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Generalized Complex Conformable Derivative and Integral Bases in Fréchet Spaces

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ABSTRACT

This paper presents an additional approach in the field of polynomial bases, utilizing generalized complex conformable fractional derivative and integral operators. These operators are applied to polynomial bases of complex conformable derivatives (GCCDB) and generalized complex conformable integrals (GCCIB) in Fréchet spaces. We also investigate their convergence properties within closed disks, open disks, open regions surrounding closed disks, origin and for all entire functions, employing the Cannon sum, order, type and T_ρ-property as convergence criteria for our study. The significance of this work lies in generalizing certain previous studies and considering them as special cases of it. Additionally, this paper concludes examples and applications that illustrate the concepts of GCCDB and GCCIB.

INTRODUCTION

Theory of basic sets of polynomials (BPs) plays an important role in different mathematical branches. It aids mathematicians and those involved in mathematical fields simplifying studies in areas such as partial differential equations, mathematical physics, and nonlinear analysis. In 20^{th} century mathematicians developed the notion of BPs ([1], see also [2–4]). Their notion depends explicity on an analytic function $f(z)$ which can be expanded by the BP $\{P_n(z)\}$ as $f(z) \sim \sum_n a_n P_n(z)$. Taylor series represents the first appearance of the BPs. It was later generalized and expanded to include polynomials such as Lagendre, Euler, Bessel, Bernoulli, Chebyshev polynomials ([5–10]).

As an early study, authors ([1, 2]) study BPs in one complex variable for open and closed disks. Previous studies ([11–19]) show that BPs extended and generalized in numerous directions. One of them is produced thanks to Abul-Ez and Constales ([13, 16, 18, 19]). It's noteworthy to say that their study contains variables of domains such as closed hyperball, open hyperball, open hyperball containing closed ball, origin, whole space in Clifford analysis. Another direction of study was pursued by Malonek ([17]), Kishka et al. ([14]), El-sayed ([11]), Kumuyi and Nassif ([15]), El-sayed and Kishka([12]), their researches explored domains such as polycylindrical, hyperspherical, and hyperelliptical regions acrossmultiple complex variables.

Hassan et al. ([20]) give a new perspective of BPs of special monogenic polynomials in Fréchet spaces. Because of the importance of the derivative of BPs (DBPs) in topology and algebra, mathematicians have been studied in different regions and spaces. DBPs one defined for open disks and closed disks (resp., polyclinderical, hyperspherical and hyperellitical regions) in one complex variable (resp., several complex variables), one can see ([21–23]) (resp., [11], [12], [14], [15], [24]).

One of the crucial issues of the present moment is the fraction calculus (that is, calculus of integrals and derivatives of arbitrary order). Fractional calculus has started to be gained significant attention in a few fields of science and engineering. It has been applied to breakdown numerous dynamical processes and complex nonlinear physical phenomena in physics, electromagnetic, engineering, anomalous diffusion, chemistry, visco-elasticity, and electro-chemistry. Due to its extensive use in the aforementioned domains, this topic has grown in relevance during the past few decades. Numerous recent efforts have been made in this area, some of which are described in [25–27].

This paper introduces two new bases, denotes (GCCDB and GCCIB). It investigate the convergence properties of GCCDB and GCCIB in the context of Fréchet spaces, employing tools such as the Cannon function, order, type and T_ρ-property. Further-more, the paper includes examples and applications related to GCCDB and GCCIB.

2 Basic concepts

This section presents previous studies on basic sets, Cannon function, order, type and the T_ρ-property. Additionally, it introduces conceptes related to complex conformable derivatives.

Definition 2.1. *The mapping* $\|.\|: X \to \mathbb{R}$, *where* X *is a vector space and* $\mathbb{R}$ *is the set of real numbers, is a seminorm if the following axioms are satisfied:*

(i) $\|f\| \geq 0 \ \forall f \in X$,

(ii) $\|f+g\| \leq \|f\| + \|g\| \ \forall f, g \in X$,

(iii) $\|af\| = |a|\|f\| \ \forall f \in X \ \forall \ a \in \mathbb{C}$.

Any norm on X *is a seminorm with the additional property that* $\|f\| = 0$ *if and only if* $f = 0$.

Definition 2.2. *A space* X *over* $\mathbb{C}$ *is an abelian group* (X, t) *with a mapping* $\mathbb{C} \times X \to X$; $(\lambda, f) \to \lambda f$ *such that for all* $\lambda, \mu \in X$ *and* $f, r \in \mathbb{C}$ *such that the following axioms hold:*

(i) $(\lambda + \mu)f = \lambda f + \mu f$;

(ii) $(\lambda\mu)f = \lambda(\mu f)$;

(iii) $\lambda(f + r) = \lambda f + \lambda r$;

(iv) $1f = f$.

Definition 2.3. *Let* W *be a family of countable proper system of seminorms on* X.

