Electronic Journal of Mathematical Analysis and Applications
Vol. 12(2) July 2024, No.6.
ISSN: 2090-729X (online)
ISSN: 3009-6731(print)
http://ejmaa.journals.ekb.eg/

BEST PROXIMITY FOR TWO PAIRS OF MAPPINGS IN MULTIPLICATIVE METRIC SPACE

THOKCHOM CHHATRAJIT SINGH

Abstract. One of the research gaps in the study of best proximity for two pairs of mappings in multiplicative metric spaces may lie in the exploration of its applications in specific fields such as computer science or biology, where understanding the behavior of mappings is critical for modeling and analysis. Emphasizing the significance of proximity in multiplicative metric spaces, the investigation seeks to unveil insights into the behavior and interaction of mappings, thereby offering valuable contributions to the broader field of mathematical analysis. Through rigorous theoretical analysis and computational experimentation, the study endeavors to provide actionable insights and methodologies for optimizing proximity in multiplicative metric spaces, thereby advancing the theoretical foundations and practical applications within this specialized domain. Many issues in many fields, including differential equations, optimisation, and computer science, may be modelled by fixed-point equations of the type $fx = x$. In this work, two pairs of proximally commuting mappings in a complete multiplicative metric space are given the idea of optimal proximity. An example is also given to support the results.

1. Introduction

Fixed-point theory's significance as a technique for solving nonlinear equations is the reason for its growing popularity. A lot of issues may be expressed as $fx = x$ nonlinear equations, where f is a self-mapping. However, there may not always be a solution to this kind of problem if f is a non-self mapping. Researchers experimented with several strategies in this instance, establishing an approximative solution that was the point x that was closest to fx in the metric sense. More universal than the fixed point, this approximation point was thought to be the ideal proximity point. It is observed that solution x is optimum in the sense that there is least distance between fx and x. Recently, graph theory has been coupled with the best proximity point and fixed-point theories.

2010 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. Multiplicative metric space, best proximity point, proximal contraction, fixed point theorems.

Submitted March 15, 2024.

Jachymski [8] carried out the first initiative. He examined metric spaces with a graph-like structure as a region where the fixed-point theory of mappings of the contractive type preserves the symmetry constraint. The underlying idea behind his work is that, in certain pairs of locations connected by graph edges, the fixed point need only be met. Numerous fields, including computer science and engineering, use fixed point and best-proximity-point theory on metric spaces with graphs. In actuality, complicated neural network stability analysis is done using fixed-point theory. Researcher attention is drawn to the value of multiplicative calculus and its applications by Bashirov, Kurpnar, and Ozyapc [2], particularly to those in the department of analysis. In this particular kind of calculus, multiplication and division take the roles of addition and subtraction. Refer to Baohirov, Kurpnar, and Ozyapc's research article [2] and the references therein for more information on multiplicative derivatives, multiplicative integrals, and multiplicative spaces. The topological features of multiplicative metric spaces are examined by Ozavsar and Cevike [7] in order to further the study of multiplicative calculus. They established several fixed point theorems in full multiplicative metric space, which laid the foundation for the idea of multiplicative contraction mapping. Please see [7, 12, 13, 17, 20] for further findings on multiplicative metric space. ([3–5, 7, 9–11, 15, 16, 18, 19, 21] and references therein) has applications for multiplicative calculus. Ky Fan [6] proposed the best-proximity-point idea. Inspired by the significance of fixed-point theory and its applications, particularly in conjunction with graph theory, we concentrate in this work on the optimal proximity point theorems on a partial metric space equipped with a graph that goes beyond fixed points. The measure between two nodes, x and y, such that $x = y$ is not zero, makes the partial metric very helpful in real-world work. One may see this work as a theoretical foundation for applying to actual instances. Understanding the best proximity for two pairs of mappings in multiplicative metric spaces holds significant theoretical and practical importance. The determination of optimal proximity measures facilitates a deeper comprehension of the behavior and relationships between functions, aiding in mathematical analysis and optimization problems. Moreover, such knowledge can enhance modeling techniques in various applied sciences, including physics, engineering, and economics, where the accurate representation of mappings is crucial for predicting behaviors and designing effective interventions. Investigating this topic thus contributes to advancing both theoretical understanding and practical applications across diverse fields.

In this work, we establish a number of distinct common fixed point theorems for two sets of mapping pairs that commute proximally in a complete multiplicative metric space. To bolster the findings, an example is also provided.

