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Representation in Fréchet Spaces of Hyperbolic Theta and Integral Operator Bases for Polynomials

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ABSTRACT

In this study, we present a novel definition for hyperbolic Theta operator bases (HTOBs) and hyperbolic integral operator bases (HIOBs) within the context of complex calculus. We employ the constructed HTOBs and HIOBs on a specific base of polynomials (BPs) across diverse convergence regions within Fréchet spaces. Consequently, we explore the correlation between the approximation properties of the resulting base and the original one. Furthermore, we derive insights into the T_ρ-property and the mode of increase of the polynomial bases as defined by HTOBs and HIOBs. The investigation extends to various bases of special polynomials, including Chebyshev, Bessel, Gontcharaff, Euler and Bernoulli, polynomials, ensuring the robustness and applicability of the obtained results.

1 INTRODUCTION

The approximation of an analytic function $f(z)$ as a basic series of the following type

$$a_0L_0(z) + a_1L_1(z) + a_2L_2(z) + \cdots, \tag{1.1}$$

where $L_0(z),\ L_1(z),\ L_2(z), \ldots,$ is a BPs, has been developed by the British Mathematician J. M. Whittaker in the early 1930$_s$ [39]. J. M. Whittaker and B. Cannon [15, 16, 40, 41] have obtained many results about approximation of analytic and entire functions by basic series (1.1).

Numerous specific instances of polynomial series have undergone thorough examination. Taylor's series stands out as the simplest case, with other notable examples including expansions from interpolation theory and series involving polynomials such as Hermite, Legendre, Legendre, Euler, Bernoulli, Chebyshev, and Gontcharoff.

Several scholars, including Makar [34], Mikhail [35], and Newns [37], have delved into the convergence properties of derivative and integral bases for a given set of BPs in a single complex variable within a disk centered at the origin. For multiple complex variables, as explored in [12, 20, 32, 33], representation domains extend to polycylindrical, hyperspherical, and hyperelliptical regions.

In [11, 46], the authors address this matter within the Clifford setting, known as hypercomplex derivative bases of Cliffordian polynomials, where approximation occurs in closed balls.
Recently, Hassan et al. [23] pioneered an exploration into the concept of BPs), primarily rooted in Clifford analysis and functional analysis. They formulated a comprehensive criterion to gauge the effectiveness, specifically the convergence properties, of BPs within Fréchet modules. Additionally, they provided practical applications showcasing the convergence properties of BPs in approximation theory. This pertains to the representation of special monogenic functions through an infinite series involving a sequence of special monogenic polynomials, within both closed and open ball contexts.
In a recent paper [24], the authors delved into the representation of analytic functions using complex conformable fractional derivative and integral bases within the framework of Fréchet space. These bases exhibit connections with special functions such as Bernoulli, Euler, Bessel, and Chebyshev polynomials. Furthermore, a separate investigation was conducted in [28], focusing on the approximation properties of monogenic functions through hypercomplex Ruscheweyh derivative bases in the Fréchet module. This exploration established links with Bernoulli special monogenic polynomials, Euler special monogenic polynomials, and Bessel special monogenic polynomials.
An intriguing research discovery is highlighted in [45], where the authors extended the well-known Whittaker-Cannon theorem in open hyperballs in $\mathbb{R}^{m+1}$by employing Hadamard's three-hyperballs theorem [7]. Specifically, the hypercomplex Cannon functions were proven to preserve the effectiveness properties of both Cannon and non-Cannon bases.
In [4], the authors presented an expansion of a specific monogenic function using generalized monogenic Bessel polynomials (GMBPs). Additionally, they demonstrated that the GMBPs serve as solutions to second-order homogeneous differential equations. However, several fundamental questions pertaining to the convergence properties of complex derivative and integral bases in Fréchet space still remain open. Similarly, crucial queries regarding the convergence properties of other hypercomplex derivative bases in Fréchet modules within Clifford analysis also persist.
Motivated by the aforementioned gaps in the literature, we introduce a novel definition of the Hyperbolic Theta operator and Hyperbolic integral operator within the realm of complex calculus. Subsequently, we apply these operators to a set of BPs to derive the HTOBPs and HIOBPs. The majority of the theorems established in this study revolve around the representation of analytic and entire functions through infinite series composed of HTOBPs and HIOBPs.

The structure of the current work is outlined as follows. Section 2 is dedicated to recalling the most commonly used notations and results. In Section 3, we introduce a new operator, the Hyperbolic Theta operator, and the Hyperbolic integral operator to derive the HTOBPs and HIOBPs. Section 4 presents the results indicating the effectiveness properties of the HTOBPs and HTOBPs. This section also includes the demonstration of some bases of special classes of polynomials that fulfill the established results. Section 6 contains the proofs for all the theorems. In Section 7, future work is discussed as an integral part of this paper.

2 Preliminaries

This section offers an overview of key definitions, notations, and results essential for providing background information in this paper. The concepts covered include: bases, basic series, , effectiveness, type, order, and Fréchet space (refer to [8, 14, 17, 18, 23]).

Definition 2.1. *Let Y be a vector space over $\mathbb{C}$. A real-valued function $\|.\|\colon Y \to \mathbb{R}$ is called a semi-norm if it satisfies the following conditions:*

(a) $\|t\| \geq 0 \quad \forall s \in Y,$

(b) $\|t+s\| \leq \|t\| + \|s\| \quad \forall t, s \in Y,$

(c) $\|bt\| = |b|\|t\| \quad \forall t \in Y$ *and* $\forall\, b \in \mathbb{C}$. *A norm on Y is a semi-norm with the addition of the condition:*

(d) $\|t\| = 0 \Rightarrow t = 0.$

A family of semi-norms in $\mathbb{C}$ is useful to enable us to define a Fréchet space (F-space) E as follows:

Definition 2.2. *Consider a family of semi-norms $S = (\|.\|_k)_{k\geq 0}$ where $k < l$ then $\|t\|_k \leq \|t\|_l$; for $t \in E$. A space E over $\mathbb{C}$ is called an F-space if E is a complete Hausdorff topological vector space such that $W \subset E$ is open if and only if $\forall s \in W$, there exist $\varepsilon > 0$, $N \geq 0$ such that $\{s \in E \colon \|s - t\|_k \leq \varepsilon\} \subset W$, for all $k \leq N$.*

Examples on Fréchet space*:*

The classes $A_{[D(R)]}$, $A_{[\overline{D}(R)]}$, $A_{[D_+(R)]}$ and $A_{[0^+]}$, of functions which are analytic in open disk $D(R)$, in closed disk, $\overline{D}(R)$, in $D_+(R)$, at the origin, respectively with the family of *semi-norm makes the classes* $A_{[D(R)]}$, $A_{[\bar{D}(R)]}$, $A_{[D_+(R)]}$ *and* $A_{[0^+]}$ *into an F-space E. Where* $D_+(R)$ *any open disk enclosing the closed disk* $\bar{D}(R)$*. Also, the class* $A_{[\infty]}$ *of entire functions on* $\mathbb{C}$ *with the family of semi-norm make* $A_{[\infty]}$ *into an F-space E* (see [23, 24, 28]).

Suppose that $\{L_n(z)\}$ *is a base of an F-space E and*

$$L_n(z) = \sum_k l_{n,k} z^k, \tag{2.1}$$

$$z^n = \sum_k \pi_{n,k} L_k(z), \tag{2.2}$$

$$\mathcal{X}(L_n, R) = \sum_k |\pi_{n,k}| \|L_k\|_R, \tag{2.3}$$

$$\|L_n\|_R = \sup_{\bar{D}(R)} |L_n(z)|, \tag{2.4}$$

$$\mathcal{X}(L, R) = \limsup_{n\to\infty} \{\mathcal{X}(L_n, R)\}^{\frac{1}{n}}. \tag{2.5}$$

The matrices $L = (l_{n,k})$ *and* $\Pi = (\pi_{n,k})$ *are called the matrix of coefficients and the matrix of operators of the base* $\{L_n(z)\}$*. The set* $\{L_n(z)\}$ *is base (see [23, 37]), iff*

$$L\Pi = \Pi L = I. \tag{2.6}$$

Cauchy's inequality [23,37] of a base (2.1) takes the form

$$|p_{n,i}| \le \frac{\|P_n\|_R}{R^i}, \tag{2.7}$$

Now, suppose That

$$\rho_L = \lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log \mathcal{X}(L_n, R)}{n \log n}, \tag{2.8}$$

and

$$\tau_L = \lim_{R\to\infty} \frac{e}{\rho_P} \limsup_{n\to\infty} \frac{\{\mathcal{X}(L_n, R)\}^{\frac{1}{n\rho}}}{n}. \tag{2.9}$$

Where ρ_L and τ_L gives the order and type of the base and bear its importance due to the fact that the base represents in the whole plane every integer function of type less than $\frac{1}{\tau_L}$ and order less than $\frac{1}{\rho_L}$ in any finite disk (see [21, 22, 39, 41).We refer to work by [10, 43, 44] regarding the type and order of BPs.

For any element $g(z) = \sum_n a_n(g)\, z^n$ of an F-space E, substitute z^n from (2.1), we get the basic series

$$g(z) \sim \sum_k \Pi_n(g) L_n(z), \tag{2.10}$$

where

$$\Pi_n(g) = \sum_k a_k(g) \pi_{k,n}. \tag{2.11}$$

Definition 2.3. *The basic series (2.10)) is said to represents* $g \in E$ *when it converges uniformly to* $g \in E$. *A base* $\{L_n(z)\}$ *is said to be effective for an F-space E when the basic series (2.10) represents* $g \in E$.

From Definition 2.5 put the F-space $E = A_{[\bar{D}(R)]}$. Thus $\{L_n(z)\}$ is effective for $A_{[\bar{D}(R)]}$ if the basic series represents in $A_{[\bar{D}(R)]}$ every analytic function $g(z) \in A_{[\bar{D}(R)]}$ which is analytic in $\bar{D}(R)$. Similar definitions can be applied for the classes $A_{[D(R)]}$, $A_{[D_+(R)]}$, $A_{[0^+]}$ and $A_{[\infty]}$.

Results on the effectiveness of bases (see [5, 8, 23, 37]).

Theorem 2.1. *Let* $\{L_n(z)\}$ *for* $n \geq 0$ *be a base.* $\{L_n(z)\}$ *is effective for* $A_{[\bar{D}(R)]}$ *or* $A_{[D(R)]}$ *or* $A_{[D_+(R)]}$ *or* $A_{[\infty]}$ *or* $A_{[0^+]}$ *if and only iff* $\mathcal{X}(L,R) = R$ *or* $\mathcal{X}(L,r) < R \ \forall r < R$ *or* $\mathcal{X}(L,R^+) = R$ *or* $\mathcal{X}(L,R) < \infty \ \forall R < \infty$ *or* $\mathcal{X}(L,0^+) = 0$.

The author in [19] defined the T_ρ-property of BPs of one complex variable in a closed disk. In addition, the T_ρ-property is defined in polycylinderical regions [31]. T_ρ-property is also known in the case of Clifford setting in [3].

Definition 2.4. *A base* $\{L_n(z)\}$ *has the property* T_ρ *in* $\bar{D}(R)$, *if the basic series (2.10) represents in* $\bar{D}(R)$ *every integer function of order less than ρ.*

Let $\mathcal{X}_L(R) = \limsup_{n\to\infty} \frac{\log \mathcal{X}(L_n,R)}{n \log n}$.

In fact [19] proved the following result on the T_ρ-property in $\bar{D}(R)$.

Theorem 2.2. *Let* $\{L_n(z)\}$ *be a* BPs. *The base* $\{L_n(z)\}$ *to have* T_ρ-property *in* $\overline{D}(R)$ *iff* $\mathcal{X}_L(R) \le \frac{1}{\rho}$.

