperature theory during the magneto-photothermal theory of semiconductors induced by laser pulses and some related studies in this direction were discussed by other researchers [18–23].

In the present work, using the photothermal model, deformation in a two-dimensional, isotropic, rotating medium subjected to ramp-type heating is investigated. The medium is assumed to be semiconductor and exposed to ramp-type heating, rotation and photo-excited. The vibrations caused by photo-excitation during photothermal theory produce changes in wave propagation of physical variables. The normal mode and approximate eigenvalue techniques were used to obtain exact solutions for the displacement, thermal stress, carrier density (plasma), and temperature distributions. To determine the complete solutions of physical quantities subject to heating ramp type, the problem boundary conditions are used. Furthermore, the MATLAB software is employed in this case to represent temperature, carrier density, displacement distributions, and thermal stress for the model under discussion graphically. This model is very useful for scientists and engineers to develop high-quality semiconductor materials that many modern industries rely on and that have multiple uses in electrical circuits and solar cells (photovoltaics).

2. Basic equations

Suppose that the medium being studied rotates in a regular manner. The angular velocity in this case is $\vec{\Omega} = \Omega\,\vec{n}$, where n is a unit vector that describes the rotation axis's direction. The rotation axis is the axis perpendicular to the plane, and the whole body rotates with a uniform angular velocity $\vec{\Omega} = \Omega\,\vec{n}$. Consequently, there are two additional requirements for the elastodynamic equations: Centripetal acceleration $(\vec{\Omega}\wedge(\vec{\Omega}\wedge\vec{u}))$ caused by just time-varying motion, and Cariole's acceleration $(2\vec{\Omega}\wedge\vec{u})$ caused by a moving reference frame, where $\vec{u} = (u, v, 0)$ is the dynamic displacement vector. Song et al. [21] present the constitutive equations for coupled plasma, thermal, and elastic transport in a medium with isotropic and homogeneous characteristics.

$$\rho\frac{\partial^2 u(\vec{r},t)}{\partial t^2} + \rho(\vec{\Omega}\wedge(\vec{\Omega}\wedge\vec{u})) + 2\rho\,\vec{\Omega}\wedge\vec{u} = \mu\nabla^2\vec{u}(\vec{r},t) + (\lambda + \mu)\nabla\big(\nabla.\vec{u}(\vec{r},t)\big) - \gamma\nabla \mathrm{T}(\vec{r},t) - \delta_n\nabla \mathrm{N}(\vec{r},t), \quad (1)$$

$$\frac{\partial \mathrm{N}(\vec{r},t)}{\partial t} = D_e\nabla^2\mathrm{N}(\vec{r},\mathrm{t}) - \frac{1}{\tau}\mathrm{N}(\vec{r},\mathrm{t}) + \mathrm{k}\mathrm{T}(\vec{r},\mathrm{t}), \quad (2)$$

$$\rho c_e\frac{\partial \mathrm{T}(\vec{r},t)}{\partial t} = k\nabla^2\mathrm{T}(\vec{r},\mathrm{t}) - \frac{E_g}{\tau}\mathrm{N}(\vec{r},\mathrm{t}) + \gamma T_\circ\nabla.\frac{\partial u(\vec{r},t)}{\partial t}, \quad (3)$$

Furthermore, we investigate the plane strain problem with all field variables dependent on (x, y, t). For 2-D, we use the displacement vector $\vec{u} = (u, v, 0)$ and $u = u(x, y, t)$, $v = v(x, y, t)$. Equations (1)-(3) are reduced to

$$\rho\left(\frac{\partial^2 u}{\partial t^2} - \Omega^2 u + 2\Omega\frac{\partial v}{\partial t}\right) = (\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + (\lambda + \mu)\frac{\partial^2 v}{\partial x\partial y} + \mu\frac{\partial^2 u}{\partial y^2} - \gamma\frac{\partial T}{\partial x} - \delta_n\frac{\partial N}{\partial x}, \quad (4)$$

$$\rho\left(\frac{\partial^2 v}{\partial t^2} - \Omega^2 v - 2\Omega\frac{\partial u}{\partial t}\right) = (\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u}{\partial x\partial y} + \mu\frac{\partial^2 v}{\partial x^2} - \gamma\frac{\partial T}{\partial y} - \delta_n\frac{\partial N}{\partial y}, \quad (5)$$

$$\frac{\partial \mathrm{N}}{\partial t} = D_e\nabla^2\mathrm{N} - \frac{1}{\tau}\,\mathrm{N} + \mathrm{k}\,\mathrm{T}, \quad (6)$$

$$\rho c_e\frac{\partial T}{\partial t} = K\nabla^2\mathrm{T} - \frac{E_g}{\tau}\mathrm{N} + \gamma T_\circ\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \quad (7)$$

The stress components are:

$$\sigma_{xx} = (\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} - (3\lambda + 2\mu)(\alpha_T T + d_n N), \quad (8)$$

$$\sigma_{yy} = (\lambda + 2\mu)\frac{\partial v}{\partial y} + \lambda\frac{\partial u}{\partial x} - (3\lambda + 2\mu)(\alpha_T T + d_n N), \quad (9)$$

$$\sigma_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad (10)$$

$$e = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \quad (11)$$

where λ and μ are Lame's constant, ρ is the density, σ_{ij} are the stress components, T is the temperature, t is the time, N is the carrier density, D_e is the carrier diffusion coefficient, E_g is the energy gap of the semiconductor, τ is the photo-generated carrier lifetime, c_e is the specific heat, u, v are the displacement components, $T_\circ$ is the medium's temperature, k is the thermal conductivity, α_T is the linear thermal expansion factor and $\gamma = (3\lambda + 2\mu)\alpha_T$, δ_n is the variation in the deformation potential between the valence and conduction bands, and $\delta_n = (3\lambda + 2\mu)d_n$.

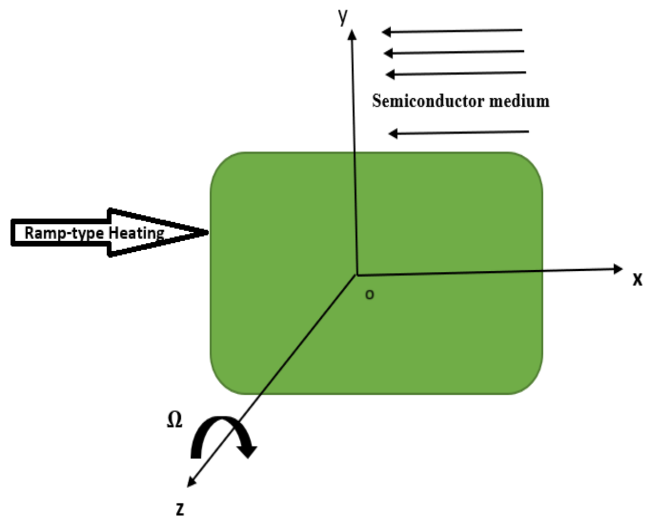


Fig.1. Schematic of the problem.