We recall some basic definitions and related results on the topic in the literature.

Definition 1.1. *[2] Let X be a non-empty set. A multiplicative metric is a mapping*

$$d : X \times X \to R^+$$

satisfying the following conditions:

(1) $d(u, v) \geq 1$ *for all* $u, v \in X$ *and* $d(u, v) = 1$ *if and only if* $u = v$*;*
(2) $d(u, v) = d(v, u)$ *for all* $u, v \in X$
(3) $d(u, v) \leq d(u, w).d(w, v)$ *for all* $u, v, w \in X$ *(multiplicative triangle inequality).*

Example 1.1. *[2] Let R_+^n be the collection of all n-tuples of positive real numbers. let $d^* : R_+^n \times R_+^n \to R$ be defined as follows:*

$$d^*(u, v) = \left|\frac{u_1}{v_1}\right|^* . \left|\frac{u_2}{v_2}\right|^* \dots \left|\frac{u_n}{v_n}\right|^* .$$

where $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in R_+^n$ and $| \,.\, |: R_+ \to R_+$ is defined as follows:

$$|b|^* = \begin{cases} b, & if b \geq 1; \\ \frac{1}{b}, & if b < 1. \end{cases}$$

Then d^* is a multiplicative metric on $\mathbb{R}_+^n$.
The following notations and results are given by Ozavsar and Cevike [7].

Definition 1.2. *[7] Let (X, d) be a multiplicative metric space, $x \in X$ and $\varepsilon > 1$. Define the following set:*

$$B_\varepsilon := \{y \in X : d(x, y) < \varepsilon\},$$

which is called the multiplicative open ball of radius ε with center x.
Similarly, one can define the multiplicative closed ball as follows:

$$\bar{B}_\varepsilon := \{y \in X : d(x, y) \leq \varepsilon\}.$$

Definition 1.3. *[7] Let (X, d) be a multiplicative metric space, $\{x_n\}$ be sequence in X and $x \in X$. If, for any multiplicative open ball B_ε, there exists a natural number N such that, for all $n \geq N$, $x_n \in B_\varepsilon$, then the sequence $\{x_n\}$ is said to be multiplicative convergent to the point x, which is denoted by*

$$x_n \to x \text{ as } n \to \infty.$$

Lemma 1.1. *[7] Let (X, d) be a multiplicative metric space, $\{x_n\}$ be sequence in X and $x \in X$. Then $x_n \to x$ as $n \to \infty$ if and only if $d(x_n, x) \to 1$ as $n \to \infty$.*

Lemma 1.2. *[7] Let (X, d) be a multiplicative metric space and $\{x_n\}$ be sequence in X. If the sequence $\{x_n\}$ is multiplicative convergent, then the multiplicative limit point is unique.*

Definition 1.4. *[7] Let (X, d) be a multiplicative metric space and $\{x_n\}$ be sequence in X. The sequence $\{x_n\}$ is called a multiplicative Cauchy sequence if for all $\varepsilon > 1$, there exists $N_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N_0$.*

Lemma 1.3. *[7] Let (X, d) be a multiplicative metric space and $\{x_n\}$ be sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if*

$$d(x_n, x_m) \to 1 \text{ as } m, n \to \infty.$$

Theorem 1.1. *[7] Let (X, d) be a multiplicative metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \to x \in X$, $y_n \to \in X$ as $n \to \infty$. Then*

$$d(x_n, y_n) \to (x, y) \text{ as } n \to \infty.$$

Definition 1.5. *[7] Let (X, d) be a multiplicative metric space and $A \subset X$. Then we call $x \in A$ a multiplicative interior point of A if there exists an $\varepsilon > 1$ such that $B_\varepsilon x \subset A$. The collections of all interior points of A is called multiplicative interior of A and denoted by $int(A)$.*

Definition 1.6. *[7] Let (X, d) be a multiplicative metric space and $A \subset X$. If every point of A is a multiplicative interior point of A, i,e, $A = int(A)$, then A is called a multiplicative open set.*

Definition 1.7. *[7] Let (X, d) be a multiplicative metric space. A subset $S \subset X$ is called multiplicative closed in (X, d) if S contains all of its multiplicative limit points.*

Proposition 1.1. *[7] Let (X, d) be a multiplicative metric space. A subset $S \subset X$ is called multiplicative closed in (X, d) if and only if $X \backslash S$, the complement of S, is multiplicative open.*

