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STARLIKE AND CONVEX FUNCTIONS ASSOCIATED WITH HYPERGEOMETRIC MATRIX FUNCTIONS

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Abstract. The purpose of the present article to obtain some sufficient conditions for the hypergeometric matrix function belonging to certain classes of starlike and convex functions. Finally, we discuss an integral operator associated with this function.

1. Introduction

Let $\mathcal{A}$ represent the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, we denote $\mathcal{S}$ by the subclass of $\mathcal{A}$ consisting of functions f of the form (1) which are also univalent in Δ.

A function $f(z) \in \mathcal{A}$ is said to be starlike if it satisfies the following analytic criteria

$$\Re\left\{\frac{z\,f'(z)}{f(z)}\right\} > 0, \quad z \in \Delta.$$

Similarly, a function $f(z) \in \mathcal{A}$ is said to be convex it satisfies the following analytic criteria

$$\Re\left\{1 + \frac{z\,f''(z)}{f'(z)}\right\} > 0, \quad z \in \Delta.$$

The classes of all starlike functions and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively and earlier studied by Robertson [14] and Silverman [16].

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The applications of hypergeometric functions [4, 11], generalized Bessel functions [1, 13], Poisson distribution series [12] etc. on certain univalent functions are interesting topics of research in geometric function theory. The hypergeometric matrix function was introduced by Jodar and Cortes [8]. Bshouty and Hengartner [2, 3] studied the linear operators and analytic injective on matrix function. Jodar and Cortes [9] studied the some basic properties of Beta and Gamma function of matrix function and obtain interesting results. Special matrix function play an important role in Mathematics and Physics for deep study one may refer [7]. Motivated with the above mentioned work we obtain some necessary and sufficient conditions for hypergeometric matrix function belonging to starlike and convex functions. First, we recall the definition of hypergeometric matrix function.

Let A, B, C be matrices in $C^{r\times r}$ and if matrices B and C commute then $F(A,B;C;z)$ is a solution of the following differential equation

$$z(1-z)\omega^{''}-zA\omega^{'}+(C-z(B+I))-AB=0.$$

Throughout this paper for a matrix A in $C^{r\times r}$ its spectrum $\sigma(A)$ denotes the set of all the eigen values of A. The 2-norm of A will be denoted by $||A||$ and defined by

$$\|A\|=Sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2},$$

where the euclidean norm of y in $C^{r\times r}$ is given by

$$\|y\|_2=\left(y^Ty\right)^{\frac{1}{2}}.$$

Let

$$\alpha(A)=\max\left\{\Re(z):z\in\sigma(A)\right\}$$

$$\beta(A)=\min\left\{\Re(z):z\in\sigma(B)\right\}.$$

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and A is a matrix in $\mathbb{C}^{r\times r}$ with $\sigma(A)\subset\Omega$, then from the properties of the matrix functional calculus [6], it follows that

$$f(A)g(A)=g(A)f(A). \tag{2}$$

Furthermore, if $B\in\mathbb{C}^{r\times r}$ is a matrix for which $\sigma(B)\subset\Omega$, and if $AB=BA$, then

$$f(A)g(B)=g(B)f(A). \tag{3}$$

The reciprocal of Gamma function denoted by $\Gamma^{-1}(z)=\frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. The image of $\Gamma^{-1}(z)$ acting on A, denoted by $\Gamma^{-1}(A)$, is a well defined matrix. If $A+nI$ is invertible for all integers $n\geq 0$, then the reciprocal gamma function is defined as [9]

$$\Gamma^{-1}(A)=A(A+I)\dots(A+(n-1)I)\Gamma^{-1}(A+nI),\ n\geq 1. \tag{4}$$

The Pochhammer symbol $(a)_n$, $a\in\mathbb{C}$, is defined as, if n is a nonnegative integer, then the Pochhammer symbol, for a complex number a, is defined by

$$(a)_n=\begin{cases}a(a+1)\cdots(a+n-1), & n\geq 1,\\ 1, & n=0.\end{cases} \tag{5}$$

The Pochhammer symbol $(a)_n$, $a \in \mathbb{C}$, in terms of gamma function is

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{6}$$

By application of the matrix functional calculus, the Pochhammer symbol for $A \in \mathbb{C}^{r\times r}$ is given by

$$(A)_n = \begin{cases} I, & \text{if } n = 0, \\ A(A+I)\dots(A+(n-1)I), & \text{if } n \geq 1. \end{cases} \tag{7}$$

This gives

$$(A)_n = \Gamma^{-1}(A)\ \Gamma(A+nI), \qquad n \geq 1. \tag{8}$$

Let S_n be class of all univalent square matrices of order n in Δ. which are normalized by

$$f(z) = Iz + \sum_{n=2}^{\infty} A_n z^n, \tag{9}$$

The class S_n is natural extension of the usual class S for $n = 1$ in Δ.