Now, the non-dimensional quantities are defined as follows:

$$(x', y') = \frac{1}{c_T t^*}(x, y),\ t' = \frac{t}{t^*},\ (u', v') = \frac{1}{c_T t^*}(u, v),\ N' = \frac{\delta_n N}{(\lambda + 2\mu)},\ \Omega' = t^*\Omega,\ c_T^2 = \frac{(\lambda + 2\mu)}{\rho},\ T' = \frac{\gamma T}{(\lambda + 2\mu)},\ \sigma'_{ij} = \frac{\sigma_{ij}}{\mu},\ t^* = \frac{K}{\rho c_e c_T^2}. \quad (12)$$

Applying the quantities in Equation (12) to Equations (4-10) yields

$$\left(\frac{\partial^2 u}{\partial t^2} - \Omega^2 u + 2\Omega\frac{\partial v}{\partial t}\right) = \frac{\partial^2 u}{\partial x^2} + a_{11}\frac{\partial^2 v}{\partial x\partial y} + a_{12}\frac{\partial^2 u}{\partial y^2} - \frac{\partial T}{\partial x} - \frac{\partial N}{\partial x}, \quad (13)$$

$$\left(\frac{\partial^2 v}{\partial t^2} - \Omega^2 v - 2\Omega\frac{\partial u}{\partial t}\right) = \frac{\partial^2 v}{\partial y^2} + a_{11}\frac{\partial^2 u}{\partial x\partial y} + a_{12}\frac{\partial^2 v}{\partial x^2} - \frac{\partial T}{\partial y} - \frac{\partial N}{\partial y}, \quad (14)$$

$$\nabla^2\mathrm{N} - a_{21}\,\mathrm{N} + a_{22}\mathrm{T} - a_{23}\frac{\partial \mathrm{N}}{\partial t} = 0, \quad (15)$$

$$\nabla^2\mathrm{T} - a_{31}\,\mathrm{N} + a_{32}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial T}{\partial t} = 0, \quad (16)$$

$$\sigma_{xx} = a_{41}\frac{\partial u}{\partial x} + a_{42}\frac{\partial v}{\partial y} - a_{41}(T + N), \quad (17)$$

$$\sigma_{yy} = a_{41}\frac{\partial v}{\partial y} + a_{42}\frac{\partial u}{\partial x} - a_{41}(T + N), \quad (18)$$

$$\sigma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \quad (19)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, a_{11} = \frac{\lambda+\mu}{\lambda+2\mu}, \quad a_{12} = \frac{\mu}{\lambda+2\mu}, \quad a_{21} = \frac{c_T^2 t^{*2}}{\tau D_e}, \quad a_{22} = \frac{k c_T^2 t^{*2}\delta_n}{\gamma D_e}, a_{31} = \frac{E_g \gamma c_T^2 t^*}{k\tau\delta_n}, a_{32} = \frac{\gamma^2 T_\circ c_T t^*}{k(\lambda+2\mu)}.$$

To keep things simple, the prime was also removed. Differentiate with respect to x Eqs. (13-19) can be written by using Eq. (11) as follow:

$$\left(\frac{\partial^3 u}{\partial x \partial t^2} - \Omega^2 \frac{\partial u}{\partial x} + 2\Omega \frac{\partial^2 v}{\partial x \partial t}\right) = a\frac{\partial^3 u}{\partial x^3} + a_{11}\frac{\partial^2 e}{\partial x^2} + a_{12}\nabla^2 \frac{\partial u}{\partial x} - \frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 N}{\partial x^2}, \tag{20}$$

$$\left(\frac{\partial^3 v}{\partial y \partial t^2} - \Omega^2 \frac{\partial v}{\partial y} - 2\Omega \frac{\partial^2 u}{\partial y \partial t}\right) = a\frac{\partial^3 v}{\partial y^3} + a_{11}\frac{\partial^2 e}{\partial y^2} + a_{12}\nabla^2 \frac{\partial v}{\partial y} - \frac{\partial^2 T}{\partial y^2} - \frac{\partial^2 N}{\partial y^2}, \tag{21}$$

$$\nabla^2 \mathrm{N} - a_{21}\,\mathrm{N} + a_{22}\mathrm{T} - a_{23}\frac{\partial \mathrm{N}}{\partial t} = 0, \tag{22}$$

$$\nabla^2 \mathrm{T} - a_{31}\,\mathrm{N} + a_{32}\frac{\partial e}{\partial t} - \frac{\partial T}{\partial t} = 0, \tag{23}$$

$$\sigma_{xx} = a_{41}\frac{\partial u}{\partial x} + a_{42}\frac{\partial v}{\partial y} - a_{41}(T+N), \tag{24}$$

$$\sigma_{yy} = a_{41}\frac{\partial u}{\partial x} + a_{42}\frac{\partial v}{\partial y} - a_{41}(T+N), \tag{25}$$

$$\sigma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{26}$$

3. Solution of the problem

In this part, the normal mode approach is used, which has the benefit of obtaining accurate solutions without any assumed limits on the field variables. The physical variable solutions can be analyzed in terms of normal modes as follows:

$$\left(u, v, T, N, \sigma_{ij}\right)(x, y, t) = \left(u^*, v^*, T^*, N^*, {\sigma_{ij}}^*\right)e^{(\omega t + iby)}. \tag{27}$$

where $\omega, i,$ *and* b refer to the angular frequency, the imaginary number and the wave number in the y - direction. Applying Eq. (27) in Eqs. (20-23), we have

$$\frac{d^2u^*}{dx^2} = \eta_1 u^* + \eta_2 v^* + \eta_3\frac{dv^*}{dx} + \eta_4\frac{dT^*}{dx} + \eta_4\frac{dN^*}{dx}, \tag{28}$$

$$\frac{d^2v^*}{dx^2} = \eta_5 u^* + \eta_6 v^* + \eta_7 T^* + \eta_7 N^* + \eta_8\frac{du^*}{dx}, \tag{29}$$

$$\frac{d^2N^*}{dx^2} = \beta N^* - a_{22}T^*, \tag{30}$$

$$\frac{d^2T^*}{dx^2} = \eta_9 v^* + \eta_{10}T^* + \eta_{11}\frac{du^*}{dx} + \eta_{12}N^*. \tag{31}$$

where

$$\eta_1 = \frac{\omega^2 + \mathrm{a}_{12}\mathrm{b}^2 - \Omega^2}{\alpha}, \eta_2 = \frac{2\omega\Omega}{\alpha}, \eta_3 = \frac{-a_{11}bi}{\alpha}, \eta_4 = \frac{1}{\alpha}, \eta_5 = \frac{-2\omega\Omega}{a_{12}}, \eta_6 = \frac{ab^2 + a_{11}b^2 + a_{12}b^2 + \omega^2 bi - \Omega^2}{a_{12}}, \ \eta_7 = \frac{ib}{a_{12}}, \ \eta_8 = \frac{-iba_{11}}{a_{12}}, \eta_9 = -a_{32}\omega bi, \ \eta_{10} = b^2 + \omega, \ \eta_{11} = -a_{32}\omega, \eta_{12} = a_{31}, a = 1 - a_{11} - a_{12}, \alpha = a + a_{11} + a_{12}, \ \beta = a_{21} + a_{23}\omega + b^2.$$

Equations (28-31), can be expressed as a differential equation with a vector-matrix as below:

$$\frac{d\tilde{L}}{dx} = A\tilde{L}. \tag{32}$$

where

$$\tilde{L} = \begin{pmatrix} u^* \\ v^* \\ T^* \\ N^* \\ \frac{du^*}{dx} \\ \frac{dv^*}{dx} \\ \frac{dT^*}{dx} \\ \frac{dN^*}{dx} \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \eta_1 & \eta_2 & 0 & 0 & 0 & \eta_3 & \eta_4 & \eta_4 \\ \eta_5 & \eta_6 & \eta_7 & \eta_7 & \eta_8 & 0 & 0 & 0 \\ 0 & 0 & -a_{22} & \beta & 0 & 0 & 0 & 0 \\ 0 & \eta_9 & \eta_{10} & \eta_{12} & \eta_{11} & 0 & 0 & 0 \end{pmatrix} \tag{33}$$

We now employ the eigenvalue approach, as in Das and Bhakta31 to solve Eq. (32), the characteristic equation of matrix A, which takes the form

$$\lambda^8 + m_1\lambda^6 + m_2\lambda^5 + m_3\lambda^4 + m_4\lambda^3 + m_5\lambda^2 + m_6\lambda + m_7 = 0. \tag{34}$$

where $m_i, i = 1,2,3,4,5,6,7$ are defined in Appendix I. The roots of Eq. (34) are as follows:

$$\lambda_i = \pm\lambda_1, \pm\lambda_2, \pm\lambda_3, \pm\lambda_4, i = 1,2,3,4.$$