An exhaustive study of BPs in Clifford and complex analysis are to be found in [6, 9 , 13, 25, 27, 29, 38, 42, 44, 47]. Also, the study of BPs in complex conformable fractional derivative is present in the papers [26, 30, 48, 49].

3 HTOBs and HIOBs

The hyperbolic Theta operator $\cosh \Theta$ (HTOC) is a differential operator defined by the power series $\sum_{j=0}^{\infty} \frac{\Theta^{2j}}{(2j)!}$, where $\Theta = \mathrm{z}\frac{\mathrm{d}}{\mathrm{dz}}$ is called the Theta operator, also called the homogeneity operator, because its eigenfunctions are the monomials in z:

$$\Theta(z^n) = \mathrm{n}z^n, \mathrm{n} = 0,1,2, \ldots,$$

$$\Theta^j(z^n) = \mathrm{n}^j z^n, \mathrm{j} = 1,2, \ldots,$$

Where $\Theta^j = \Theta\Theta^{\mathrm{j}-1}$

Now, we would like to know its effect on the function $f(z)$. First let's see its effect in the function z^n.

$$\cosh \mathrm{a}\Theta\, z^n = \sum_{j=0}^{\infty} \frac{a^{2j}\Theta^{2j}}{(2j)!} z^n = \sum_{j=0}^{\infty} \frac{(an)^{2j}}{(2j)!} z^n = \cosh \mathrm{an}\, z^n.$$

Now, we can use this result to see the effect on a function $f(z)$. If the function $f(z)$ has a power series at $z = 0$, then

$\cosh \mathrm{a}\Theta\, f(z) = \cosh \mathrm{a}\Theta \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n \cosh \mathrm{a}\Theta\, \mathrm{z}^{\mathrm{n}} = \sum_{n=0}^{\infty} a_n \cosh \mathrm{an}\, \mathrm{z}^{\mathrm{n}} = \frac{f(e^{a}z)+f(e^{-a}z)}{2}$.

The HTOC of certain functions

(1) $\cosh \mathrm{a}\Theta(\mathrm{z}^{\mathrm{n}}) = \cosh \mathrm{anz}^{\mathrm{n}}$

(2) $\cosh \mathrm{a}\Theta(1) = 1$

(3) $\cosh \mathrm{a}\Theta(\sin z) = \frac{\sin(e^{a}z)+\sin(e^{-a}z)}{2}$

(4) $\cosh \mathrm{a}\Theta(\cos z) = \frac{\cos(e^{a}z)+\cos(e^{-a}z)}{2}$.

The HTOC is linear i.e.

(i) $\cosh \Theta(f + g) = \cosh \Theta(f) + \cosh \Theta(g)$

(ii) $\cosh \Theta(cf) = c \cosh \Theta(f)$.

By applying the operator $\cosh \Theta$ into (2.1), we define the HTOBs as follows:

Definition 3.1. Suppose that $\{L_n(z)\}$ *is a base. The* HTOBs *is defined as:*

$$\cosh \Theta\, (L_n(\mathrm{z})) = \sum_k l_{n,k} \cosh k\, z^k. \tag{3.1}$$

For short, we write $\{\cosh \Theta\, (L_n(z))\} = \{\mathcal{H}_n^{\Theta}(z)\}$.

The following definition for hyperbolic integral operator $\cosh I$ (HIOC) of a function f, where the integral operator I is defined by

$$I = \frac{1}{z}\int_0^z dz$$

and

$$I(z^n) = \frac{1}{n+1} z^n, \qquad n = 0,1,2,\ldots,$$

$$I^j(z^n) = \left(\frac{1}{n+1}\right)^j z^n, \qquad j = 1,2,\ldots,$$

where $I^j = II^{j-1}$.

Let us elaborate on the HIOC. It is defined by the power series $\sum_{j=0}^{\infty} \frac{I^{2j}}{(2j)!}$. First let's see its effect in the function z^n.

$$\cosh aIz^n = \sum_{j=0}^{\infty} \frac{a^{2j} I^{2j}}{(2j)!} z^n = \sum_{j=0}^{\infty} \frac{\left(\frac{a}{n+1}\right)^{2j}}{(2j)!} z^n = \cosh\left(\frac{a}{n+1}\right) z^n.$$

Now, we would like to know its effect on the function $f(z)$, where

$$\cosh aI\, f(z) = \cosh aI \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n\, cosh\left(\frac{a}{n+1}\right) z^n,$$

where the series is uniformly convergent.

By applying the operator cosh I into (2.1), we define the HIOBs as follows:

Definition 3.2. *Let* $\{L_n(z)\}$ *be a base. The* HIOBs *is defined as:*

$$\cosh I\, (L_n(z)) = \sum_k l_{n,k} \cosh\left(\frac{1}{k+1}\right) z^k. \tag{3.2}$$

For short, we write $\{\cosh I\ (L_n(z))\} = \{\mathcal{H}_n^I(z)\}$.

Now, we give two examples that explain the representation of analytic functions by basic series of polynomials:

Example 3.1. *Let* $L_n(z) = \begin{cases} 1, & n \geq 0, \\ 1 + z^n, & n \geq 1, \end{cases}$

we can write z^n *as follows* $z^n = L_n(z) - L_0(z)$

i.e. the representation is available and

$$\pi_{n,0} = 1, \qquad \pi_{n,k} = 0 \ \ for\ k \neq 0, n. \tag{3.3}$$

Hence $\mathcal{X}(L_n, R) = 2 + R^n$.

Taking $R = 1, \mathcal{X}(L_n, 1) = 3,$ and $\mathcal{X}(L, 1) = \operatorname{limsup}_{n\to\infty}\{\mathcal{X}(L_n, 1)\}^{\frac{1}{n}} = 1$.

Therefore $\mathcal{X}(L, 1) = 1$ and $\{L_n(z)\}$ is effective for $H_{[\overline{D}(1)]}$.