A function $f(z)$ of the form (9) is said to be starlike if it satisfies the following analytic criteria

$$\Re\left\{z\ f'(z)(f(z)^{-1})\right\} > 0, \quad z \in \Delta.$$

The equivalent form of the above condition is given below

$$\|zf'(z)(f(z)^{-1}) - I\|| \leq \|zf'(z)(f(z)^{-1}) + I\|.$$

Similarly, a function $f(z)$ of the form (9)is said to be convex it satisfies the following analytic criteria

$$\Re\left\{I + z\ f''(z)(f'(z))^{-1}\right\} > 0, \quad z \in \Delta.$$

The equivalent form of the above condition is given below

$$\|z\ f''(z)(f'(z))^{-1}\|| \leq \|z\ f''(z)(f'(z))^{-1} + 2.I\||.$$

The classes of all starlike functions and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively.

The hypergeometric matrix function $F(A, B; C; z)$ is defined by

$$F(A, B; C; z) = \sum_{n=0}^{\infty} \frac{(A)_n\,(B)_n\,(C)_n^{-1}}{n!} z^n, \tag{10}$$

for matrices A, B, C in $C^{r\times r}$ such that $C + nI$ is invertible for all integer $n \geq 0$.

Jodar and Sastre [10] show that the above series is convergent for $|z| < 1$. Further, Jodar and Cortes [8] proved that if A, B, C be positive stable matrices in $C^{r\times r}$ such that

$$\beta(C) > \alpha(A) + \alpha(B),$$

then the series is convergent for $|z| = 1$. The study of univalent matrix function plays an important role in the theory of ordinary differential equation. Schwarz and Bshouthy obtained some intresting resuts in this direction one can prefer[2, 3, 15].

2. Preliminary Results

To prove our main results we shall require the following lemmas.

Lemma 2.1. *([3]) A function $f(z)$ of the form (9) and A_n be a sequence of square matrix of order n and satisfy the condition*

$$\sum_{n=2}^{\infty} n||A_n|| \leq 1 \tag{11}$$

then $f \in \mathcal{S}^*$.

Lemma 2.2. *([3]) A function $f(z)$ of the form (9) and satisfy the condition*

$$\sum_{n=2}^{\infty} n^2||A_n|| \leq 1 \tag{12}$$

then $f \in \mathcal{C}$.

3. Main Results

Theorem 3.1. *Let A, B, C be matrices in $C^{r\times r}$ such that $CB = BC$ and C, $C-A$, $C-B$, $C-A-B$, $C-A-B-I$ are positive stable and satisfy the condition*

$$\left\|\Gamma(C)\Gamma(C-B-A-I)\Gamma^{-1}(C-A)\Gamma^{-1}(C-B)\left\{AB+\right.\right.$$

$$\left.\left.(C-B-A-I)\right\}-I\right\| \leq 1 \tag{13}$$

then $zF(A, B; C; z) \in \mathcal{S}^*$

Proof. From (10) we may write

$$zF(A, B; C; z) = Iz + \sum_{n=2}^{\infty} \frac{(A)_{n-1}\,(B)_{n-1}\,(C)_{n-1}^{-1}}{(n-1)!} z^n. \tag{14}$$

To prove $zF(A, B; C; z) \in \mathcal{S}^*$ it is sufficient to show that

$$\sum_{n=2}^{\infty} n \left\| \frac{(A)_{n-1}\,(B)_{n-1}\,(C)_{n-1}^{-1}}{(n-1)!} \right\| \leq 1$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n \left\| \frac{(A)_{n-1}(B)_{n-1}(C)_{n-1}^{-1}}{(n-1)!} \right\| \\
&= \left\| \sum_{n=2}^{\infty} n \frac{(A)_{n-1}(B)_{n-1}(C)_{n-1}^{-1}}{(n-1)!} \right\| \\
&= \left\| \sum_{n=2}^{\infty} (n-1) \frac{(A)_{n-1}(B)_{n-1}(C)_{n-1}^{-1}}{(n-1)!} + \sum_{n=2}^{\infty} \frac{(A)_{n-1}(B)_{n-1}(C)_{n-1}^{-1}}{(n-1)!} \right\| \\
&= \left\| \sum_{n=2}^{\infty} AB(C)^{-1} \frac{(A+I)_{n-2}(B+I)_{n-2}(C+I)_{n-2}^{-1}}{(n-2)!} \right. \\
&\left. + \left\{ \Gamma(C)\Gamma(C-B-A)\Gamma^{-1}(C-A)\Gamma^{-1}(C-B) - I \right\} \right\| \\
&= \| AB\Gamma(C)\Gamma(C-B-A-I)\Gamma^{-1}(C-A)\Gamma^{-1}(C-B) + \\
&\left\{ \Gamma(C)(C-A-B-I)\Gamma(C-B-A-I)\Gamma^{-1}(C-A)\Gamma^{-1}(C-B) - I \right\} \| \\
&= \Gamma(C)\Gamma(C-B-A-I)\Gamma^{-1}(C-A)\Gamma^{-1}(C-B) \| AB + \\
&\{(C-A-B-I)\} - I \| \\
&\leq 1, \text{ from (13).}
\end{aligned}$$