The appropriate eigenvector $\chi = [\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8]^T$, which corresponds to eigenvalue, may be defined as

$$\chi = \begin{pmatrix} \frac{\beta(\eta_6-\lambda^2)[\beta\lambda^2\eta_7 + (a_{22}+\lambda^2)\lambda^2\eta_7 + \beta\eta_{10}(\eta_5-\lambda^2) + (\eta_{12}-\lambda^2)(\eta_5-\lambda^2)(a_{22}+\lambda^2)]}{\beta\lambda(\eta_5-\lambda^2)} - \epsilon \\ \frac{-[\beta\lambda^2\eta_7 + (a_{22}+\lambda^2)\lambda^2\eta_7 + \beta\eta_{10}(\eta_5-\lambda^2)]}{\lambda} \\ \frac{\beta\lambda^2(\eta_6-\lambda^2) + \beta\eta_9(\eta_5-\lambda^2)}{\lambda} \\ \frac{(a_{22}+\lambda^2)[\beta\lambda^2(\eta_6-\lambda^2) + \beta\eta_9(\eta_5-\lambda^2)]}{\beta\lambda} \\ \frac{\beta(\eta_6-\lambda^2)[\beta\lambda^2\eta_7 + (a_{22}+\lambda^2)\lambda^2\eta_7 + \beta\eta_{10}(\eta_5-\lambda^2) + (\eta_{12}-\lambda^2)(\eta_5-\lambda^2)(a_{22}+\lambda^2)]}{\beta(\eta_5-\lambda^2)} - \epsilon \\ -[\beta\lambda^2\eta_7 + (a_{22}+\lambda^2)\lambda^2\eta_7 + \beta\eta_{10}(\eta_5-\lambda^2)] \\ \beta\lambda^2(\eta_6-\lambda^2) + \beta\eta_9(\eta_5-\lambda^2) \\ \frac{(a_{22}+\lambda^2)}{\beta}[\beta\lambda^2(\eta_6-\lambda^2) + \beta\eta_9(\eta_5-\lambda^2)] \end{pmatrix} \tag{35}$$

where,

$$\epsilon = \frac{[\beta\eta_7 + (a_{22}+\lambda^2)\eta_7][\beta\lambda^2(\eta_6-\lambda^2) + \beta\eta_9(\eta_5-\lambda^2)]}{\beta(\eta_5-\lambda^2)}.$$

The eigenvector v that corresponds to the eigenvalue from Eq. (35) is easily determined. The following notations are used in the remainder of the work:

$$\chi_1 = [\chi]_{\lambda=\lambda_1}, \ \chi_2 = [\chi]_{\lambda=\lambda_2}, \ \chi_3 = [\chi]_{\lambda=\lambda_3}, \ \chi_4 = [\chi]_{\lambda=\lambda_4}, \ \chi_5 = [\chi]_{\lambda=\lambda_5}, \ \chi_6 = [\chi]_{\lambda=\lambda_6}, \chi_7 = [\chi]_{\lambda=\lambda_7}, \chi_8 = [\chi]_{\lambda=\lambda_8}.$$

Taking the regularity criteria at infinity into account, the solution to equation (33) is as follows:

$$V = A_1\chi_1 e^{-\lambda_1 x} + A_2\chi_2 e^{-\lambda_2 x} + A_3\chi_3 e^{-\lambda_3 x} + A_4\chi_4 e^{-\lambda_4 x}, (x \geq 0) \tag{36}$$

where A_1, A_2, A_3, A_4 are constants determined by the problem's boundary conditions. From Eqs. (33), (34) and (36), we have

$$u^*(x) = A_1\chi_{11}e^{-\lambda_1 x} + A_2\chi_{12}e^{-\lambda_2 x} + A_3\chi_{13}e^{-\lambda_3 x} + A_4\chi_{14}e^{-\lambda_4 x},$$

$$v^*(x) = A_1\chi_{21}e^{-\lambda_1 x} + A_2\chi_{22}e^{-\lambda_2 x} + A_3\chi_{23}e^{-\lambda_3 x} + A_4\chi_{24}e^{-\lambda_4 x},$$

$$T^*(x) = A_1\chi_{31}e^{-\lambda_1 x} + A_2\chi_{32}e^{-\lambda_2 x} + A_3\chi_{33}e^{-\lambda_3 x} + A_4\chi_{34}e^{-\lambda_4 x},$$

$N^*(x) = A_1\chi_{41}e^{-\lambda_1 x} + A_2\chi_{42}e^{-\lambda_2 x} + A_3\chi_{43}e^{-\lambda_3 x} + A_4\chi_{44}e^{-\lambda_4 x}$, $\sigma_{xx}{}^*(x) = -a_{41}\big(A_1\lambda_1\chi_{11}e^{-\lambda_1 x} + A_2\lambda_2\chi_{12}e^{-\lambda_2 x} + A_3\lambda_3\chi_{13}e^{-\lambda_3 x} + A_4\lambda_4\chi_{14}e^{-\lambda_4 x}\big) + iba_{42}\big(A_1\chi_{21}e^{-\lambda_1 x} + A_2\chi_{22}e^{-\lambda_2 x} + A_3\chi_{23}e^{-\lambda_3 x} + A_4\chi_{24}e^{-\lambda_4 x}\big) - a_{41}\big(A_1\chi_{31}e^{-\lambda_1 x} + A_2\chi_{32}e^{-\lambda_2 x} + A_3\chi_{33}e^{-\lambda_3 x} + A_4\chi_{34}e^{-\lambda_4 x}\big) - a_{41}(A_1\chi_{41}e^{-\lambda_1 x} + A_2\chi_{42}e^{-\lambda_2 x} + A_3\chi_{43}e^{-\lambda_3 x} + A_4\chi_{44}e^{-\lambda_4 x})$,

$\sigma_{yy}{}^*(x) = -a_{41}\big(A_1\lambda_1\chi_{21}e^{-\lambda_1 x} + A_2\lambda_2\chi_{22}e^{-\lambda_2 x} + A_3\lambda_3\chi_{23}e^{-\lambda_3 x} + A_4\lambda_4\chi_{24}e^{-\lambda_4 x}\big) + iba_{42}\big(A_1\chi_{11}e^{-\lambda_1 x} + A_2\chi_{12}e^{-\lambda_2 x} + A_3\chi_{13}e^{-\lambda_3 x} + A_4\chi_{14}e^{-\lambda_4 x}\big) - a_{41}\big(A_1\chi_{31}e^{-\lambda_1 x} + A_2\chi_{32}e^{-\lambda_2 x} + A_3\chi_{33}e^{-\lambda_3 x} + A_4\chi_{34}e^{-\lambda_4 x}\big) - a_{41}(A_1\chi_{41}e^{-\lambda_1 x} + A_2\chi_{42}e^{-\lambda_2 x} + A_3\chi_{43}e^{-\lambda_3 x} + A_4\chi_{44}e^{-\lambda_4 x})$,

$\sigma_{xy}{}^*(x) = ib\big(A_1\chi_{11}e^{-\lambda_1 x} + A_2\chi_{12}e^{-\lambda_2 x} + A_3\chi_{13}e^{-\lambda_3 x} + A_4\chi_{14}e^{-\lambda_4 x}\big) - \big(A_1\lambda_1\chi_{21}e^{-\lambda_1 x} + A_2\lambda_2\chi_{22}e^{-\lambda_2 x} + A_3\lambda_3\chi_{23}e^{-\lambda_3 x} + A_4\lambda_4\chi_{24}e^{-\lambda_4 x}\big)$. (37)

where χ_{ij} , $i = 1,2,3,4$ are defined in Appendix II.

4. Boundary conditions

In this part, we will apply these boundary conditions to our problem. We assume that the free surface in our suggested model is traction-free.

(1) The mechanical normal stress that is traction free can be rewritten in the following form: $\sigma_{xx} = 0$ at $x = 0$,

(2) Also, the mechanical shear stress takes the form $\sigma_{xy} = 0$ at $x = 0$,

(3) The thermal boundary condition due to ramp-type heating at the free surface is as follows $T = T_1 \frac{t}{t_o}$ at $x = 0$, where T_1 is an arbitrary constant. (38)

(4) The plasma condition (carrier density) can be expressed as during the photo-thermal $\frac{\partial N}{\partial x} = \frac{c}{D_e}N$ at $x = 0$.