(i) W *is called a proper system of seminorms on* X, *if* $\forall w_1, w_2, \dots, w_s \in W (s > 0), \exists w \in W$ *and* $P > 0$ *such that, for all* $f \in X$, $\sup_{k=1,\dots,s} W_k(f) \le P_w(f)$;

(ii)X *is a Fréchet space, if* (X, W) *is a seminormed Hausdorff topological space such that* $j < s \Rightarrow W_j(f)$ $(\forall f \in X$ *and* $f \in U \subseteq X)$, *there are* $\epsilon > 0$ *and* $N > 0$ *such that* $\{g \in X: W_j(f - g) \le \epsilon\} \subset U$ $(\forall j \le N)$, *and* X *is complete metric topology;*

(iii)Let $(W_n)_{n\ge0}$ be a sequence in a Fréchet space X, then $(W_n)_{n\ge0}$ converges to an element $f \in X$ iff $\forall W_j \in W \Rightarrow \lim_{n\to\infty} W_j(f_n - f) = 0$.

Definition 2.4. *A Banach space over* X *is an* X *space over* $\mathbb{C}$, *which equipped with a function* $\|.\|_{\mathbb{C}}: \mathbb{C} \to [0, \infty)$ *such that* $\forall f, g \in \mathbb{C}$ *and* $\lambda \in X$:

(i) $\|f + g\|_{\mathbb{C}} \le \|f\|_{\mathbb{C}} + \|g\|_{\mathbb{C}}$;

(ii) $\|\lambda f\|_{\mathbb{C}} = |\lambda| \|f\|_{\mathbb{C}}$;

(iii) $\mathbb{C}$ *is complete with respect to the metric* $d(f, g) = \|f - g\|_{\mathbb{C}}$.

Given an open disk $D(R)$, with it's closure $\overline{D}(R)$, $D_+(R)$ (any open disk enclosing the closed disk $\overline{D}(R)$) and $\overline{D}(\epsilon)$ (some disk surrounding 0). Then, we consider: $\square[D(R)] = \{g \in \square[D(R)] : g(z)$ is an analytic function $\forall z \in D(R)\}$; $\square[\overline{D}(R)] = \{g \in \square[\overline{D}(R)] : g(z)$ is an analytic function $\forall z \in \overline{D}(R)\}$; $\square[D_+(R)] = \{g \in \square[D_+(R)] : g(z)$ is an analytic function $\forall z \in D_+(R)\}$; $\square[0] = \{g \in \square[0] : g(z)$ is an analytic function $\forall z \in \overline{D}(\epsilon)\}$; $\square[\infty] = \{g \in \square[\infty] : g(z)$ is an entire function $\forall z \in \overline{D}(n),\ where\ n < \infty\}$.
Then the Fréchet spaces of the sets which mentioned above are given respectively by:
$\|g\|_r = \sup_{\overline{D}(r)} |g(z)|, z \in \mathbb{C}\ \forall r < R, g \in \square[D(R)]$; $\|g\|_R = \sup_{\overline{D}(R)} |g(z)|, z \in \mathbb{C}, g \in \square[\overline{D}(R)]$; $\|g\|_\epsilon = \sup_{\overline{D}(\epsilon)} |g(z)|, \epsilon > 0, z \in \mathbb{C}, \forall g \in \square[0]$; $\|g\|_n = \sup_{\overline{D}(n)} |g(z)|, z \in \mathbb{C}, g \in \square[\infty]$.

Definition 2.5. *Let* $\{P_n(z)\}$ *be a sequence of Fréchet space* X. *Then* $\{P_n(z)\}$ *is a base, if it can be expressed in the form*

$$z^n = \sum_k \pi_{n,k} P_k(z), \quad \pi_{n,k} \in \mathbb{C}, \tag{2.1}$$

where

$$P_n(z) = \sum_k p_{n,k} z^k, \quad p_{n,k} \in \mathbb{C}, \tag{2.2}$$

$\Pi = (\pi_{n,k})$ *and* $P = (p_{n,k})$ *are the matrices of operators and coefficients of the base* $\{P_n(z)\}$ *in* $\mathbb{C}$.

Remark 2.1 *A simple base is a base of degree n. Also, Cannon set is a base if* N_n *(i.e., the number of non zero coefficients* $\pi_{n,k}$ *in* (2.1)) *satisfies* $\lim_{n\to\infty}\{N_n\}^{\frac{1}{n}} = 1$.

Theorem 2.1 *A necessary and sufficient condition for a sequence* $\{P_n(z)\}$ *to be a base in* $\overline{D}(r)$ *is* $P\Pi = \Pi P = I$, *where I is unite matrix.*

Remark 2.2. *The Cauchy inequality for the base* (2.2) *is given by*

$$|p_{n,k}| \leq \frac{\|P_n\|_R}{R^k}. \tag{2.3}$$

If $g(z) = \sum_n P_n(z) a_n(z)$ is an analytic function on a Fréchet space X, then

$$g(z) \sim \sum_k P_n(z) \Pi_n(g), \tag{2.4}$$

where

$$\Pi_n(g) = \sum_k \pi_{k,n} a_k(g). \tag{2.5}$$

Definition 2.6. *A base* $\{P_n(z)\}$ *is effective for Fréchet space* X, *if the basic series* (2.2) *converges uniformally to* $g(z)$ *on* X.