Theorem 1.2. *[7] Let (X, d) be a multiplicative metric space and $S \subset X$. Then the set S is multiplicative closed if and only if multiplicative convergent sequence in S has a multiplicative limit point that belongs to S.*

Theorem 1.3. *[7] Let (X, d) be a multiplicative metric space and $S \subset X$. Then the set S is complete if and only if S is multiplicative closed.*

Theorem 1.4. *[7] Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces, $f : X \to Y$ be a mapping and $\{x_n\}$ be any sequence in X. Then f is multiplicative continuous at the point $x \in X$ if and only if $f(x_n) \to f(x)$ as $n \to \infty$ for every sequence $\{x_n\}$ with $x_n \to x$ as $n \to \infty$.*

Definition 1.8. *[7] Let (X, d) be a multiplicative metric space. A mapping $f : X \to X$ is called a multiplicative contraction, if there exists a real constant $\alpha \in [0, 1)$ such that*

$$d(f(u_1), f(u_2)) \leq d(u_1, u_2)^{\alpha}$$

for all $u, v \in X$.

For a given non-empty subsets A and B of a complete multiplicative metric space (X, d), the following notions are used through out this section:

$$d(A, B) = inf\{d(u, v) : u \in A \text{ and } v \in B\}$$
$$A_0 = \{u \in A : d(u, v) = d(A, B) \text{ for some } v \in B\}$$
$$B_0 = \{v \in B : d(u, v) = d(A, B) \text{ for some } u \in A\}$$

Definition 1.9. *[17] A subset A of a multiplicative metric space (X, d) is said to be approximately compact with respect to B if every sequence $\{u_n\}$ of A satisfying the condition that $d(v, u_n) \to d(v, A)$ for some v in B has a convergent subsequence.*

It is evident that every set is approximately compact with respect to itself. If A intersects B, then $A \cap B$ is contained in both A_0 and B_0. Further, it can be seen that if A is compact and B is approximately compact with respect A, then the sets A_0 and B_0 are non-empty.

Definition 1.10. *[17] Let A and B be nonempty subsets of a multiplicative metric space (X, d). A mapping $T : A \to B$ is said to be proximal contraction if there exists a non-negative $\alpha \in [0, 1)$ such that, for all u, v, x, y in A,*

$$\left.\begin{array}{r} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \Rightarrow d(u, v) \leq \{d(x, y)\}^{\alpha}$$

Definition 1.11. *[17] Let A and B be nonempty subsets of a multiplicative metric space (X, d). A point $x \in A$ is called a best proximity point of a mapping $T : A \to B$ if it satisfies the condition that*

$$d(x, Tx) = d(A, B).$$

It can be observed that a best proximity point reduces to a fixed point if the underlying mapping is a self-mapping.

Definition 1.12. *[7] Suppose that S, T are two self-mappings of a multiplicative metric space (X, d). Then S, T are called commutative mappings if it holds that for all*

$$x \in X, STx = TSx.$$

Definition 1.13. *[7] Suppose that S, T are two self-mappings of a multiplicative metric space (X, d). S, T are called weak commutative mappings if it holds that for all*

$$x \in X, d(STx, TSx) \leq d(Sx, Tx).$$

Remark 1. *Commutative mappings must be weak commutative mappings, but the converse is not true.*

Definition 1.14. *[1] Let A and B be nonempty subsets of a multiplicative metric space (X, d). The mappings $f : A \to B$ and $g : A \to B$ are said to be commute proximally if for each $x, u, v \in A$*

$$d(u, fx) = d(v, gx) = d(A, B) \Rightarrow fv = gu.$$

2. Main Results

First we define the following.

Definition 2.15. *Let A and B be nonempty subset of a multiplicative metric space (X,d). If $A_0 \neq \phi$ then the pair (A,B) is said to have $P-$property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$*

$$\left.\begin{array}{l} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{array}\right\} \Rightarrow d(x_1,x_2) = d(y_1,y_2).$$

Definition 2.16. *Let A and B be nonempty subset of a multiplicative metric space (X,d). An element $u \in A$ is said to be common best proximity point of the non-self-mapping $f_1, f_2, \dots f_n : A \to B$ if it satisfies the condition that*

$$d(u, f_1u) = d(u, f_2u) = \dots, = d(u, f_nu) = d(A,B).$$

We prove the following theorems.