Combining equations (37) and (38) yields four equations for the constants A_1, A_2, A_3 and A_4.

$t_1A_1 + t_2A_2 + t_3A_3 + t_4A_4 = 0$,

$t_5A_1 + t_6A_2 + t_7A_3 + t_8A_4 = 0$,

$\chi_{31}A_1 + \chi_{32}A_2 + \chi_{33}A_3 + \chi_{34}A_4 = t_9$,

$t_{10}A_1 + t_{11}A_2 + t_{12}A_3 + t_{13}A_4 = 0$. (39)

where,

$t_1 = -a_{41}\lambda_1\chi_{11} + iba_{42}\chi_{21} - a_{41}\chi_{31} - a_{41}\chi_{41}$, $t_2 = -a_{41}\lambda_2\chi_{12} + iba_{42}\chi_{22} - a_{41}\chi_{32} - a_{41}\chi_{42}$, $t_3 = -a_{41}\lambda_3\chi_{13} + iba_{42}\chi_{23} - a_{41}\chi_{33} - a_{41}\chi_{43}$, $t_4 = -a_{41}\lambda_4\chi_{14} + iba_{42}\chi_{24} - a_{41}\chi_{34} - a_{41}\chi_{44}$,

$t_5 = ib\chi_{11} - \lambda_1\chi_{21}$, $t_6 = ib\chi_{12} - \lambda_2\chi_{22}$, $t_7 = ib\chi_{13} - \lambda_3\chi_{23}$, $t_8 = ib\chi_{14} - \lambda_4\chi_{24}$, $t_9 = T_1\frac{t^*t\gamma}{t_o(\lambda+2\mu)}e^{-(\omega t+iby)}$, $t_{10} = -\lambda_1\chi_{41} - \frac{c}{D_e}\chi_{41}$, $t_{11} = -\lambda_2\chi_{42} - \frac{c}{D_e}\chi_{42}$, $t_{12} = -\lambda_3\chi_{43} - \frac{c}{D_e}\chi_{43}$, $t_{13} = -\lambda_4\chi_{44} - \frac{c}{D_e}\chi_{44}$.

To calculate the constants A_1, A_2, A_3 and A_4, Cramer's method is applied as there is one non-homogeneous equation in Eq. (39).

$$A_1 = \frac{\Delta A_1}{\Delta}, A_2 = \frac{\Delta A_2}{\Delta}, A_3 = \frac{\Delta A_3}{\Delta}, A_4 = \frac{\Delta A_4}{\Delta}. \quad (40)$$

Where

$\Delta = -t_{13}t_3t_6\chi_{31} + t_{12}t_4t_6\chi_{31} + t_{13}t_2t_7\chi_{31} - t_{11}t_4t_7\chi_{31} - t_{12}t_2t_8\chi_{31} + t_{11}t_3t_8\chi_{31} + t_{13}t_3t_5\chi_{32} - t_{12}t_4t_5\chi_{32} - t_1t_{13}t_7\chi_{32} + t_{10}t_4t_7\chi_{32} + t_1t_{12}t_8\chi_{32} - t_{10}t_3t_8\chi_{32} - t_{13}t_2t_5\chi_{33} + t_{11}t_4t_5\chi_{33} + t_1t_{13}t_6\chi_{33} - t_{10}t_4t_6\chi_{33} - t_1t_{11}t_8\chi_{33} + t_{10}t_2t_8\chi_{33} + t_{12}t_2t_5\chi_{34} - t_{11}t_3t_5\chi_{34} - t_1t_{12}t_6\chi_{34} + t_{10}t_3t_6\chi_{34} + t_1t_{11}t_7\chi_{34} - t_{10}t_2t_7\chi_{34}$,

$\Delta A_1 = -t_{13}t_3t_6t_9 + t_{12}t_4t_6t_9 + t_{13}t_2t_7t_9 - t_{11}t_4t_7t_9 - t_{12}t_2t_8t_9 + t_{11}t_3t_8t_9$,

$\Delta A_2 = t_{13}t_3t_5t_9 - t_{12}t_4t_5t_9 - t_1t1_3t_7t_9 + t_{10}t_4t_7t_9 + t_1t_{12}t_8t_9 - t_{10}t_3t_8t_9$,

$\Delta A_3 = -t_{13}t_2t_5t_9 + t_{11}t_4t_5t_9 + t_1t_{13}t_6t_9 - t_{10}t_4t_6t_9 - t_1t_{11}t_8t_9 + t_{10}t_2t_8t_9$,

$\Delta A_4 = t_{12}t_2t_5t_9 - t_{11}t_3t_5t_9 - t_1t_{12}t_6t_9 + t_{10}t_3t_6t_9 + t_1t_{11}t_7t_9 - t_{10}t_2t_7t_9$.

Using Eqs. (27) and (37), one may calculate the dimensionless temperature T, carrier density N, displacements u and v, and stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$.

$u(x,y,t) = \big[A_1\chi_{11}e^{-\lambda_1 x} + A_2\chi_{12}e^{-\lambda_2 x} + A_3\chi_{13}e^{-\lambda_3 x} + A_4\chi_{14}e^{-\lambda_4 x}\big]e^{(\omega t+iby)}$,

$v(x,y,t) = \big[A_1\chi_{21}e^{-\lambda_1 x} + A_2\chi_{22}e^{-\lambda_2 x} + A_3\chi_{23}e^{-\lambda_3 x} + A_4\chi_{24}e^{-\lambda_4 x}\big]e^{(\omega t+iby)}$,

$T(x,y,t) = \big[A_1\chi_{31}e^{-\lambda_1 x} + A_2\chi_{32}e^{-\lambda_2 x} + A_3\chi_{33}e^{-\lambda_3 x} + A_4\chi_{34}e^{-\lambda_4 x}\big]e^{(\omega t+iby)}$, (41)

$N(x,y,t) = \big[A_1\chi_{41}e^{-\lambda_1 x} + A_2\chi_{42}e^{-\lambda_2 x} + A_3\chi_{43}e^{-\lambda_3 x} + A_4\chi_{44}e^{-\lambda_4 x}\big]e^{(\omega t+iby)}$,

$\sigma_{xx}(x,y,t) = \big[-a_{41}\big(A_1\lambda_1\chi_{11}e^{-\lambda_1 x} + A_2\lambda_2\chi_{12}e^{-\lambda_2 x} + A_3\lambda_3\chi_{13}e^{-\lambda_3 x} + A_4\lambda_4\chi_{14}e^{-\lambda_4 x}\big) + iba_{42}\big(A_1\chi_{21}e^{-\lambda_1 x} + A_2\chi_{22}e^{-\lambda_2 x} + A_3\chi_{23}e^{-\lambda_3 x} + A_4\chi_{24}e^{-\lambda_4 x}\big) - a_{41}\big(A_1\chi_{31}e^{-\lambda_1 x} + A_2\chi_{32}e^{-\lambda_2 x} + A_3\chi_{33}e^{-\lambda_3 x} + A_4\chi_{34}e^{-\lambda_4 x}\big) - a_{41}(A_1\chi_{41}e^{-\lambda_1 x} + A_2\chi_{42}e^{-\lambda_2 x} + A_3\chi_{43}e^{-\lambda_3 x} + A_4\chi_{44}e^{-\lambda_4 x})\big]e^{(\omega t+iby)}$,

$\sigma_{yy}(x,y,t) = \big[-a_{41}\big(A_1\lambda_1\chi_{21}e^{-\lambda_1 x} + A_2\lambda_2\chi_{22}e^{-\lambda_2 x} + A_3\lambda_3\chi_{23}e^{-\lambda_3 x} + A_4\lambda_4\chi_{24}e^{-\lambda_4 x}\big) + iba_{42}\big(A_1\chi_{11}e^{-\lambda_1 x} + A_2\chi_{12}e^{-\lambda_2 x} + A_3\chi_{13}e^{-\lambda_3 x} + A_4\chi_{14}e^{-\lambda_4 x}\big) - a_{41}\big(A_1\chi_{31}e^{-\lambda_1 x} +$