The Cannon function $\lambda_P(r)$, which determines the effectiveness of bases in X, given by

$$\lambda_P(r) = \limsup_{n\to\infty} \{\omega_{P_n}(r)\}^{\frac{1}{n}}, \tag{2.6}$$

$$\omega_{P_n}(r) = \sum_k |\pi_{n,k}| \|P_k\|_r, \tag{2.7}$$

$$\|P_n\|_r = \sup_{\overline{D}(r)} |P_n(z)|. \tag{2.8}$$

Theorem 2.2. *A necessary and sufficient condition for a base* $\{P_n(z)\}$ *to be effective for Fréchet spaces* □$[D(R)]$, □$[\overline{D}(R)]$, □$[D_+(R)]$, □$[0]$, □$[\infty]$ *is* $\lambda_P(r) < R, \forall r < R$; $\lambda_P(R) = R$; $\lambda_P(R^+) = R$; $\lambda_P(0^+) = 0$; $\lambda_P(R) < \infty, \forall R < \infty$, *respectively.*

Definition 2.7. *The order ρ_P and type τ_P of a base $\{P_n(z)\}$ of analytic functions are given by*

$$\rho_P = \lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log \omega_{P_n}(R)}{n \log n}, \qquad (2.9)$$

$$\tau_P = \lim_{R\to\infty} \frac{e}{\rho_P} \limsup_{n\to\infty} \frac{\left\{\omega_{P_n}(R)\right\}^{\frac{1}{n\rho}}}{n}. \qquad (2.10)$$

Let a base $\{P_n(z)\}$ is of order ρ_P (resp., type τ_P), then it represents every analytic function in $\overline{D}(R)$ of order $\frac{1}{\rho_P}$ (resp., type $\frac{1}{\tau_P}$).

Definition 2.8. *T_ρ-property of a base $\{P_n(z)\}$ in □$\left[\overline{D}(R)\right]$ and □$[D(R)]$ (where $0 < \rho < \infty$) means that $\{P_n(z)\}$ represents all entire functions of order less than ρ in the same domain.*

Theorem 2.3. *A base $\{P_n(z)\}$ has T_ρ-property in □$\left[\overline{D}(R)\right]$ and □$[D(R)]$ iff $\omega_P(r) \le \frac{1}{\rho}$ $\forall r < R$, respectively, where*

$$\omega_P(R) = \limsup_{n\to\infty} \frac{\log \omega_{P_n}(R)}{n \log n}. \qquad (2.11)$$

In [28], the complex conformable derivative of a function f is defined as follows:

Definition 2.9. *Let $f:\mathbb{C}\to\mathbb{C}$ be a function. The complex conformable fractional derivative of f of order $\alpha \in (0,1]$ is given by*

$$T_\alpha(f)(z) = \lim_{\epsilon\to 0} \frac{f(z+\epsilon z^{1-\alpha}) - f(z)}{\epsilon}, \qquad (2.12)$$

Theorem 2.4. *Let f_1, f_2 be two complex conformable derivative of order $\alpha \in (0,1]$ at a point $z \in \mathbb{C}$. Then*

(i) $T_\alpha(a_1 f_1 + a_2 f_2) = a_1 T_\alpha(f_1) + a_2 T_\alpha(f_2)$ *for* $a_1, a_2 \in \mathbb{R}$,

(ii) $T_\alpha(z^p) = p z^{p-\alpha}$ *for* $p \in \mathbb{R}$,

(iii) $T_\alpha(f) = 0$, *where* $f(z)$ *is a constant function,*

(iv) $T_\alpha(f_1 f_2) = f_1 T_\alpha(f_2) + f_2 T_\alpha(f_1)$,

(v) $T_\alpha\left(\frac{f_1}{f_2}\right) = \frac{f_2 T_\alpha(f_1) - f_1 T_\alpha(f_2)}{{f_2}^2}$, $f_2(z) \neq 0$,

(vi) $T_\alpha(f_1)(z) = z^{1-\alpha}\frac{df_1}{dz}$, f_1 *is differentiable.*

Example 2.1. *Let* $\alpha \in (0,1]$. *Then*

(i) $T_\alpha(e^{cz}) = cz^{1-\alpha}e^{cz}, c \in \mathbb{C}$,

(ii) $T_\alpha(\sin bz) = bz^{1-\alpha}\cos bz\,, b \in \mathbb{C}$,

(iii) $T_\alpha(\cos bz) = -bz^{1-\alpha}\sin bz\,, b \in \mathbb{C}$.

3 Generating new bases using GCCD and GCCI

This section provides two new bases using GCCD and GCCI.