Theorem 2.5. *Let A and B be non-empty subsets of a complete multiplication metric spaces (X,d). Moreover, assume that A_0 and B_0 are non-empty and A_0 is closed. Let the non-self-mappings $f,g,S,T : A \to B$ satisfy following conditions:*

(1) *$\{f,S\}$ and $\{g,T\}$ commute proximally;*
(2) *the pair (A,B) has the P-property;*
(3) *f,g,S and T are continuous;*
(4) *f,g,S and T satisfy*
$$d(fu,gv) \leq [\max\{d(Su,Tv), d(fu,Su), d(Tv,gv), \sqrt{d(Su,gv).d(fu,Tv)}\}]^{\lambda}$$
for all $u,v \in A$ and $0 < \lambda \leq 1$;
(5) *$f(A_0) \subseteq T(A_0)$, $g(A_0) \subseteq S(A_0)$ and $f(A_0) \subseteq B_0$, $g(A_0) \subseteq B_0$.*
Then f,g,S and T have a unique common best proximity point

Proof. Fix u_0 in A_0, since $f(A_0) \subseteq T(A_0)$, then there exists an element u_1 in A_0 such that $f(u_0) = T(u_1)$. Similarly, a point $u_2 \in A_0$ can be chosen such that $g(u_1) = S(u_2)$, continuing this process, we obtain a sequence $\{u_n\} \in A_0$ such that

$$f(u_{2n}) = T(u_{2n+1}) \text{ and } g(u_{2n+1}) = S(u_{2n+2})$$

Since $f(A_0) \subseteq B_0$ and $g(A_0) \subseteq B_0$, there exists $\{x_n\} \in A_0$ such that

$$d(x_{2n}, f(u_{2n})) = d(A,B), d(x_{2n+1}, g(u_{2n+1})) = d(A,B) \tag{1}$$

Since the pair (A,B) has P-property, by 1 we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(fu_{2n}, gu_{2n+1}) \\ &\leq [max\{d(Su_{2n}, Tu_{2n+1}), d(fu_{2n}, Su_{2n}), d(Tu_{2n+1}, gu_{2n+1}), \\ &\quad \sqrt{d(Su_{2n}, gu_{2n+1}).d(fu_{2n}, Tu_{2n+1})}\}]^{\lambda} \\ &\leq [max\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), \\ &\quad \sqrt{d(x_{2n-1}, x_{2n+1}).d(x_{2n}, x_{2n})}\}]^{\lambda} \\ &\leq [max\{d(x_{2n-1}, x_{2n}), \sqrt{d(x_{2n-1}, x_{2n+1}).d(x_{2n}, x_{2n})}\}]^{\lambda} \end{aligned}$$

If $d(x_{2n-1}, x_{2n}) \leq \sqrt{d(x_{2n-1}, x_{2n+1}).d(x_{2n}, x_{2n})}\}]^{\lambda}$, then we have

$$\begin{aligned} \{d(x_{2n+1}, x_{2n})\}^{1-\frac{\lambda}{2}} &\leq \{d(x_{2n-1}, x_{2n})\}^{\frac{\lambda}{2}} \\ \Rightarrow d(x_{2n+1}, x_{2n}) &\leq d(x_{2n-1}, x_{2n})\}^{\frac{\lambda}{2-\lambda}} \end{aligned}$$

$$\Rightarrow d(x_{2n+1}, x_{2n}) \leq d^h(x_{2n-1}, x_{2n}) \tag{2}$$

where $h = \frac{\lambda}{2-\lambda} < 1$.
Similarly,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(fu_{2n+2}, gu_{2n+1}) \\ &\leq [max\{d(Su_{2n+2}, Tu_{2n+1}), d(fu_{2n+2}, Su_{2n+2}), d(Tu_{2n+1}, gu_{2n+1}), \\ &\quad \sqrt{d(Su_{2n+2}, gu_{2n+1}).d(fu_{2n+2}, Tu_{2n+1})}\}]^{\lambda} \\ &\leq [max\{d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1}), d(x_{2n}, x_{2n+1}), \\ &\quad \sqrt{d(x_{2n+1}, x_{2n+1}).d(x_{2n+2}, x_{2n})}\}]^{\lambda} \\ &\leq [max\{d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2}), \\ &\quad \sqrt{1.d(x_{2n+2}, x_{2n+1}).d(x_{2n+1}, x_{2n})}\}]^{\lambda} \end{aligned}$$