$A_2\chi_{32}e^{-\lambda_2 x} + A_3\chi_{33}e^{-\lambda_3 x} + A_4\chi_{34}e^{-\lambda_4 x}) - a_{41}(A_1\chi_{41}e^{-\lambda_1 x} + A_2\chi_{42}e^{-\lambda_2 x} + A_3\chi_{43}e^{-\lambda_3 x} + A_4\chi_{44}e^{-\lambda_4 x})]e^{(\omega t+iby)}$,

$\sigma_{xy}(x,y,t) = [ib(A_1\chi_{11}e^{-\lambda_1 x} + A_2\chi_{12}e^{-\lambda_2 x} + A_3\chi_{13}e^{-\lambda_3 x} + A_4\chi_{14}e^{-\lambda_4 x}) - (A_1\lambda_1\chi_{21}e^{-\lambda_1 x} + A_2\lambda_2\chi_{22}e^{-\lambda_2 x} + A_3\lambda_3\chi_{23}e^{-\lambda_3 x} + A_4\lambda_4\chi_{24}e^{-\lambda_4 x})]e^{(\omega t+iby)}$.

5. Results and Discussion:

In the frame of this model, we will introduce some numerical data in an effort to demonstrate the analytical technique that was previously described. The results indicate changes in the distribution of temperature, carrier density, stress, and displacement components using MATLAB programming software. The silicon (Si) element (example of semiconductor material) is used for this problem's numerical simulation [24-27].

$\rho = 2330$ kg/m^3, $\lambda = 3.64 \times 10^{10}$N/$m^2$, $\mu = 5.46 \times 10^{10}$N/$m^2$,

$T_\circ = 300$ K , $K = 150$ W $m^{-1}k^{-1}$, $c_e = 695\ m^2$/K,

$E_g = 1.11\text{eV}, D_e = 2.5x10^{-3}m^2s^{-1}, \alpha_t = 4.14 \times 10^{-6}k^{-1}$,

$c = 2ms^{-1}, t = 0.1s, d_n = 9x10^{-31}, \tau = 5x10^{-5}$s.

Figures 2–5 provide numerical and graphical computations of the temperature, carrier density, thermal stress, and displacement components with respect to distance.

5.1. The effect of rotation parameter Ω.

Fig.2 illustrates how the temperature T, carrier density N, thermal stress σ_{xx}, σ_{xy}, and displacement u, change in relation to axial x over a range of rotation Ω values. It is observed that as the axial x at $\Omega = 0{,}0.1{,}0.2{,}0.3$ increases, all the quantities T, N, σ_{xx}, u and σ_{xy} decrease. When we take the special case of rotation i.e. ($\Omega = 0$), we can reach to ramp-type heating in a semiconductor medium under photothermal theory [21].

5.2. The effect of angular frequency parameter ω.

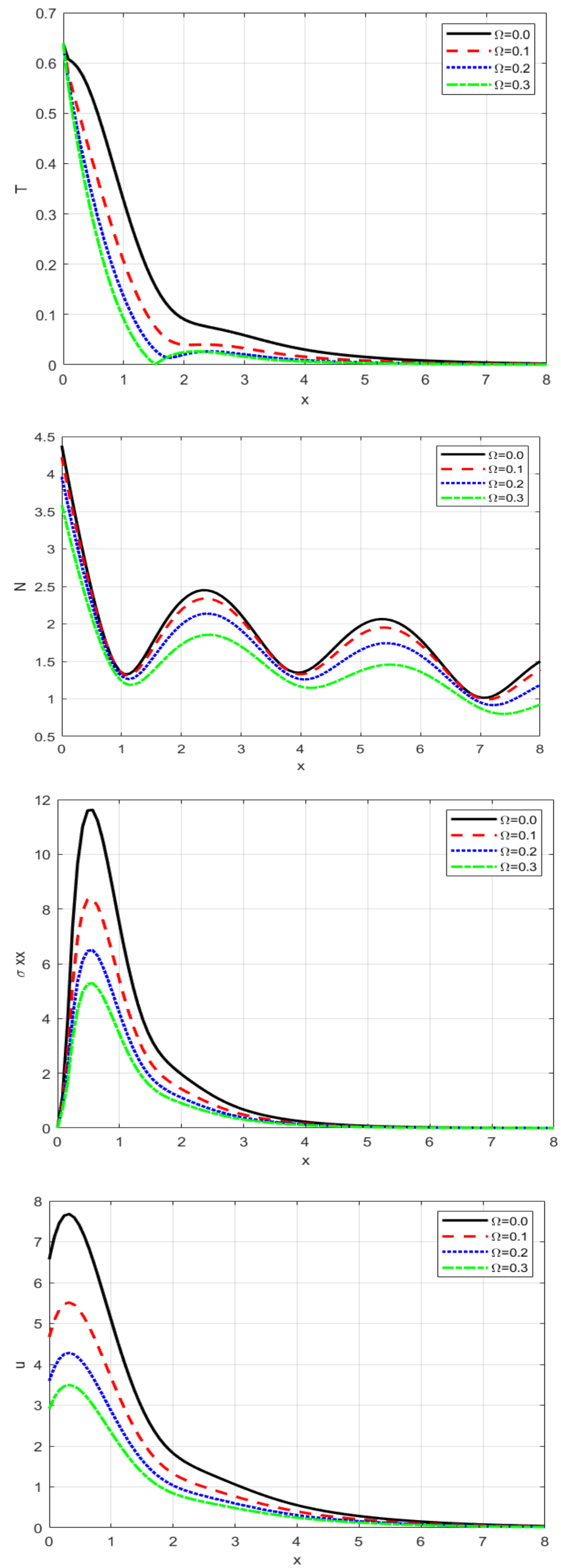
The influence of angular frequency parameter on the wave propagation of some fundamental physical field quantity distributions such as temperature T, carrier density N, thermal stress σ_{xx}, σ_{xy}, and displacement u with the horizontal distance x is shown in the third category (Fig. 3). They have the same decreasing behavior for all values of the angular frequency parameter ω.

5.3. The effect of time parameter t.

Figure 4 represents the representation of some physical quantities under investigation against the horizontal distance x. The category is carried out under the effect of time t, the magnitude of the physical components T, N, σ_{xx}, u and σ_{xy} increases by increasing the value of time.

5.4. The effect of ramp-type heating parameter t_o.

The impact of the ramp-type heating parameter t_o is plotted due to the main components T, N, σ_{xx}, u and σ_{xy} with the distance x as seen in Figure 5. All physical distributions T, N, σ_{xx}, u and σ_{xy} have the same variance. Their values rise dramatically as horizontal distance increases.