Definition 3.1. *Let* T_α *be a complex conformable derivative of order* $\alpha \in (0,1]$. *Then the generalized complex conformable derivative is given by*

$$G_N(D_\alpha) = \sum_{j=1}^{N} a_j D_\alpha^j, \tag{3.1}$$

where

$$D_\alpha^N = z^{(N-1)\alpha}T_\alpha^N, \quad \forall z \in \mathbb{C}. \tag{3.2}$$

As $T_\alpha^N = T_\alpha^{N-1}T_\alpha$, *it follows by Theorem 2.4 that*

$$\begin{aligned} T_\alpha z^n &= nz^{n-\alpha}, \\ T_\alpha^2 z^n &= n(n-\alpha)z^{n-2\alpha}, \\ T_\alpha^2 z^n &= n(n-\alpha)(n-2\alpha)z^{n-3\alpha}, \end{aligned}$$

□

$$T_\alpha^N z^n = \delta_{n,N} z^{n-N\alpha}, \tag{3.3}$$

where $\delta_{n,N} = n(n-\alpha)(n-2\alpha)\ldots[n-(N-1)\alpha]z^{n-N\alpha}$. *So*

$$G_N(D_\alpha)z^n = \left(\sum_{j=1}^{N} a_j \delta_{n,j}\right) z^{n-\alpha}. \tag{3.4}$$

Definition 3.2. *Let* I_α $(I_\alpha z^n = \frac{z^{n+\alpha}}{n+\alpha})$ *be a special case of Riemann improper integral* $(I_\alpha^\alpha(f)(z) = \int_a^z \frac{f(t)}{t^{1-\alpha}} dt)$ *and* $\alpha \in (0,1]$. *Then the generalized complex conformable integral is given by*

$$G_N(L_\alpha) = \sum_{j=1}^{N} b_j L_\alpha^j, \qquad (3.5)$$

where

$$L_\alpha^N = z^{(1-N)\alpha} I_\alpha^N, \quad \forall z \in \mathbb{C}. \qquad (3.6)$$

Since $I_\alpha^N = I_\alpha^{N-1} I_\alpha$, *we deduce that*

$$I_\alpha z^n = \frac{z^{n+\alpha}}{n+\alpha},$$

$$I_\alpha^2 z^n = \frac{z^{n+2\alpha}}{(n+\alpha)(n+2\alpha)},$$

$$I_\alpha^3 z^n = \frac{z^{n+3\alpha}}{(n+\alpha)(n+2\alpha)(n+3\alpha)},$$

□

$$I_\alpha^N z^n = \eta_{n,N} z^{n+N\alpha}, \qquad (3.7)$$

where $\eta_{n,N} = \frac{1}{(n+\alpha)(n+2\alpha)\dots(n+N\alpha)}$. *So*

$$G_N(L_\alpha) z^n = \left(\sum_{j=1}^{N} b_j \eta_{n,j} \right) z^{n+\alpha}. \qquad (3.8)$$

Definition 3.3. *Let* $\{P_n(z)\}$ *be a set of polynomials defined in (2.2). Then the corresponding generalized conformable derivative set* $\{P_n^{GD}(z)\}$ *is given by*

$$P_n^{GD}(z) = \sum_k p_{n,k}^{GD} z^{k-\alpha}, \qquad (3.9)$$

$$z^{n-\alpha} = \sum_k \pi_{n,k}^{GD} P_k^{GD}(z), \qquad (3.10)$$

where $p_{n,k}^{GD} = p_{n,k}\beta_{k,N}$, $\pi_{n,k}^{GD} = \frac{\pi_{n,k}}{\beta_{n,N}}$ *and* $\beta_{k,N} = \sum_{j=1}^{N} a_j \delta_{k,j}$ *for all* $k \in \mathbb{N}$.

Definition 3.4. *Let $\{P_n(z)\}$ be a set of polynomials defined in (2.2). Then the corresponding generalized conformable integral set $\{P_n^{GI}(z)\}$ is given by*

$$P_n^{GI}(z) = \sum_k p_{n,k}^{GI} z^{k+\alpha}, \tag{3.11}$$

$$z^{n+\alpha} = \sum_k \pi_{n,k}^{GI} P_k^{GI}(z), \tag{3.12}$$

where $p_{n,k}^{GI} = p_{n,k}\gamma_{k,N}$, $\pi_{n,k}^{GI} = \frac{\pi_{n,k}}{\gamma_{n,N}}$ and $\gamma_{k,N} = \sum_{j=1}^{N} b_j \eta_{k,j}$ for all $k \in \mathbb{N}$.

Theorem 3.1. *If the set $\{P_n(z)\}$ is a base of polynomials. Then $\{P_n^{GD}(z)\}$ and $\{P_n^{GI}(z)\}$ are bases of polynomials.*

Proof. The matrix of coefficients (resp., operators) of $\{P_n^{GD}(z)\}$ is given by $P^{GD} = \left(p_{n,k}^{GD}\right) = (p_{n,k}\beta_{k,N})$ (resp., $\Pi^{GD} = (\frac{\pi_{n,k}}{\beta_{n,N}})$). Then

$$P^{GD}\Pi^{GD} = \left(\sum_j p_{n,j}^{GD}\pi_{j,k}^{GD}\right) = \left(\sum_j p_{n,j}\pi_{j,k}\right) = I.$$

Similarly

$$\Pi^{GD}P^{GD} = \left(\sum_j \pi_{n,j}^{GD}p_{j,k}^{GD}\right) = \left(\frac{\beta_{k,N}}{\beta_{n,N}}\sum_j \pi_{n,j}p_{j,k}\right) = I.$$

Using Theorem 2.1, we conclude that $\{P_n^{GD}(z)\}$ is also a base.