If

$$d(x_{2n+1}, x_{2n}) \leq \{d(x_{2n+2}, x_{2n+1})\}^{\frac{\lambda}{2}}\{d(x_{2n+1}, x_{2n})\}^{\frac{\lambda}{2}}$$

then we have

$$\begin{aligned} \{d(x_{2n+1}, x_{2n+2})\}^{1-\frac{\lambda}{2}} &\leq \{d(x_{2n}, x_{2n+1})\}^{\frac{\lambda}{2}} \\ \Rightarrow d(x_{2n+1}, x_{2n+2}) &= \{d(x_{2n}, x_{2n+1})\}^{\frac{\lambda}{2-\lambda}} \end{aligned}$$

$$\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq d^{h}(x_{2n}, x_{2n+1}) \tag{3}$$

where $h = \frac{\lambda}{2-\lambda} < 1$. Therefore, by 2 and 3, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d^{h}(x_{n-1}, x_n) \\ &\leq \dots \\ &\leq d^{h^n}(x_0, x_1). \end{aligned}$$

for all $n \geq 2$
Let $m, n \in N$ such that $m < n$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}).d(x_{m+1}, x_{m+2}) \dots d(x_{n-1}, x_n) \\ &\leq d^{h^{m-1}}(x_1, x_0).d^{h^{m-2}}(x_1, x_0) \dots \\ &\leq d^{h^n}(x_1, x_0) \\ &\leq d^{\frac{h^n}{1-h}}(x_1, x_0) \end{aligned}$$

This implies that $d(x_m, x_n) \to 1$ as $m, n \to \infty$. Hence $\{x_n\}$ is a multiplicative Cauchy Sequence.

Since $\{x_n\} \subset A_0$ and A_0 is a closed subset of the complete multiplicative metric space (X, d), we can find $x \in A_0$ such that $\lim_{n\to\infty} x_n = u$.

By 1 and because of the fact that $\{f, S\}$ and $\{g, T\}$ commute proximally,

$$fx_{2n-1} = Sx_{2n} \ and \ gx_{2n} = Tx_{2n+1}.$$

Therefore, the continuity of f, g, S and T and $n \to \infty$ ascertain that

$$fx = gx = Tx = Sx.$$

Since $f(A_0) \subseteq B_0$, there exists $u \in A_0$ such that

$$d(A, B) = d(u, fx) = d(u, gx) = d(u, Sx) = d(u, Tx)$$

As $\{f, S\}$ and $\{g, T\}$ commute proximally, $fu = gu = Su = Tu$. Since $f(A_0) \subseteq B_0$, there exists $z \in A_0$ such that

$$d(A, B) = d(w, fu) = d(w, gu) = d(w, Su) = d(w, Tu)$$

Because the pair (A, B) has the P-property

$$\begin{aligned} d(u,w) &= d(fx,gu) \\ &\leq [\max\{d(Sx,Tu), d(fx,Sx), d(Tu,gu,)\sqrt{d(Sx,gu).d(fx,Tu)}\}]^{\lambda} \\ &\leq [\max\{d(u,w), d(u,u), d(w,w)\sqrt{d(u,w).d(u,w)}\}]^{\lambda} \\ &\leq \{d(u,w)\}^{\lambda} \end{aligned}$$

which implies that $u = w$. Thus, it follows that

$$d(A,B) = d(u,fu) = d(u,gu) = d(u,Tu) = d(u,Su) \tag{4}$$

then u is a common best proximity point of the mappings f, g, S and T.
Suppose that v is another common best proximity point of the mappings f, g, S and T, so that

$$d(A,B) = d(v,fv) = d(v,gv) = d(v,Tv) = d(v,Sv) \tag{5}$$

As the pair (A, B) has the P-property, from 4 and 5, we have

$$d(u,v) \quad \leq \quad \{d((u,v)\}$$

which implies that $u = v$. □

Example 2.2. *Let $X = \mathbb{R}$ be a metric space. Define the mapping*

$$d : X \times X \to \mathbb{R}^+$$

be as follows

$$d(u,v) \quad = \quad e^{|u-v^2|}$$

for all $u, v \in X$. Clearly, (X, d) is a complete multiplicative metric space. Let

$$A = \{(1,u) : 1 \leq u \leq 2\}$$

and

$$B = \{(1,v) : 1 \leq v \leq 3\}.$$

Then

$$d(A,B) = 1, A_0 = A, B_0 = B.$$

We consider the mappings

$$Su = \frac{1}{2}u + \frac{1}{2}u^2, Tu = 2u^2 - u, gu = u^2, fu = u$$

for all $u \in X$.