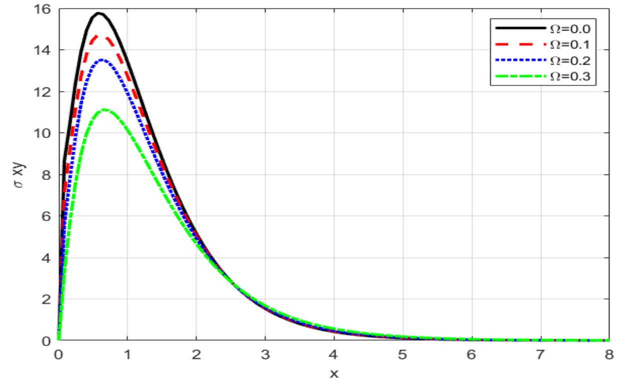
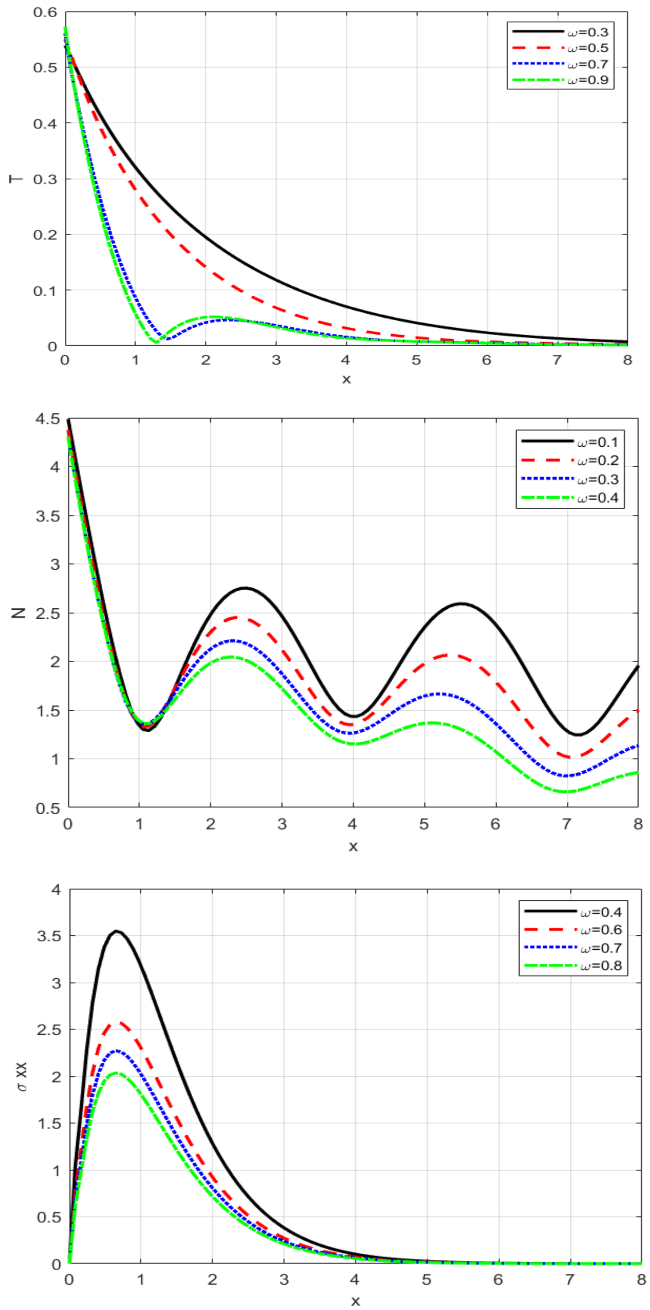


Fig.2. The variations of the main physical quantities T, N, σ_{xx}, u *and* σ_{xy} against x −axis under the effect of rotation.



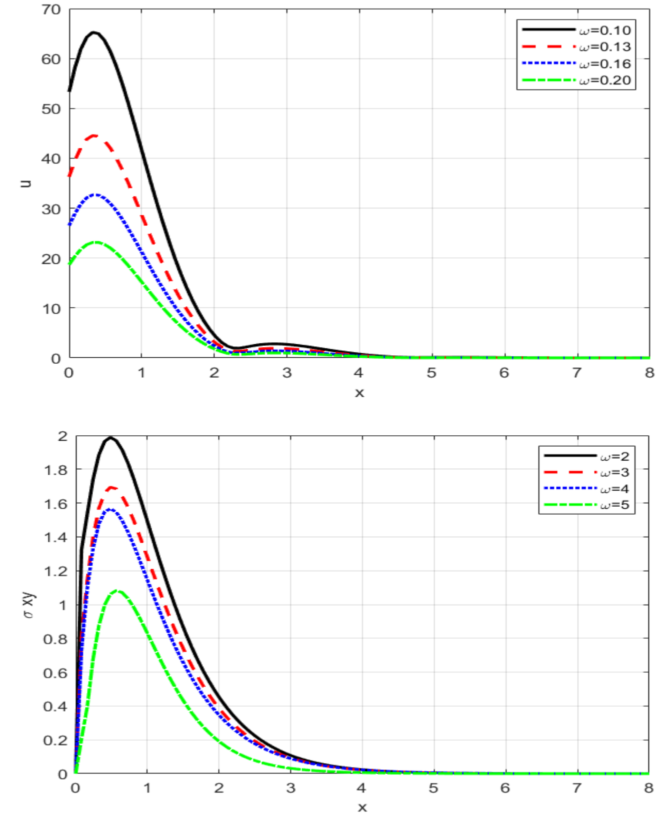
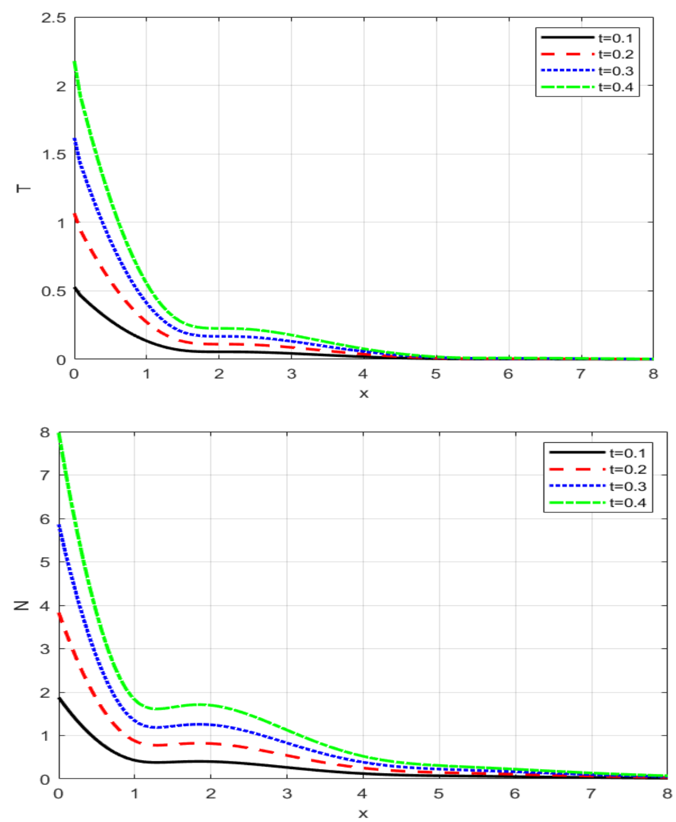


Fig.3. The variations of the main physical quantities T, N, σ_{xx}, u *and* σ_{xy} against x −axis under the effect of angular frequency.