Similarly, one can deduce that $\{P_n^{GI}(z)\}$ is a base, as $\{P_n(z)\}$ is a base.

4 Effectiveness of $P_n^{GD}(z)$ and $P_n^{GI}(z)$

This section discusses the effectiveness of $\{P_n^{GD}(z)\}$ and $\{P_n^{GI}(z)\}$ in Fréchet spaces.

Theorem 4.1. *If the base $\{P_n(z)\}$ is efffective for* □$[D(R)]$*, then the bases $\{P_n^{GD}(z)\}$ and $\{P_n^{GI}(z)\}$ are also, effective for* □$[D(R)]$.

Proof. Since $\{P_n(z)\}$ is a polynomial base, for all $r < R$ we have $\|P_n\|_r = \sup_{\bar{D}(r)}|P_n(z)|$ and $\|P_n^{GD}\|_r = sup_{\bar{D}(r)}|P_n^{GD}(z)|$.

As

$$\begin{aligned}\|P_n^{GD}\|_r &= \sup_{\bar{D}(r)} |P_n^{GD}(z)| \\ &= \sup_{\bar{D}(r)} \left| \sum_k p_{n,k} \beta_{k,N} z^{k-\alpha} \right| \\ &\leq \frac{\|P_n\|_r}{r^\alpha} \sum_k \beta_{k,N} \left(\frac{r}{R}\right)^k \\ &= \psi \|P_n\|_R, \qquad (4.1)\end{aligned}$$

where $\psi = \frac{1}{r^\alpha} \sum_k \beta_{k,N} \left(\frac{r}{R}\right)^k < \infty$, it follows by (2.7) and (4.1), that

$$\begin{aligned}\omega_{P_n^{GD}}(r) &= \sum_k \left|\pi_{n,k}^{GD}\right| \left\|P_k^{GD}\right\|_r \\ &\leq \frac{\psi}{\beta_{n,N}} \sum_k \left|\pi_{n,k}\right| \|P_k\|_R \\ &= \frac{\psi}{\beta_{n,N}} \omega_{P_n}(R). \qquad (4.2)\end{aligned}$$

This, together with (2.7), implies that

$$\lambda_{P^{GD}}(r) \leq \lambda_P(R). \qquad (4.3)$$

Since $\{P_n(z)\}$ is an effective base for □$[D(R)]$, we must have $\lambda_{P^{GD}}(r) < R$ for all $r < R$.

Now by Theorem 2.2, we also have that $\{P_n^{GD}(z)\}$ is also effective for □$[D(R)]$.

In a similar way, one can prove that $\{P_n^{GI}(z)\}$ is also effective for □$[D(R)]$.

Theorem 4.2. *The base* $\{P_n^{GD}(z)\}$ *is effective for* □ $[\bar{D}(R)]$, *if the following conditions hold:*

(i) $\{P_n(z)\}$ *is effective for* □ $[\bar{D}(R)]$,

(ii) $\lim_{n\to\infty} \{U_n\}^{\frac{1}{n}} = 1$, *where* U_n *is the highest degree of (2.1).*

Proof. Like in the proof of Theorem 4.1, we have $\|P_n\|_R = \sup_{\bar{D}(R)} |P_n(z)|$ and

$$\|P_n^{GD}\|_R = \sup_{\bar{D}(R)} |P_n^{GD}(z)|$$

$$= \sup_{\bar{D}(R)} \left| \sum_{k=0}^{u_n} p_{n,k} \beta_{k,N} z^{k-\alpha} \right|$$

$$\leq \frac{\|P_n\|_R}{R^\alpha} \sum_{k=0}^{u_n} \beta_{k,N}$$

$$\leq \frac{\|P_n\|_R}{R^\alpha} \beta_{u_n,N}(u_n + 1), \qquad (4.4)$$

where $u_n \leq U_n$.

This, together with (2.7) and (4.4), implies that

$$\omega_{P_n^{GD}}(R) = \sum_{k=0}^{U_n} \left|\pi_{n,k}^{GD}\right| \left\|P_k^{GD}\right\|_R$$

$$\leq \frac{1}{\beta_{n,N} R^\alpha} \sum_{k=0}^{U_n} \left|\pi_{n,k}\right| \|P_k\|_R \beta_{U_k,N}(U_k + 1)$$

$$\leq \frac{\beta_{U_n,N}}{\beta_{n,N} R^\alpha} (U_n + 1) \omega_{P_n}(R), \qquad (4.5)$$

and hence $\lambda_{P^{GD}}(R) \leq R$. But, since $\lambda_{P^{GD}}(R) \geq R \Rightarrow \lambda_{P^{GD}}(R) = R$, we have $\{P_n^{GD}(z)\}$ is effective for □ $[\bar{D}(R)]$.

The next example shows that the condition of (ii) in Theorem 4.2 is necessary.