Suppose now that $f(z) = e^z$ is function analytic in $\overline{D}(1)$, then the basic series $\sum_{n=0}^{\infty} \Pi_n(g) L_n(z)$ represents this function in $\overline{D}(1)$. To find the actual form of the basic series we substitute from (3.3) in (2.10) to obtain $e^z \sim 2 - e + \sum_{n=1}^{\infty} \frac{L_n(z)}{n!}$.

Example 3.2. *Consider the BPs* $\{L_n(z)\}$ *defined by*

$$L_n(z) = \begin{cases} 1, & n \geq 1 \\ z^n - nz^n, & n \geq 1. \end{cases}$$

It is clear that $z^n = n! \sum_{j=0}^{n} L_j(z)$.

Hence $\mathcal{X}(L_n, R) = n! \sum_{j=0}^{n} \|L\|_j \geq n! \, \|L\|_0 = n!$.

Therefore $\mathcal{X}(L, R) = \operatorname{limsup}_{n\to\infty}\{\mathcal{X}(L_n, R)\}^{\frac{1}{n}} = \operatorname{limsup}_{n\to\infty}\{n!\}^{\frac{1}{n}} = \infty$,

and $\{L_n(z)\}$ is cannot effective for $A_{[\bar{D}(R)]}$ for any value of R. Then, we can define a function analytic in $\bar{D}(R)$ not represented by the basic series (2.10) in $\bar{D}(R)$. The formula (2.11) for the coefficients of basic series (2.10) takes the form

$$\Pi_n(g) = -\sum_{k=0}^{\infty} f^{(k)}(0).$$

Applying this to e^z an analytic function in $\bar{D}(R)$, we obtain $\Pi_n(g) = -(1 + 1 + \cdots)$. Whereas relation (2.11) fails to know the basic series coefficients (2.10). This leads to the fact that the basic series does not exist, and is not represent by e^z.

Now, we give an example that explains the approximation of analytic functions by series of HTOBPs:

Example 3.3. *Consider the HTOBPs* $\{C_n^{\Theta}(z)\}$ *of base of polynomials* $\{L_n(z)\}$ *defined in Example 3.1 as follows*

$$C_n^{\Theta}(z) = \begin{cases} 1, & n = 0, \\ 1 + \cosh n \, z^n, & n \geq 1. \end{cases}$$

We may proceed very similar as above in Example 3.1 to obtain that the basic series approximate by $\frac{e^{ez}+e^{z/e}}{2}$ as follows: $\frac{e^{ez}+e^{z/e}}{2} \sim 2-e+\sum_{n=1}^{\infty}\frac{C_n^{\Theta}(z)}{n!}$,

4 Main results

The main results of the present work are formulated in the following theorems.

Theorem 4.1. *If* $\{L_n(z)\}$ *is a base, then the sets* $\{\mathcal{H}_n^{\Theta}(z)\}$ *and* $\{\mathcal{H}_n^{I}(z)\}$ *are also bases.*

Theorem 4.2. *If* $\{L_n(z)\}$ *is a base effective for the spaces* $A_{[D(R)]}$ *or* $A_{[D_+(R)]}$ *or* $A_{[\infty]}$ *or* $A_{[0^+]}$. *Then its HTOBs* $\{\mathcal{H}_n^{\Theta}(z)\}$ *and HIOBs* $\{\mathcal{H}_n^{I}(z)\}$ *are effective in the same spaces of effectiveness of the original base*$\{L_n(z)\}$.

Suppose that $\{L_n(z)\}$ *is a BPs satisfying the condition:*

$$\lim_{n\to\infty}\frac{D_n}{n}=1, \tag{4.1}$$

where D_n *is the highest degree in representation (2.2).*

Theorem 4.3.

If $\{L_n(z)\}$ *is a BPs effective for the space* $A_{[\bar{D}(R)]}$ *, and the condition (4.1) is satisfied. Then its HTOBPs* $\{\mathcal{H}_n^{\Theta}(z)\}$ *is effective in the same space of effectiveness of the original base* $\{L_n(z)\}$.

The fact that the condition (4.1) cannot be dropped is illustrated below by Example 6.1 in Section 6.

Let $\{L_n(z)\}$ be a BPs of type τ_P and order ρ_P and HTOBPs $\{\mathcal{H}_n^{\Theta}(z)\}$ is of type $\tau_{\mathcal{H}^{\Theta}}$ and order $\rho_{\mathcal{H}^{\Theta}}$.

Suppose that $\{L_n(z)\}$ *satisfying the condition:*

$$D_n = O[n]. \tag{4.2}$$

Theorem 4.4. Let $\{L_n(z)\}$ *is a BPs of type* τ_L *and order* ρ_L *and satisfying the condition (4.2). Then HTOBPs* $\{\mathcal{H}_n^{\Theta}(z)\}$ *is of type* $\tau_{\mathcal{H}^{\Theta}} \leq \tau_P$ *and order* $\rho_{\mathcal{H}^{\Theta}} \leq \rho_P$ *whenever* $\rho_{\mathcal{H}^{\Theta}} = \rho_P$, *both bounds of the inequality are attainable.*

The fact that the condition (4.2) cannot be dropped is illustrated below by Example 6.3 in Section 6.

Suppose that $\{L_n(z)\}$ *satisfying the condition:*

$$D_n = o[\mathrm{n}\log n] \tag{4.3}$$

Theorem 4.5. *If* $\{L_n(z)\}$ *is a BPs have property* T_ρ *in* $\overline{D}(R), R > 0$ *and satisfying the condition (4.3). Then its HTOBPs* $\{\mathcal{H}_n^{\Theta}(z)\}$ *have the same property.*

The fact that the condition (4.3) cannot be dropped is illustrated below by Example 6.4 in Section 6.