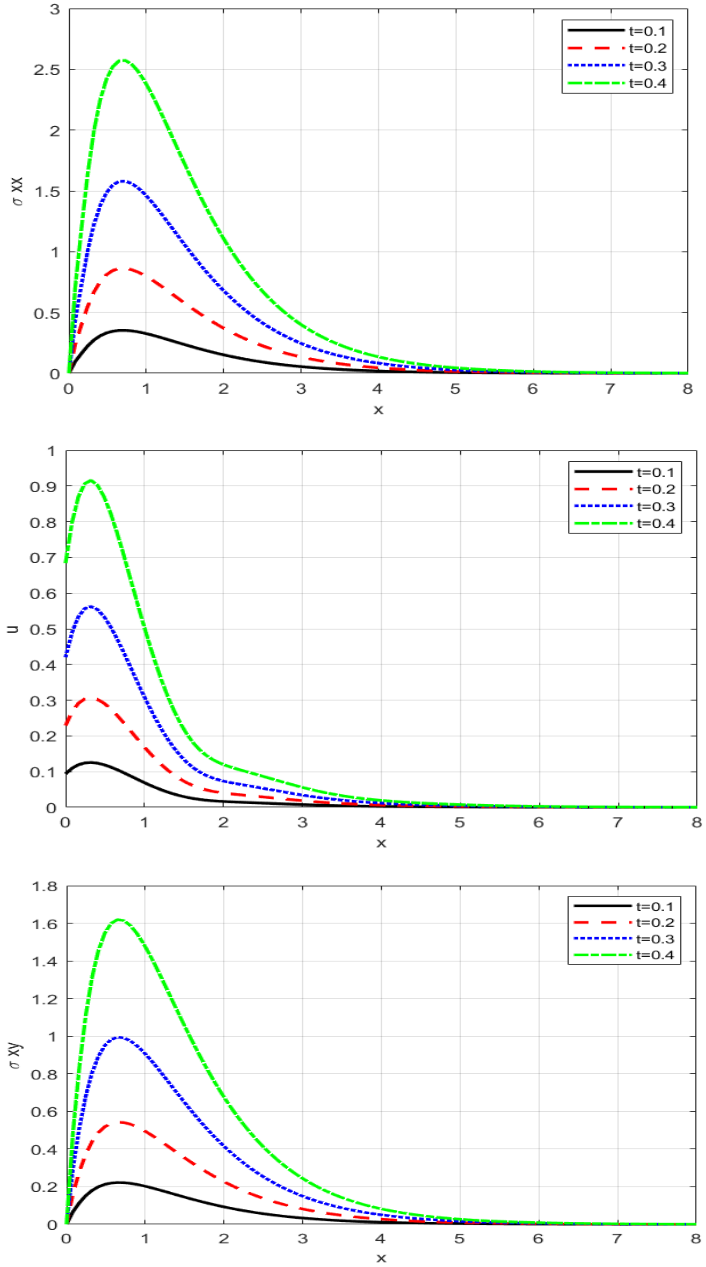
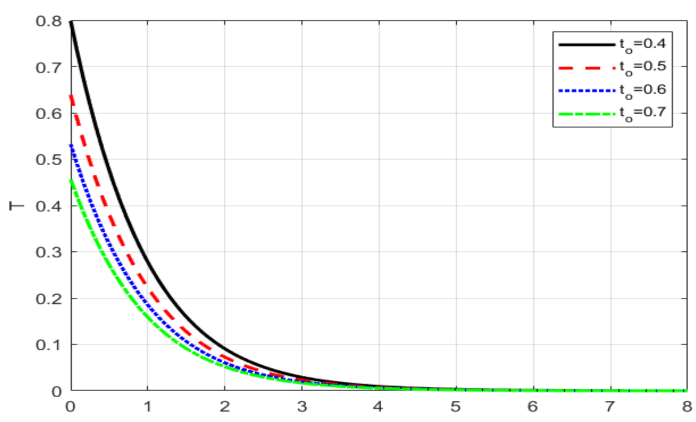


Fig.4. The variations of the main physical quantities $T, N, \sigma_{xx}, u \; and \; \sigma_{xy}$ against x −axis under the effect of time.



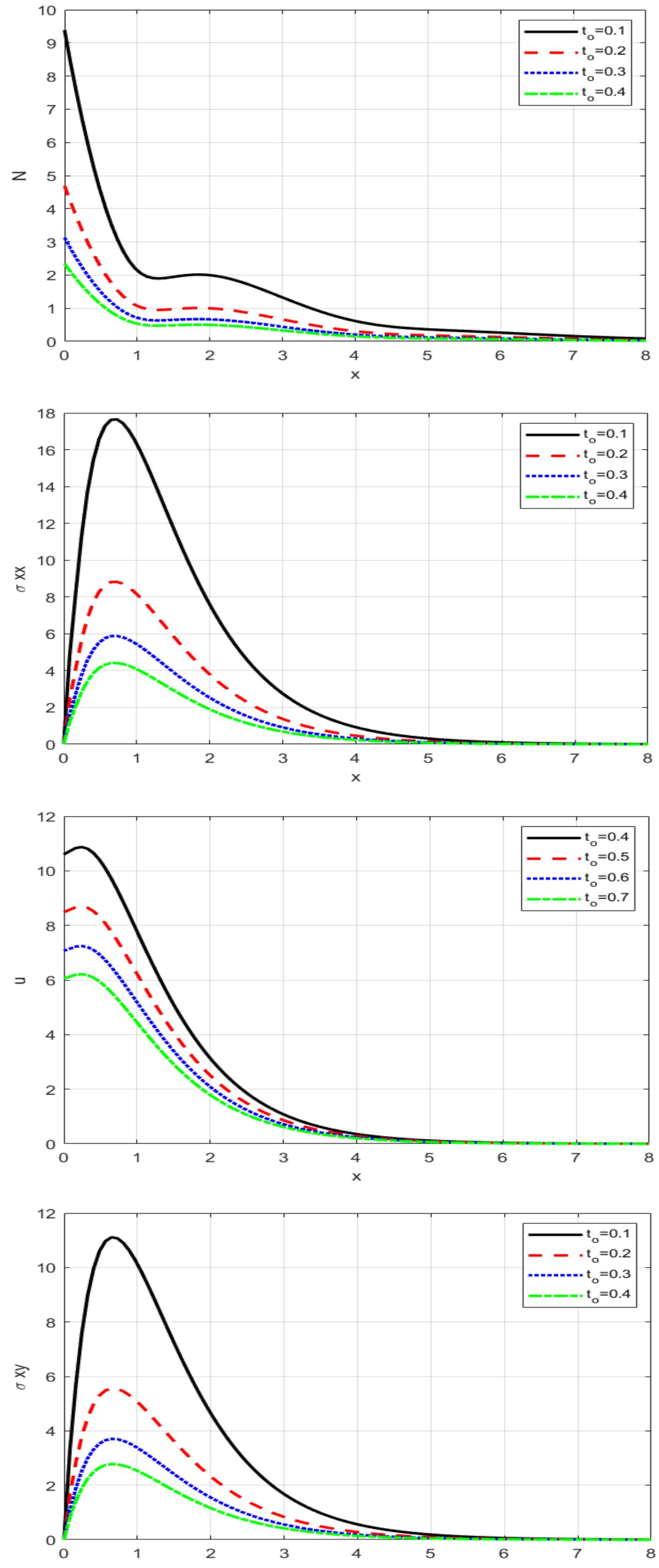


Fig.5. The variations of the main physical quantities $T, N, \sigma_{xx}, u \; and \; \sigma_{xy}$ against x −axis under the effect of ramp-type heating parameter.

6. Conclusion

This study presents a novel model that characterizes the photo-thermoelastic processes under the effect of rotation in a semiconductor material with ramp-type heating. The main governing equations in 2D electronic-elastic deformation are presented. The system was solved using the eigenvalue technique and normal mode analysis. Numerical calculations of the basic physical quantities were performed and graphed. This investigation leads us to the conclusion that the wave distributions of the primary physical field parameters are significantly influenced by the rotation. In the absence of rotation, we notice that the curve has a large value with increasing amplitude compared to the rest of the curves in its presence. The magnitude values of physical field quantities distributions are affected clearly in the presence of rotation, angular frequency, time, and ramp-type heating parameter. In addition, any minor changes in these parameters tend to cause changes in the propagations of waves with continuous distribution behavior. As expected, it can be found that the principal quantities satisfied the boundary conditions. The examined problem has several significant uses. Modern technology and plasma physics (such as integrated circuits, solar cells, and the electronic industry) make use of rotator elastic semiconductor materials in the context of photo-thermoelastic processes during ramp-type heating.

Author's contributions

All authors contributed equally to the writing and editing of this article. All authors read and approved the final version of the manuscript.