Example 4.1. *Let* $\{P_n(z)\}$ *be a base of polynomials, defined as*

$$P_n(z) = \begin{cases} z^n, & n \text{ is even}, \\ z^n + z^\theta, \theta = 2^n, & n \text{ is odd}. \end{cases}$$

<u>If n is odd, then</u>

$$z^n = P_n(z) - P_\theta(z) \Rightarrow \omega_{P_n}(R) = R^n + 2R^\theta \Rightarrow \omega_{P_n}(1) = 3 \Rightarrow \limsup_{n\to\infty}\{\omega_{P_n}(1)\}^{\frac{1}{n}} = 1.$$

Hence $\{P_n(z)\}$ *is effective for* □ $[\bar{D}(1)]$.

The corresponding generalized complex conformable derivative base $\{P_n^{GD}(z)\}$, *is given by*

$$P_n^{GD}(z) = \begin{cases} \beta_{n,N} z^{n-\alpha}, & n \text{ is even}, \\ \beta_{n,N} z^{n-\alpha} + \beta_{\theta,N} z^{\theta-\alpha}, \theta = 2^n, & n \text{ is odd}. \end{cases}$$

If n is odd, then

$z^{n-\alpha}=\frac{1}{\beta_{n,N}}\left\{P_n^{GD}(z)-P_\theta^{GD}(z)\right\}\Rightarrow \omega_{P_n^{GD}}(R)=R^{n-\alpha}+2\left(\frac{\beta_{\theta,N}}{\beta_{n,N}}\right)R^{\theta-\alpha}$. Taking $R=1,N$

$=1,a_j=1\Rightarrow \omega_{P_n^{GD}}(1)=1+\frac{2^{n+1}}{n}\Rightarrow \text{limsup}_{n\to\infty}\left\{\omega_{P_n^{GD}}(1)\right\}^{\frac{1}{n}}$=2.

Hence $\{P_n^{GD}(z)\}$ *is not effective for* □ $[\bar{D}(1)]$.

Remark 4.1. *Theorem 4.2 is still true if we replace the base* $\{P_n^{GD}(z)\}$ *by* $\{P_n^{GI}(z)\}$.

Theorem 4.3. *The base* $\{P_n^{GD}(z)\}$ *and* $\{P_n^{GI}(z)\}$ *are effective for* □ $[D_+(R)]$, □ $[0]$ *and* □ $[\infty]$, *whenever the base* $\{P_n(z)\}$ *is effective for* □ $[D_+(R)]$, □ $[0]$ *and* □ $[\infty]$.

The proof of Theorem 4.3 is similar to [29] and the proof of Theorem 4.1.

Authors in [5,9] have shown that the Bessel polynomials $\{P_n(z)\}$ and general Bessel polynomials $\{Q_n(z)\}$ are effective for □ $[\bar{D}(R)]$. Furthermore, in [10] the authors proved the effectiveness of Chebyshev polynomials $\{T_n(z)\}$ for □ $[\bar{D}(1)]$. As an application of Theorem 4.2 and Remark 4.1, we obtain the following:

(i) The GCCD and GCCI of Bessel ($\{P_n^{GD}(z)\}$ *and* $\{P_n^{GI}(z)\}$) are effective for □ $[\bar{D}(R)]$.

(ii) The GCCD and GCCI of of general Bessel ($\{Q_n^{GD}(z)\}$ *and* $\{Q_n^{GI}(z)\}$) are effective for □ $[\bar{D}(R)]$.

(iii) The GCCD and GCCI of Chebyshev polynomial $\{T_n^{GD}(z)\}$ is effective for □ $[\bar{D}(1)]$.

5 Order, Type and T_ρ-property of the generalized bases

This section provides a study of the order $\rho_{P^{GD}}$ (resp., type $\tau_{P^{GD}}$) of $\{P_n^{GD}(z)\}$. It also investigates the T_ρ-property of $\{P_n^{GD}(z)\}$.

Theorem 5.1. *Let the base* $\{P_n(z)\}$ *be of order* ρ_P *and type* τ_P, *satisfying* $U_n=O[n]$. *Then the following axioms achieved:*

(i) $\{P_n^{GD}(z)\}$ *is of order* $\rho_{P^{GD}}\le\rho_P$ *and type* $\tau_{P^{GD}}\le\tau_P$ *whenever* $\rho_{P^{GD}}=\rho_P$,

(ii) $\{P_n^{GI}(z)\}$ *is of order* $\rho_{P^{GI}}\le\rho_P$ *and type* $\tau_{P^{GI}}\le\tau_P$ *whenever* $\rho_{P^{GI}}=\rho_P$.

Proof. From (4.5) and (2.9), we deduce that:

$$\lim_{R\to\infty}\limsup_{n\to\infty}\frac{\log\omega_{P_n^{GD}}(R)}{n\log n}\le\lim_{R\to\infty}\limsup_{n\to\infty}\frac{\log\frac{\beta_{U_n,N}}{\beta_{n,N}R^\alpha}(U_n+1)+\log\omega_{P_n}(R)}{n\log n},\qquad(5.1)$$

which shows that $\rho_{P^{GD}}\le\rho_P$.