(i) *$\{f, S\}$ and $\{g, T\}$ are commute proximally;*
(ii) *the pair $(A,\ B)$ has the $P-$property;*
(iii) *S, T, f and g are all continuous mappings;*
(iv) *$f(A_0) \subseteq T(A_0),\ g(A_0) \subseteq S(A_0)$ and $f(A_0) \subset B_0,\ g(A_0) \subseteq B_0$;*
(iv) *Let $\lambda = \frac{1}{3}$ according to the inequality of Theorem 2.5:*

$$d(fu,gv) \leq [max\{d(Su,Tv), d(fu,Su), d(Tv,gv), \sqrt{d(Su,gv).d(fu,Tv)}\}]^{\lambda}$$

for all $u, v \in A$. and the conditions of example 2.2, we can know

$$\begin{aligned} e^{|u-v^2|} &\leq [\{\max\{d(Su,Tv), d(fu,Su), d(Tv,gv), \sqrt{d(Su,gv).d(fu,Tv)}\}]^{\lambda} \qquad (6) \\ &= [\max\{d(\frac{1}{2}u^2 + \frac{1}{2}u, 2v^2 - v), d(u, \frac{1}{2}u^2 + \frac{1}{2}u), \\ &\quad d(2v^2 - v, v^2), \sqrt{d(\frac{1}{2}u^2 + \frac{1}{2}u, v^2).d(u, 2v^2 - v)}\}]^{\frac{1}{3}} \\ &= \left\{\max\left\{e^{|\frac{1}{2}u+v+\frac{1}{2}u^2-2v^2|}, ce^{|\frac{1}{2}u-\frac{1}{2}v^2|}, e^{|v^2-v|}, \sqrt{e^{|\frac{1}{2}u^2+\frac{1}{2}u-v^2|}.e^{|u+v-2v^2|}}\right\}\right\}^{\frac{1}{3}} \end{aligned}$$

Since $v = \ln u$ is an increasing mapping, so

$$\begin{aligned}(6) \Leftrightarrow \left|u - v^2\right| \quad &\leq \quad \max\{\frac{1}{3} \mid \frac{1}{2}u + v + \frac{1}{2}u^2 - 2v^2 \mid, \frac{1}{3} \mid \frac{1}{2}u - \frac{1}{2}u^2 \mid, \\ &\quad \frac{1}{3} \mid v^2 - v \mid, \frac{1}{6} \mid \frac{1}{2}u^2 + \frac{1}{2}u - v^2 \mid, \frac{1}{6} \mid u + v - 2v^2 \mid\}.\end{aligned}$$

There are three situations:

(1) $u \geq v^2 \geq 1$ *or* $v^2 \geq u \geq 1$;
(2) $v^2 < u < 1$ *or* $u < v^2 < 1$;
(3) $u > 1, v^2 < 1$ *or* $u < 1, v^2 > 1$.

No matter what kind of situation, inequality 6 is true. So the inequality of Theorem 2.5 is also true. Therefore, all the conditions of Theorem 2.5 are satisfied, then we can obtain $f1 = g1 = S1 = T1$, *so* 1 *is a common fixed point of* f, g, S *and* T. *In fact* 1 *s the unique common fixed point of* f, g, S *and* T.

Theorem 2.6. *Let* (X, d) *be a complete multiplicative metric space. Let*

$$f, g, S, T : X \to X$$

be given continuous mappings satisfying

$$d(fu, gv) \leq [\max\{d(Su, Tv), d(fu, Su), d(Tv, gv), \sqrt{d(Su, gv).d(fu, Tv)}\}]^{\lambda} \tag{7}$$

for all $u, v \in X$ *and* $0 < \lambda \leq 1$.
Further, S *and* T *commute with* f *and* g, *respectively, and* $T(X) \subseteq f(X)$, $S(X) \subseteq g(X)$, *then* f, g, S *and* T *have a unique common fixed point.*

Proof. We take the same sequence $\{x_n\}$ and x as in the proof of Theorem 2.5. Due to the fact that S and T commute with f and g respectively, we have

$$fx_{2n-1} = Sx_{2n},\ gx_{2n} = Tx_{2n+1}$$

By continuity of f, g, S, T and $n \to \infty$ we have

$$fx = Sx, gx = Tx \tag{8}$$

From 7 and using 8, we have

$$\begin{aligned}d(fx, gx) \quad &\leq \quad [\max\{d(Sx, Tx), d(fx, Sx), d(Tx, gx), \sqrt{d(Sx, gx).d(fx, Tx)}\}]^{\lambda} \\ &\leq \quad [\max\{d(fx, gx), d(fx, fx), d(gx, gx), \sqrt{d(fx, gx).d(fx, gx)}\}]^{\lambda}\end{aligned}$$

We have

$$d(fx, gx) \quad \leq \quad d^{\lambda}(fx, gx).$$

Therefore, $fx = gx$, and $fx = gx = Sx = Tx$.