Data availability statement

The data used to support the findings of this study are available from the corresponding author upon request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Nomenclature

ρ the material density
t the time
$T_\circ$ absolute temperature
u, v the displacement components
c_e the specific heating at constant strain
N the carrier density
λ, μ the Lame's constants
D_e the coefficient of carrier diffusions
α_T the coefficient of the linear thermal expansion
τ the photogenerated carrier lifetime
d_n the coefficient of electronic deformation
E_g the energy gap
δ_n the difference of deformation potential of conduction and valence band
k the thermal conductivity of the sample
σ_{ij} the stress components

Appendix I

$m_1 = a_{22} - \eta_1 - \eta_{12} - \eta_{11}\,\eta_4 - \eta_6 - \eta_3\,\eta_8,$
$m_2 = -\eta_3\,\eta_5 - \eta_2\,\eta_8,$

$m_3 = -a_{22}\eta_1 - \beta\eta_{10} - a_{22}\eta_{12} + \eta_1\eta_{12} - a_{22}\eta_{11}\eta_4 -$

$\beta\eta_{11}\eta_4 - \eta_2\eta_5 - a_{22}\eta_6 + \eta_1\eta_6 + \eta_{12}\eta_6 + \eta_{11}\eta_4\eta_6 -$
$\eta_{11}\eta_3\eta_7 - a_{22}\eta_3\eta_8 + \eta_{12}\eta_3\eta_8 - \eta_7\eta_9 - \eta_4\eta_8\eta_9,$

$m_4 = -a_{22}\eta_3\,\eta_5 + \eta_{12}\eta_3\,\eta_5 - \eta_{11}\eta_2\eta_7 - a_{22}\eta_2\eta_8 +$
$\eta_{12}\eta_2\eta_8 - \eta_4\,\eta_5\eta_9,$

$m_5 = \beta\eta_1\eta_{10} + a_{22}\eta_1\eta_{12} - a_{22}\eta_2\,\eta_5 + \eta_{12}\eta_2\,\eta_5 + a_{22}\eta_1\eta_6 +$
$\beta\eta_{10}\eta_6 + a_{22}\eta_{12}\eta_6 - \eta_1\eta_{12}\eta_6 + a_{22}\eta_{11}\eta_4\eta_6 +$
$\beta\eta_{11}\eta_4\eta_6 - a_{22}\eta_{11}\eta_3\eta_7 - \beta\eta_{11}\eta_3\eta_7 + \beta\eta 10\eta_3\eta_8 +$
$a_{22}\eta_{12}\eta_3\eta_8 - a_{22}\eta_7\eta_9 - \beta\eta_7\eta_9 + \eta_1\eta_7\eta_9 - a_{22}\eta_4\eta_8\eta_9 -$
$\beta\eta_4\eta_8\eta_9,$

$m_6 = \beta\eta_{10}\eta_3\,\eta_5 + a_{22}\eta_{12}\eta_3\,\eta_5 - a_{22}\eta_{11}\eta_2\eta_7 - \beta\eta_{11}\eta_2\eta_7 +$
$\beta\eta_{10}\eta_2\eta_8 + a_{22}\eta_{12}\eta_2\eta_8 - a_{22}\eta_4\,\eta_5\eta_9 - \beta\eta_4\,\eta_5\eta_9,$

$m_7 = \beta\eta_{10}\eta_2\eta_5 + a_{22}\eta_{12}\eta_2\eta_5 - \beta\eta_1\eta_{10}\eta_6 - a_{22}\eta_1\eta_{12}\eta_6 +$
$a_{22}\eta_1\eta_7\eta_9 + \beta\eta_1\eta_7\eta_9,$

Appendix II

$\chi_{11} = \frac{\beta(\eta_6-\lambda_1{}^2)[\beta\lambda_1{}^2\eta_7+(a_{22}+\lambda_1{}^2)\lambda^2\eta_7+\beta\eta_{10}(\eta_5-\lambda_1{}^2)+(\eta_{12}-\lambda_1{}^2)(\eta_5-\lambda_1{}^2)(a_{22}+\lambda_1{}^2)]}{\beta\lambda_1(\eta_5-\lambda_1{}^2)} - \epsilon$,,

$\chi_{12} = \frac{\beta(\eta_6-\lambda_2{}^2)[\beta\lambda_2{}^2\eta_7+(a_{22}+\lambda_2{}^2)\lambda_2{}^2\eta_7+\beta\eta_{10}(\eta_5-\lambda_2{}^2)+(\eta_{12}-\lambda_2{}^2)(\eta_5-\lambda_2{}^2)(a_{22}+\lambda_2{}^2)]}{\beta\lambda_2(\eta_5-\lambda_2{}^2)} - \epsilon,$

$\chi_{13} = \frac{\beta(\eta_6-\lambda_3{}^2)[\beta\lambda_3{}^2\eta_7+(a_{22}+\lambda_3{}^2)\lambda_3{}^2\eta_7+\beta\eta_{10}(\eta_5-\lambda_3{}^2)+(\eta_{12}-\lambda_3{}^2)(\eta_5-\lambda_3{}^2)(a_{22}+\lambda_3{}^2)]}{\beta\lambda_3(\eta_5-\lambda_3{}^2)} - \epsilon,$

$\chi_{14} = \frac{\beta(\eta_6-\lambda_4{}^2)[\beta\lambda_4{}^2\eta_7+(a_{22}+\lambda_4{}^2)\lambda_4{}^2\eta_7+\beta\eta_{10}(\eta_5-\lambda_4{}^2)+(\eta_{12}-\lambda_4{}^2)(\eta_5-\lambda_4{}^2)(a_{22}+\lambda_4{}^2)]}{\beta\lambda_4(\eta_5-\lambda_4{}^2)} - \epsilon,$

$\chi_{21} = \frac{-[\beta\lambda_1{}^2\eta_7+(a_{22}+\lambda_1{}^2)\lambda_1{}^2\eta_7+\beta\eta_{10}(\eta_5-\lambda_1{}^2)]}{\lambda_1},$

$\chi_{22} = \frac{-[\beta\lambda_2{}^2\eta_7+(a_{22}+\lambda_2{}^2)\lambda_2{}^2\eta_7+\beta\eta_{10}(\eta_5-\lambda_2{}^2)]}{\lambda_2},$

$\chi_{23} = \frac{-[\beta\lambda_3{}^2\eta_7+(a_{22}+\lambda_3{}^2)\lambda_3{}^2\eta_7+\beta\eta_{10}(\eta_5-\lambda_3{}^2)]}{\lambda_3},$

$\chi_{24} = \frac{-[\beta\lambda_4{}^2\eta_7+(a_{22}+\lambda_4{}^2)\lambda_4{}^2\eta_7+\beta\eta_{10}(\eta_5-\lambda_4{}^2)]}{\lambda_4},$

$\chi_{31} = \frac{\beta\lambda_1{}^2(\eta_6-\lambda_1{}^2)+\beta\eta_9(\eta_5-\lambda_1{}^2)}{\lambda_1},$

$\chi_{32} = \frac{\beta\lambda_2{}^2(\eta_6-\lambda_2{}^2)+\beta\eta_9(\eta_5-\lambda_2{}^2)}{\lambda_2},$

$\chi_{33} = \frac{\beta\lambda_3{}^2(\eta_6-\lambda_3{}^2)+\beta\eta_9(\eta_5-\lambda_3{}^2)}{\lambda_3},$

$\chi_{34} = \frac{\beta\lambda_4{}^2(\eta_6-\lambda_4{}^2)+\beta\eta_9(\eta_5-\lambda_4{}^2)}{\lambda_4},$

$\chi_{41} = \frac{(a_{22}+\lambda_1{}^2)[\beta\lambda_1{}^2(\eta_6-\lambda_1{}^2)+\beta\eta_9(\eta_5-\lambda_1{}^2)]}{\beta\lambda_1},$

$\chi_{42} = \frac{(a_{22}+\lambda_2{}^2)[\beta\lambda_2{}^2(\eta_6-\lambda_2{}^2)+\beta\eta_9(\eta_5-\lambda_2{}^2)]}{\beta\lambda_2},$

$\chi_{43} = \frac{(a_{22}+\lambda_3{}^2)[\beta\lambda_3{}^2(\eta_6-\lambda_3{}^2)+\beta\eta_9(\eta_5-\lambda_3{}^2)]}{\beta\lambda_3},$

$\chi_{44} = \frac{(a_{22}+\lambda_4{}^2)[\beta\lambda_4{}^2(\eta_6-\lambda_4{}^2)+\beta\eta_9(\eta_5-\lambda_4{}^2)]}{\beta\lambda_4}.$