Now using (2.10) and taking into account $\rho_{P^{GD}}=\rho_P$, we get

$$\lim_{R\to\infty}\frac{e}{\rho_{P^{GD}}}\limsup_{n\to\infty}\frac{\left\{\omega_{P_n^{GD}}(R)\right\}^{\frac{1}{n\rho_{P^{GD}}}}}{n}\le\lim_{R\to\infty}\frac{e}{\rho_P}\limsup_{n\to\infty}\frac{\left\{\omega_{P_n}(R)\right\}^{\frac{1}{n\rho_P}}}{n}. \tag{5.2}$$

This implies that $\tau_{P^{GD}}\le\tau_P$. So (i) is proved.
In a similar way, one can prove (ii).

The following example gives a base which has the same order and type of its corresponding GCCD base.

Example 5.1. *Let $\{P_n(z)\}$ be a base of polynomials in □ $[\overline{D}(R)]$ such that*

$$P_n(z)=\begin{cases}(5n+3)^n+z^n, & n\ne0,\\ 1, & n=0.\end{cases}$$

$z^n=P_n(z)-(5n+3)^nP_0(z)$, *so* $\omega_{P_n}(R)=2(5n+3)^n+R^n\Rightarrow$

$\lim_{R\to\infty}\limsup_{n\to\infty}\frac{\log\omega_{P_n}(R)}{n\log n}=1$ *and* $=\lim_{R\to\infty}\frac{e}{\rho}\limsup_{n\to\infty}\frac{\{\omega_{P_n}(R)\}^{\frac{1}{n\rho}}}{n}=5e$.

Constructing the corresponding GCCDB $\{P_n^{GD}(z)\}$ as follows:

$$P_n^{GD}(z)=\begin{cases}(5n+3)^n+\beta_{n,N}z^{n-\alpha}, & n\ne0,\\ 1, & n=0.\end{cases}$$

$z^{n-\alpha}=\frac{1}{\beta_{n,N}}\{P_n^{GD}(z)-(5n+3)^nP_0^{GD}(z)\}$, *so* $\omega_{P_n^{GD}}(R)=\frac{2(5n+3)^n}{\beta_{n,N}}+R^{n-\alpha}$. *Taking* $N=1, a_j=1$. *Then* $\omega_{P_n^{GD}}(R)=\frac{2(5n+3)^n}{n}+R^{n-\alpha}\Rightarrow\rho_{P^{GD}}=1$ *and* $\tau_{P^{GD}}=5e$.

We conclude that the two bases $\{P_n(z)\}$ and $\{P_n^{GD}(z)\}$ have the same order and type.

Next example shows that the condition $U_n=O[n]$ in Theorem 5.1 is necessary.

Example 5.2. *Let $\{P_n(z)\}$ be a base of polynomials in □ $[\overline{D}(R)]$ such that*

$$P_n(z)=\begin{cases}z^n, & n \text{ is even},\\ z^n+\frac{\kappa}{l^{2\kappa}}z^{2\kappa},\kappa=n^n,R=l, & n \text{ is odd}.\end{cases}$$

<u>If n is odd, then</u>

$z^n=P_n(z)-\frac{\kappa}{l^{2\kappa}}P_{2\kappa}(z)\Rightarrow\omega_{P_n}(R)=R^n+2\kappa\left(\frac{R}{l}\right)^{2\kappa}$, *but* $R=l\Rightarrow\omega_{P_n}(R)=R^n+2\kappa\Rightarrow\rho_P=\lim_{R\to\infty}\limsup_{n\to\infty}\frac{\log\{R^n+2\kappa\}}{n\log n}=1$.

Hence $\{P_n(z)\}$ is of order 1 in □ $[\overline{D}(R)]$.

Constructing the corresponding GCCDB $\{P_n^{GD}(z)\}$ as follows:

$$P_n^{GD}(z)=\begin{cases}\beta_{n,N}z^{n-\alpha}, & n \text{ is even},\\ \beta_{n,N}z^{n-\alpha}+\frac{\kappa}{l^{2\kappa}}\beta_{2\kappa,N}z^{2\kappa-\alpha},\kappa=n^n,R=l, & n \text{ is odd}.\end{cases}$$

<u>If n is odd, then</u>

$z^{n-\alpha}=\frac{1}{\beta_{n,N}}\left\{P_n^{GD}(z)-\frac{\kappa}{l^{2\kappa}}P_{2\kappa}^{GD}(z)\right\}\Rightarrow\omega_{P_n^{GD}}(R)=\beta_{n,N}R^{n-\alpha}+\left(\frac{2\kappa}{l^{2\kappa}}\right)\left(\frac{\beta_{2\kappa,N}}{\beta_{n,N}}\right)R^{2\kappa-\alpha}$.

Taking $R=l, N=1, a_1=1\Rightarrow\omega_{P_n^{GD}}(R)=nR^{n-\alpha}+4n^{2n-1}R^{-\alpha}\Rightarrow\rho_{P^{GD}}=\lim_{R\to\infty}\limsup_{n\to\infty}\frac{\log\{nR^{n-\alpha}+4n^{2n-1}R^{-\alpha}\}}{n\log n}=2$.