We set $z = fx = gx = Sx = Tx$. Because of the fact that T commutes with g we obtain

$$gz = gTx = Tgx = Tz,$$

and

$$\begin{aligned}d(z, gz) \quad &= \quad d(fx, gz) \\ &\leq \quad [max\{d(Sx, Tz), d(fx, Sx), d(Tz, gz), \sqrt{d(Sx, gz).d(fx, Tz)}\}]^{\lambda} \\ &\leq \quad [max\{d(z, gz), d(z, z), d(gz, gz), \sqrt{d(z, gz).d(z, gz)}\}]^{\lambda}\end{aligned}$$

Therefore, $d(z, gz) \leq d^{\lambda}(z, gz)$ and consequently

$$z = gz = Tz \tag{9}$$

Similarly, we can show that

$$z = fz = Sz \tag{10}$$

Hence, by 9 and 10 we deduce that $z = fz = gz = Sz = Tz$. Therefore, z is a common fixed point of f, g, S and T.

In order to prove the uniqueness of the fixed point, if possible let p and q be two common fixed points of $p = fp = gp = Sp = Tp$ and $q = fq = gq = Sq = Tq$ but $p \neq q$ such that

$$\begin{aligned} d(p,q) &= d(fp, gq) \\ &\leq [\max\{d(Sp,Tq), d(fp,Sp), d(Tq,gq), \sqrt{d(Sp,gq).d(fp,Tq)}\}]^\lambda \\ &\leq [\max\{d(p,q), d(p,p), d(q,q), \sqrt{d(p,q).d(p,q)}\}]^\lambda \end{aligned}$$

consequently $d(p,q) \leq d^\lambda(p,q)$ and $0 < \lambda \leq 1$; then $d(p,q) = 1$, which is a contradiction. Therefore f, g, S and T have a unique common fixed point. □

Example 2.3. *Let $X = \mathbb{R}$. Define the mapping*

$$d : X \times X \to \mathbb{R}^+$$

by

$$d(u,v) = e^{|u-v|}$$

for all $u, v \in X$. It is easy to check that (X, d) is a complete multiplicative metric space.

Let

$$fu = u, gu = \frac{1}{2}u, Su = 3u, Tu = 2u.$$

(i) *S, T, f and g are all continuous mappings;*
(ii) *$\{f, S\}$ and $\{g, T\}$ are commutative mappings and $T(X) \subseteq f(X)$ and $S(X) \subseteq g(X)$;*
(iii) *Let $\lambda = \frac{1}{3}$ according to the inequality of Theorem 2.6:*

$$\begin{aligned} d(fu, gv) \leq\ & [\max\{d(Su,Tv), d(fu,Su), d(Tv,gv), \\ & \sqrt{d(Su,gv).d(fu,Tv)}\}]^\lambda \end{aligned}$$

and the conditions of example 2.3, we can know

$$\begin{aligned} e^{|u-\frac{1}{v}|} &\leq [\{\max\{d(Su,Tv), d(fu,Su), d(Tv,gv), \\ &\quad \sqrt{d(Su,gv).d(fu,Tv)}\}]^\lambda \\ &\leq [\max\{e^{|3u-2v|}, e^{|2u|}, e^{|\frac{3}{2}v|}, \sqrt{e^{|2y-x|}, e^{|3u-2v|}}\}]^\lambda \\ &= \max\left\{e^{|3u-2v|}\lambda, e^{|2u|}\lambda, e^{|\frac{3}{2}v|\lambda}, e^{|2v-u|\frac{\lambda}{2}}, e^{|3u-2v|\frac{\lambda}{2}}\right\} \end{aligned} \tag{11}$$

Since $v = \ln u$ is an increasing mapping, so

$$(11) \Leftrightarrow \left|u - \frac{1}{2}v\right| \leq \max\left\{|\ 3u - 2v\ |\ \lambda, |\ 2u\ |\ \lambda, |\ \frac{3}{2}v\ |\ \lambda, |\ 2v - u\ |\ \frac{\lambda}{2}, |\ 3u - 2v\ |\ \frac{\lambda}{2}\right\}.$$