Now, suppose that $\{L_n(z)\}$ *be a simple BPs* $(D_n = n)$. *A*pplying Theorems 4.3, 4.4 and 4.5, we get the following results:

Corollary 4.1. If $\{L_n(z)\}$ *is effective for the space* $A_{[\bar{D}(R)]}$. *Then its HTOBPs* $\{\mathcal{H}_n^{\Theta}(z)\}$ *is effective in the same space of effectiveness of the original base.*

Corollary 4.2. *If* $\{L_n(z)\}$ *is of type* τ_P *and order* ρ_P. *Then its HTOBPs* $\{\mathcal{H}_n^{\Theta}(z)\}$ *is of type* $\tau_{\mathcal{H}^{\Theta}} \leq \tau_P$ *and order* $\rho_{\mathcal{H}^{\Theta}} \leq \rho_P$ *whenever* $\rho_{\mathcal{H}^{\Theta}} = \rho_P$

Corollary 4.3. *If* $\{L_n(z)\}$ *have property* T_ρ *in* $\bar{D}(R), R > 0$. *Then its HTOBPs* $\{\mathcal{H}_n^{\Theta}(z)\}$ *al have the same* T_ρ*- property.*

Important note

If the base $\{\mathcal{H}_n^{\Theta}(z)\}$ is replaced by the base $\{\mathcal{H}_n^{I}(z)\}$, we confirm that Theorems 4.3, 4.4 and 4.5 will be still true with not necessary conditions (4.1), (4.2) and (4.3) in Theorems 4.3, 4.4 and 4.5, respectively for the base $\{\mathcal{H}_n^{I}(z)\}$.

5 Applications

Firstly: As an applications of Theorem 4.3

(1) We consider the Bessel polynomials $\{L_n(z)\}$ and the general Bessel polynomials$\{M_n(z)\}$. The authors [2, 4] show that the bases $\{P_n(z)\}$ and $\{Q_n(z)\}$ are effective for $A_{[\bar{D}(R)]}$.

(2) We consider the Chebychev polynomials $\{\mathfrak{I}_n(z)\}$. The authors [5] show that the Chebyshev polynomial $\{\mathfrak{I}_n(z)\}$ is effective for $A_{[\bar{D}(1)]}$.

(3) The authors [1], considered the base of Gontcharaff polynomials $\{\mathcal{G}_n(z)\}$ (B$\mathcal{G}$Ps), associated with a given set (at^n) of points, are the polynomials defined by

$$\mathcal{G}_n(z) = \begin{cases} 1, & n = 0 \\ \mathcal{G}_n(z; a, at, at^2, \dots, at^{n-1}) = \int_a^z ds_1 \int_{at}^{s_1} ds_2 \dots \int_{at^{n-1}}^{s_{n-1}} ds_n\,, & n \geq 1. \end{cases}$$

Where a and t are given complex numbers.

When $|t| < 1$, The authors [1] has shown that the B$\mathcal{G}$Ps $\{\mathcal{G}_n(z)\}$ is effective for $A_{[\bar{D}(R)]},\ R \geq |a|$.

According to Theorem 4.3, the following results follows:

Corollary 5.1. *The HTOC and HIOC of Bessel polynomials* $\{\cosh \Theta L_n(z)\}$ *and* $\{cosh\, I L_n(z)\}$ *are effective in the same space of effectiveness of the original* $\{L_n(z)\}$.

Corollary 5.2. *The HTOC and HIOC of general Bessel polynomials* $\{cosh\, \Theta M_n(z))\}$ *and* $\{cosh\, I M_n(z)\}$ *are effective in the same space of effectiveness of the original* $\{M_n(z)\}$.

Corollary 5.3. *The HTOC and HIOC of Chebyshev polynomials* $\{\cosh\ \Theta \mathfrak{I}_n(z)\}$ *and* $\{\cosh I \mathfrak{I}_n(z)\}\}$ *are effective in the same space of effectiveness of the original* $\{\mathfrak{I}_n(z)\}$.

Corollary 5.4. *The HTOC and HIOC of the Gontcharaff polynomials* $\{cosh\, \Theta \mathcal{G}_n(z)\}$ *and* $\{\cosh I \mathcal{G}_n(z)\}$ *are effective in the same space of effectiveness of the original* $\{\mathcal{G}_n(z)\}$.

Secondly: As an application of Theorems 4.4

The authors [22] show that the Euler polynomials $\{E_n(z)\}$ is of order 1 and type $\frac{1}{\pi}$ and the Bernoulli polynomials $\{B_n(z)\}$ is of order 1 and type $\frac{1}{2\pi}$.

When $|t| = 1$, The author [36] has show that the B$\mathcal{G}$Ps $\{\mathcal{G}_n(z)\}$ will be of order 1 and type $\frac{|a|}{\tau}$, where τ is the modulus of a zero of the function

$$u(z) = \sum_{n=0}^{\infty} t^{\frac{1}{2}n(n-1)} \frac{z^n}{n!},$$

of least modulus.

According to Theorem 4.4, the following results follows:

Corollary 5.5. *The HTOC and HIOC of Bernoulli polynomials* $\{cosh\, \Theta B_n(z)\}$ *and* $\{cosh\, I B_n(z)\}$ *have the type and order of the original base* $\{B_n(z)\}$.

Corollary 5.6. *The HTOC and HIOC of Euler polynomials* $\{cosh\, \Theta E_n(z)\}$ *and* $\{cosh\, I E_n(z)\}$ *have the type and order of the original base* $\{E_n(z)\}$.

Corollary 5.7. *The HTOC and HIOC of Gontcharaff polynomials* $\{cosh\, \Theta \mathcal{G}_n(z)\}$ *and* $\{cosh\, I \mathcal{G}_n(z)\}$ *have the type and order of the original base* $\{\mathcal{G}_n(z)\}$.

Remark 5.1. *When* $t = 1$, *it is clear that* $\mathcal{G}_n(z) = \frac{(z-a)^n}{n!}$.

Thirdly: As an application of Theorem 4.5

The Bernoulli polynomials $\{B_n(z)\}$, Euler polynomials $\{E_n(z)\}$ and the Gontcharaff polynomials $\{\mathcal{G}_n(z)\}$ have property T_1 (see [22, 36]).