Hence $\{P_n^{GD}(z)\}$ is of order 2 in □ $[\overline{D}(R)]$.

Theorem 5.2. *The bases $\{P_n^{GD}(z)\}$ and $\{P_n^{GI}(z)\}$ have T_ρ-property in $\overline{D}(R)$, $R>0$, if the following conditions hold:*

(i) The base $\{P_n(z)\}$ *has* T_ρ*-property in* $\overline{D}(R),\ R>0$,

(ii) $\lim_{n\to\infty}\frac{\log U_n}{n\log n}=0$.

Proof. If $\omega_{P^{GD}}(R)$ is given by

$$\omega_{P^{GD}}(R)=\limsup_{n\to\infty}\frac{\log\omega_{P_n^{GD}}(R)}{n\log n}, \tag{5.3}$$

then (4.5), (2.11) and Theorem 5.2. we have

$$\omega_{P^{GD}}(R)\le\omega_P(R)\le\frac{1}{\rho}. \tag{5.4}$$

Likewise, one can prove the T_ρ-property for the base $\{P_n^{GI}(z)\}$.

The following example shows that the condition (ii) in Theorem (5.2) is necessary:

Example 5.3. *Let* $\{P_n(z)\}$ *be a base of polynomials in* □ $[\overline{D}(R)]$ **,** *such that*

$$P_n(z)=\begin{cases} z^n, & n\ is\ even,\\ z^n+hz^h, h=(2n)^{2n}, & n\ is\ odd.\end{cases}$$

If n is odd, then

$$z^n=P_n(z)-hP_h(z)\Rightarrow\omega_{P_n}(R)=R^n+2hR^h\Rightarrow\omega_{P_n}(1)=1+2h\Rightarrow\omega_P(1)=\limsup_{n\to\infty}\frac{\log\{1+2h\}}{n\log n}=2.$$

The base $\{P_n(z)\}$ *has* $T_{\frac{1}{2}}$ *for* □ $[\overline{D}(1)]$.

Constructing the corresponding GCCDB as follows:

$$P_n^{GD}(z)=\begin{cases}\beta_{n,N}\,z^{n-\alpha}, & n\ is\ even,\\ \beta_{n,N}z^{n-\alpha}+h\beta_{h,N}z^{h-\alpha}, h=(2n)^{2n}, & n\ is\ odd.\end{cases}$$

$z^{n-\alpha}=\frac{1}{\beta_{n,N}}\{P_n^{GD}(z)-hP_h^{GD}(z)\}\Rightarrow\omega_{P_n^{GD}}(R)=R^{n-\alpha}+2h\left(\frac{\beta_{h,N}}{\beta_{n,N}}\right)R^{h-\alpha}$.

Taking $R=1, N=1, a_j=1\Rightarrow\omega_{P_n^{GD}}(1)=1+\frac{2h^2}{n}=1+2^{4n+1}n^{4n-1}\Rightarrow\omega_{P^{GD}}(1)=\limsup_{n\to\infty}\frac{\log\{1+2^{4n+1}n^{4n-1}\}}{n\log n}=4$.

The base $\{P_n^{GD}(z)\}$ *has* $T_{\frac{1}{4}}$ *for* □ $[\overline{D}(1)]$.

This example shows that the two bases have not the same T_ρ*-property.*

In [8], the authors establish that Bernoulli polynomials $\{B_n(z)\}$ are of order 1 and type $\frac{1}{2\pi}$. Furthermore, Euler polynomial $\{E_n(z)\}$ are order 1 and type $\frac{1}{\pi}$. As an

Application of Theorem 5.1, we obtain the following:

(i) The GCCD and GCCI of Bernoulli ($\{B_n^{GD}(z)\}$ and $\{B_n^{GI}(z)\}$) are of order 1 and type $\frac{1}{2\pi}$.

(ii) The GCCD and GCCI of Euler ($\{E_n^{GD}(z)\}$ and $\{E_n^{GI}(z)\}$) are of order 1 and type $\frac{1}{\pi}$.

In [8], the authors demonstrate that Bernoulli polynomials $\{B_n(z)\}$ and Euler polynomials $\{E_n(z)\}$ exhibit the T_1-property. According to Theorem 5.2, we obtain:

(i) The GCCD and GCCI of Bernoulli ($\{B_n^{GD}(z)\}$ and $\{B_n^{GI}(z)\}$) have T_1-property.

(ii) The GCCD and GCCI of Euler ($\{E_n^{GD}(z)\}$ and $\{E_n^{GI}(z)\}$) have T_1-property.

6 Conclusion

In this study, we have established new polynomial bases in Fréchet spaces by employing generalized complex conformable derivatives and integrable mappings. These generated bases are regraded as a generalization of the work in [29]. Notably, when we set $j = 1, a_1 = 1, N = 1$, the previous studies become special cases of our work. We also have convergence criteria such as effectiveness, order, type and T_ρ-property are used for our study. Additionaly, this paper includes applications of Bessel, Bernoulli, Euler and Chebyshev polynomials are added to the paper.

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