Thus, the proof of Theorem 3.1 is established. □

Theorem 3.2. *Let A, B, C be matrices in $C^{r\times r}$ such that $CB = BC$ and C, $C-A$, $C-B$, $C-A-B$, $C-A-B-I$ are positive stable and satisfy the condition*

$$\begin{aligned}
&= \| A(A+I)B(B+I)(C)^{-1}(C+I)^{-1}\Gamma(C+2I)\Gamma(C-B-A-2I)\Gamma^{-1}(C-A)\Gamma^{-1}(C-B) \\
&+ 3AB(C)^{-1}\Gamma(C+I)\Gamma(C-B-A-I)\Gamma^{-1}(C-A)\Gamma^{-1}(C-B) \\
&+ \left\{ \Gamma(C)\Gamma(C-B-A)\Gamma^{-1}(C-A)\Gamma^{-1}(C-B) - I \right\} \| \\
&\leq 1
\end{aligned}$$

then $zF(A, B; C; z) \in \mathcal{C}$

Proof. To prove $zF(A, B; C; z)$ defined by (14) in $\mathcal{C}$, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n^2 \left\| \frac{(A)_{n-1}(B)_{n-1}(C)_{n-1}^{-1}}{(n-1)!} \right\| \leq 1.$$

Now

$$\sum_{n=2}^{\infty} n^2 \left\| \frac{(A)_{n-1}(B)_{n-1}(C)_{n-1}^{-1}}{(n-1)!} \right\|$$

$$= \left\| \sum_{n=2}^{\infty} n^2 \frac{(A)_{n-1}(B)_{n-1}(C)_{n-1}^{-1}}{(n-1)!} \right\|$$

$$= \left\| \sum_{n=2}^{\infty} [(n-1)(n-2)+3(n-1)+1] \frac{(A)_{n-1}(B)_{n-1}(C)_{n-1}^{-1}}{(n-1)!} \right\|$$

$$= \left\| \sum_{n=3}^{\infty} \frac{(A)_{n-1}(B)_{n-1}(C)_{n-1}^{-1}}{(n-3)!} + 3\sum_{n=2}^{\infty} \frac{(A)_{n-1}(B)_{n-1}(C)_{n-1}^{-1}}{(n-2)!} \right.$$

$$\left. + \sum_{n=2}^{\infty} \frac{(A)_{n-1}(B)_{n-1}(C)_{n-1}^{-1}}{(n-1)!} \right\|$$

$$= \left\| \sum_{n=3}^{\infty} A(A+I)B(B+I)(C)^{-1}(C+I)^{-1} \frac{(A+2I)_{n-3}(B+2I)_{n-3}(C+2I)_{n-3}^{-1}}{(n-3)!} \right.$$

$$+ 3\sum_{n=2}^{\infty} AB(C)^{-1} \frac{(A+I)_{n-2}(B+I)_{n-2}(C+I)_{n-2}^{-1}}{(n-2)!}$$

$$\left. + \left\{ \Gamma(C)\Gamma(C-B-A)\Gamma^{-1}(C-A)\Gamma^{-1}(C-B) - I \right\} \right\|$$

$$= \| A(A+I)B(B+I)(C)^{-1}(C+I)^{-1}\Gamma(C+2I)\Gamma(C-B-A-2I)\Gamma^{-1}(C-A)\Gamma^{-1}(C-B)$$

$$+ 3AB(C)^{-1}\Gamma(C+I)\Gamma(C-B-A-I)\Gamma^{-1}(C-A)\Gamma^{-1}(C-B)$$

$$+ \left\{ \Gamma(C)\Gamma(C-B-A)\Gamma^{-1}(C-A)\Gamma^{-1}(C-B) - I \right\} \|$$

$$\leq 1$$

Thus, the proof of Theorem 3.2 is established.

□

4. An Integral Operator

In this section we define an integral operator $G(A, B; C; z)$ associated with hypergeometric matrix function $F(A, B; C; z)$ as follows

$$G(A, B; C; z) = \int_0^z F(A, B; C; t)dt. \tag{15}$$

Theorem 4.3. *Let A, B, C be matrices in $C^{r \times r}$ such that $CB = BC$ and C, $C-A$, $C-B$, $C-A-B$, $C-A-B-I$ are positive stable and satisfy the condition* (13) *then $G(A, B; C; z)$ defined by* (15) *in the class $\mathcal{C}$.*

Proof. The representation of $G(A, B; C; z)$ given by (15) defined as

$$G(A, B; C; z) = Iz + \sum_{n=2}^{\infty} \frac{(A)_{n-1}(B)_{n-1}(C)_{n-1}^{-1}}{(n-1)!} z^n. \tag{16}$$

Now, applying the same reasoning as in Theorem 3.2 we obtain the required result. Hence, we omit the details. □

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