According to Theorem 4.5, the following results follows:

Corollary 5.8. *The HTOC and HIOC of Bernoulli polynomials* $\{cosh\,\Theta B_n(z)\ \}$ *and* $\{cosh\,I B_n(z)\ \}\ \}$ *have the same property* T_1 *of the original base* $\{B_n(z)\}$..

Corollary 5.9. *The HTOC and HIOC of Euler polynomials* $\{cosh\,\Theta E_n(z)\ \}$ *and* $\{cosh\,I E_n(z)\ \}$ *have the same property* T_1 *of the original base* $\{E_n(z)\}$.

Corollary 5.10. *The HTOC and HIOC of Gontcharaff polynomials* $\{cosh\,\Theta \mathcal{G}_n(z)\ \}$ *and* $\{cosh\,I \mathcal{G}_n(z)\ \}$ *have the same property* T_1 *of the original base* $\{\mathcal{G}_n(z)\}$.

Now, suppose that $g_N(\Theta)$ and $g_N(I)$ are polynomials of the Theta operator Θ and the integral operator I such that $g_N(\Theta) = \sum_{j=0}^{N} g_j \Theta^j$ and $g_N(I) = \sum_{j=0}^{N} g_j I^j$, j be a finite positive integer and g_j are constant is not equal zero.

It is clear that Theorems 4.3, 4.4 and 4.5 will be still true when we replace the bases $\{cosh\,\Theta(L_n(z))\ \}$ and $\{cosh\,I(L_n(z))\ \}$ by the bases $\{cosh\ g_N(\Theta)(L_n(z))\}$ and $\{cosh\ g_N(I)(L_n(z))\}$, respectively.

6 Proofs of results

proof of Theorem 4.1. At first construct the coefficient matrix $\mathcal{H}^{\Theta}$. Differentiation (2.1) by $\cosh\Theta$, we obtain

$$\cosh \Theta \left(L_n(z)\right) = \mathcal{H}^{\Theta}(z) = \sum_k cosh\, k\; l_{n,k} z^k,$$

where

$$\mathcal{H}^{\Theta} = \left(\mathcal{H}^{\Theta}_{n,k}\right) = \left(\cosh k\; l_{n,k}\right).$$

To get the operator matrix Π^{Θ}. By effecting by $\cosh \Theta$ on both sides of of (2.2), we get

$$z^n = \frac{1}{\cosh n} \sum_k \pi_{n,k} \mathcal{H}^{\Theta}(z),$$

where

$$\Pi^{\Theta} = \left(\Pi^{\Theta}_{n,k}\right) = \left(\frac{1}{\cosh n} \pi_{n,k}\right).$$

Subsequently,

$$\mathcal{H}^{\Theta} \Pi^{\Theta} = \left(\sum_k \mathcal{H}^{\Theta}_{n,k} \Pi^{\Theta}_{k,h}\right) = \left(\sum_k l_{n,k} \pi_{k,h}\right) = \left(\delta_{n,h}\right) = I.$$

Also, we have

$$\Pi^{\Theta} \mathcal{H}^{\Theta} = \left(\sum_k \Pi^{\Theta}_{k,h} \mathcal{H}^{\Theta}_{n,k}\right) = \left(\sum_k \frac{1}{\cosh n} \pi_{n,k} \cosh h\, l_{n,k}\right) = \left(\frac{\cosh h}{\cosh n} \delta_{n,h}\right) = I.$$

From (2.6), we infer that the set $\{\mathcal{H}^{\Theta}(z)\}$ is a base. Similarly, we can also prove that the set $\{\mathcal{H}^{I}\,(z)\}$ is a base.

Proof of Theorem 4.2. Let $\| \mathcal{H}_n^{\Theta} \|_r$ is the maximum moduli of $\{\mathcal{H}^{\Theta}(z)\}$ on the closed disk $\overline{D}(r)$, then

$$\| \mathcal{H}_n^{\Theta} \|_r = \sup_{\overline{D}(r)} |\mathcal{H}_n^{\Theta}(z)|$$

$$\leq \sup_{\overline{D}(r)} \left| \sum_j l_{n,j} \cosh j\, z^j \right|$$

$$\leq \sum_j \frac{\|L_n\|_R}{R^j} \cosh j\, r^j$$

$$\leq \|L_n\|_R \sum_j e^j \left(\frac{r}{R}\right)^j$$

$$\leq e^{d_n} \|L_n\|_R \sum_j \left(\frac{r}{R}\right)^j$$

$$= \emptyset(r, R) e^{d_n} \|L_n\|_R$$

$$= K_1 e^{d_n} \|L_n\|_R \quad for\ all\ r < R \tag{6.1}$$

where $K_1 = \emptyset(r, R) = \sum_{j=0}^{\infty} \left(\frac{r}{R}\right)^j < \infty$ and $\mathrm{d_n}$ is the degree of BP $L_n(z)$, $d_n \leq D_n$. It follows from (2.3) and (6.1) that

$$\mathcal{X}(\mathcal{H}_n^{\Theta}, r) = \sum_k \left|\Pi_{n,k}^{\emptyset}\right| \|\mathcal{H}_k^{\Theta}\|_r$$

$$\leq \frac{K_1}{\cosh n} \sum_k \left|\pi_{n,k}\right| e^{d_k} \|L_k\|_R$$

$$\leq \frac{K_1}{cosh\, n} e^{D_n} \sum_k \left|\pi_{n,k}\right| \|L_k\|_R$$

$$\leq 2K_1 e^{D_n - n}\mathcal{X}(L_n, R). \qquad (6.2)$$

Thus (2.5) and (6.2) yield

$$\mathcal{X}(\mathcal{H}^{\Theta}, r) \leq \mathcal{X}(L, R) \qquad for\ all\ r < R. \qquad (6.3)$$

Suppose now that $\{L_n(z)\}$ is effective for $A_{[D(R)]}$. From Theorem 2.1, we get

$$\mathcal{X}(L, r) < R \qquad for\ all\ r < R. \qquad (6.4)$$

There exists a number r_1 such that $r < r_1 < R$. In view of (6.3) and (6.4), we have

$$\mathcal{X}(\mathcal{H}^{\Theta}, r) \leq \mathcal{X}(L, r_1) < R \qquad for\ all\ r < R,$$

and $\{\mathcal{H}^{\Theta}(\mathrm{z})\}$ is effective for $A_{[D(R)]}$. We can proceed very similar as above to prove that $\{\mathcal{H}^{\Theta}(\mathrm{z})\}$ is effective for $A_{[D_+(R)]}$ *or* $A_{[\infty]}$ *or* $A_{[0^+]}$ *(see [24, 28]). Also,* using the same steps, we can show that $\{\mathcal{H}^{I}(z)\}$ is effective for the same spaces.

Proof of Theorem 4.3. The maximum moduli of $\{\mathcal{H}^{\Theta}(z)\}$ on the closed disk $\overline{D}(R)$, is given by

$$\begin{aligned}\| \mathcal{H}_n^{\Theta} \|_R &= \sup_{\overline{D}(R)} |\mathcal{H}_n^{\Theta}(z)| \\ &\leq \sup_{\overline{D}(R)} \left| \sum_j l_{n,j} \cosh j\, z^j \right| \\ &\leq \sum_j \frac{\|L_n\|_R}{R^j} \cosh j\, R^j\end{aligned}$$

$$\leq \|L_n\|_R \sum_j e^j$$

$$\leq e^{d_n}\|L_n\|_R. \tag{6.7}$$

Now, using (2.3) and (6.7), we get

$$\mathcal{X}(\mathcal{H}_n^{\Theta}, R) = \sum_k \left|\Pi_{n,k}^{\emptyset}\right| \|\mathcal{H}_k^{\Theta}\|_R$$

$$\leq \frac{1}{\cosh n} \sum_k \left|\pi_{n,k}\right| e^{d_k} \|L_k\|_R$$

$$\leq \frac{1}{cosh\, n} e^{D_n} \sum_k \left|\pi_{n,k}\right| \|L_k\|_R$$

$$\leq 2e^{D_n - n}\mathcal{X}(L_n, R). \tag{6.8}$$

A combination between (2.5), (6.8) and condition (4.1), yields $\mathcal{X}(\mathcal{H}^{\Theta}, R) \leq \mathcal{X}(L, R) = R$. But $\mathcal{X}(\mathcal{H}^{\Theta}, R) \geq R$. Therefore

$$\mathcal{X}(\mathcal{H}^{\Theta}, R) = R, \tag{6.9}$$

and the HTOBP $\{\mathcal{H}^{\Theta}(z)\}$ is effective for $A_{[\bar{D}(R)]}$. Similarly, we can also prove that the base $\{\mathcal{H}^{I}(z)\}$ is effective for $A_{[\bar{D}(R)]}$.

The following example show that the condition (4.1) cannot be dropped.

Example 6.1. Let the BPs $\{L_n(z)\}$ be given by

$$L_n(z) = \begin{cases} z^n, & n \text{ is even} \\ z^n + z^{2n}, & n \text{ is odd}. \end{cases}$$

In this base, we have $\mathcal{X}(L_n, R) = R^n + 2R^{2n}$. Taking $\mathrm{R} = 1$, hence $\mathcal{X}(L, 1) = 1$ and the base $\{L_n(z)\}$ is effective for $A_{[\bar{D}(1)]}$.

The HTOBPs $\{\mathcal{H}_n^{\Theta}(z)\}$ as follows:

$$\mathcal{H}^{\Theta}(z) = \begin{cases} \cosh n\, z^n, & n \; is \; even \; and \geq 2 \\ \cosh n\, z^n + \cosh 2n\, z^{2n}, & n \; is \; odd. \end{cases}$$

We easily obtain $\mathcal{X}(\mathcal{H}_n^{\Theta}, R) = \left(1/_{cos\square\, n}\right)[\cosh n\, R^n + 2\cosh 2n\, R^{2n}]$.

When $R = 1$, hence $\mathcal{X}(\mathcal{H}^{\Theta}, 1) = 2 > 1$. So that the HTOBPs $\{\mathcal{H}_n^{\Theta}(z)\}$ is not effective for $A_{[\bar{D}(1)].}$

Proof of Theorem 4.4. From (6.8), $\mathcal{X}(\mathcal{H}_n^{\Theta}, R) \leq 2e^{D_n - n}\mathcal{X}(L_n, R)$. Hence

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log \mathcal{X}(\mathcal{H}_n^{\Theta}, R)}{n \log n} \leq \lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log 2 + D_n - n + \log \mathcal{X}(L_n, R)}{n \log n},$$

and the order of the HTOBPs is at most ρ_L. If $\rho_{\mathcal{H}^{\Theta}} = \rho_L$, then

$$\lim_{R\to\infty} \frac{e}{\rho_{\mathcal{H}^{\Theta}}} \limsup_{n\to\infty} \frac{\{\mathcal{X}(\mathcal{H}_n^{\Theta}, R)\}^{\overline{n\rho_{\mathcal{H}^{\Theta}}}^{\,1}}}{n} \leq \lim_{R\to\infty} \frac{e}{\rho} \limsup_{n\to\infty} \frac{\{\mathcal{X}(L_n, R)\}^{\overline{n\rho_L}^{\,1}}}{n},$$

and the type of the HTOBPs is at most τ_L. Using similar steps, we can show that the base $\{\mathcal{H}_n^{I}(z)\}$ is at most ρ_L and τ_L.

The following example show that the $\{L_n(z)\}$ and $\{\mathcal{H}_n^{\Theta}(z)\}$ have the same order and type.

Example 6.2. It is easily verified that the base $\{L_n(z)\}$ *given by* $L_n(z) = n^n + z^n$,

$L_0(z) =$

is of type $\tau_L = e$ and order $\rho_L = 1$. For the base $\{\mathcal{H}_n^{\Theta}(z)\}$ such that:

$$\mathcal{H}_n^{\Theta}(z) = n^n + \cosh n\, z^n\, , \mathcal{H}_0^{\Theta}(z) = 1.$$

We can show that it is of order $\rho_{\mathcal{H}^{\Theta}} = 1$ and type $\tau_{\mathcal{H}^{\Theta}} = e$.

The following example show that the condition (4.2) cannot be dropped.

Example 6.3. *Consider the BPs* $\{L_n(z)\}$ *given by*

$$L_n(z) = \begin{cases} z^n, & n \text{ is even, and } \geq 2 \\ z^n + \nu\left(\frac{z}{b}\right)^{2\nu}, & n \text{ is odd and } \nu = n^n, b > 1. \end{cases}$$

This base is of order $\rho_L = 1$ *(see [24]). For the HTOBP* $\{\mathcal{H}_n^{\Theta}(z)\}$ *such that:*

$$\mathcal{H}_n^{\Theta}(z) = \begin{cases} \cosh n\, z^n\, , & n \text{ is even} \\ \cosh n\, z^n + \frac{\nu}{b^{2\nu}} \cosh 2\nu\, z^{2\nu}\, , & n \text{ is odd } \nu = n^n, b > 1. \end{cases}$$

It is easily seen that $\rho_{\mathcal{H}^{\Theta}} = \infty$ and $\rho_{\mathcal{H}^{\Theta}} > \rho_L$. Therefore Theorem 4.4 is not verified.

Proof of Theorem 4.5. Let $\mathcal{X}_{\mathcal{H}^{\Theta}}(R)$ given by

$$\mathcal{X}_{\mathcal{H}^{\Theta}}(R) = \limsup_{n\to\infty} \frac{\log \mathcal{X}(\mathcal{H}_n^{\Theta}, R)}{n \log n}. \tag{6.10}$$

In view of (4.3), (6.8) and (6.10) we have

$$\mathcal{X}_{\mathcal{H}^\Theta}(R) \le \limsup_{n\to\infty} \frac{\log \mathcal{X}(\mathcal{H}_n^\Theta, R)}{n\log n} \le \limsup_{n\to\infty} \frac{\log 2 + D_n - n + \log \mathcal{X}(L_n, R)}{n\log n}$$

$$\le \mathcal{X}_L(R). \quad (6.11)$$

Suppose now that $\{L_n(z)\}$ have property T_ρ in $\overline{D}(R),\ R > 0$. It follows from Theorem 2.2 and inequality (6.11) that $\mathcal{X}(\mathcal{H}^\Theta, R) \le \mathcal{X}(L, R) \le \frac{1}{\rho}$, and $\{\mathcal{H}_n^\Theta(z)\}$ have property T_ρ in $\overline{D}(R),\ R > 0$. Using similar steps we can prove that $\{\mathcal{H}_n^I(z)\}$ have property T_ρ in $\overline{D}(R),\ R > 0$.

The following example show that the condition (4.3) cannot be dropped.

Example 6.4. *It is easily verified that the BPs* $\{L_n(z)\}$ *given by*

$$L_n(z) = \begin{cases} z^n, & n \textit{ is even} \\ z^n + \dfrac{z^{t(n)}}{2^{n^n}}, & n \textit{ is odd}, \end{cases}$$

have property $\mathrm{T}_{\frac{1}{\log 2}}$ in $\overline{D}(2)$ (see [24]). W*here* $t(n)$ *is the nearest even integer to* $n^n + n\log n$. For the HTOBPs $\{\mathcal{H}_n^\Theta(z)\}$ is

$$\mathcal{H}_n^\Theta(z) = \begin{cases} \cosh n\, z^n, & n \textit{ is even} \\ \cosh n\, z^n + \cosh t(n)\ \dfrac{z^{t(n)}}{2^{n^n}}, & n \textit{ is odd}. \end{cases}$$

It can be verified that $\mathcal{X}(\mathcal{H}^\Theta, 2) \le 1 + \log 2$ and the HTOBPs $\{\mathcal{H}_n^\Theta(z)\}$, have not $\mathrm{T}_{\frac{1}{\log 2}}$-property in $\overline{D}(2)$.

7 Conclusion.

In this conclusion, we will discuss some ideas for future works that contain new research ideas for the reader. The future works are divided into seven categories:

1. Previous studies [8, 21, 32, 37, 38, 42, 43] have extensively explored the convergence properties of associated bases, such as product base, the inverse base, transposed inverse base, transpose base, similar base, square root base, and Hadamard product base. These bases are comprised of simple monic bases $(P_{n,n} = 1 \; for \; all \; n)$.. Future endeavors may involve generalizing the theorems for the aforementioned associated bases when their constituents are non-simple monic bases.

2. The forthcoming research may focus on the convergence properties of bases, specifically the generalization of Kronecker product bases of polynomials to higher dimensions in Clifford analysis. Additionally, extending the representation of Clifford-valued functions in open and closed balls by infinite series with Legendre, Laguerre, Bessel, Hermite, and Gontcharoff polynomials to hypercomplex analysis is a potential avenue. Moreover, the investigation into the effectiveness, growth type, and order of the above sets could be extended to several complex variables, covering regions such as polycylindrical, hyperelliptical, spherical, and Faber regions, for all entire functions and at the origin.

3. Exploring new types of bases like q-Bernoulli and q-Euler in both Complex and Clifford analysis is a prospective area of study.

4. Generalizing the theory of BPs in Clifford analysis to regions of representation such as polycylindrical, spherical, hyperelliptical, and Faber regions is an avenue worth exploring.

5. The extension of basic sets of polynomials from complex matrices to Fréchet spaces or Fréchet modules is another potential research direction.

6. Investigating the idea of fractional derivatives of basic sets in several complex variables and Clifford analysis is an intriguing prospect.

7. Considering the potential generalization of HTOBs and HIOBs to the case of several complex variables and Clifford analysis would be a significant contribution. If feasible, generalizing the theorems proven in this paper to accommodate this extension is an important consideration.

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