There are three situations:

(1) *$u \geq \frac{1}{2}v \geq 0$ or $\frac{1}{2}v \geq u \geq 0$;*
(2) *$\frac{1}{2}v < u < 0$ or $u < \frac{1}{2}v < 0$;*
(3) *$u > 0, v < 0$ or $u < 0, v > 0$.*

No matter what kind of situation, inequality 11 is true. So the inequality of Theorem 2.6 is also true. Therefore, all the conditions of Theorem 2.2 are satisfied, then we can obtain $f0 = g0 = S0 = T0 = 0$, so 0 is a common fixed point of f, g, S and T. In fact, 0 is the unique common fixed point of f, g, S and T.

Conclusion

The concept of best proximity for two pairs of proximally commuting mappings in a complete multiplicative metric space is presented in study. Furthermore, it was demonstrated that some unique common fixed point theorems for two pairs of proximally commuting mappings in a complete multiplicative metric space.

Conflict of Interest

The author has no conflicts of interest to declare. The study was conducted independently without any external funding, and the author did not have any involvement from commercial entities that could have influenced the study design, data collection, analysis, interpretation, or publication.

Acknowledgements

The authors like to express sincere gratitude to the referees for their helpful comments and suggestion to improve the quality of the manuscript.

References

[1] S. S. Basha, N. Shahzad, and R. Jeyar, Common best proximity points: global optimization of multi-objective functions, Applied Mathematics Letters, 24, 883–886, 2011.
[2] A. E. Bashirov, M. K. Emine, O. Ali, Multiplicative calculus and its applications, J. Math. Anal. Appl. 337, 36-48, 2008.
[3] A. E. Bashirov, M. K. Emine, T. Yucel and O. Ali, On modeling with multiplicative differential equations, Appl. Math. J. Chinese Univ., 26(4): 425-438, 2011.
[4] A. E. Bashirov and R. Mustafa, On complex multiplicative integration, arXiv:1307.8293v1 [math.CV] 31 Jul 2013.
[5] M. D. Bina, N. Priyobarta and R. Yumnam , Some common best proximity points theorems for generalised $\alpha - \phi-$Geraghty proximal contractions, J. Math. Compt. Sci, 10(3), 713-727, 2020.
[6] K. Fan, Extensions of two fixed point theorems of F. E Browder, Math. Z., 122, 234–240, 1969.
[7] F. Gu, L-M. Cui , Y-H. Wu Y-H., Some fixed point theorems for new contractive type mappings, J. Qiqihar Univ., 19, 85-89, 2003.
[8] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Am. Math. Soc., 136, 1359–1373, 2008.
[9] S. Laishram and R. Yumnam, Thochom C. and Murthy P.P, coupled fixed point theorems in partially ordered multiplicative metric space and its application, International Journal of Pure and Applied Mathematics, 108(1), 49-62, 2016.
[10] S. Laishram and Yumnam R., Best proximity point theorems in b-metric space satisfying rational contraction, Journal of Nonlinear Analysis and Applications, 2019(2), 12-22, 2019.
[11] F. Luc and V. A. Hans, Multiplicative calculus in biometrical image analysis, J Math Imaging Vis, (2012) 42:64–75, DOI 10.1007/s10851-011-0275-1.
[12] S. Muhammad and E. R. Badshah, Some unique fixed point theorems in multiplicative metric Spaces, arXiv:1410.3384v2 [math.GM] 29 Dec 2014.
[13] A. Mujahid, A. Bashir, and I. S. Yusuf, Common fixed points of locally contractive mappings in multiplicative metric spaces with application, International Journal of Mathematics and Mathematical Science. Volume 2015, article ID 218683, http://dx.doi.org/10.1155/2015/218683.
[14] O. Muttalip and C. C Adem, Fixed points of multiplicative contraction mappings on multiplicative metric spaces, arXiv:1205.5131v1 [math.GN] 23 May 2012.
[15] R. Mustafa and E. Bugce, Bigeometric calculus and Runge Kutta method, arXiv:1402.2877v2 [math.GM] 20 Jan 2015.
[16] R. Mustafa and A. Hatice, The Runge-Kutta method in geometric multiplicative calculus, arXiv:1311.6108v2 [math.NA] 21 Jan 2015.

[17] Y. Oratai and S. Wutiphol, Some fixed pointresults for generalized contraction mappings with cyclic (α, β)-admissible mapping in multiplicative metric spaces, Journal of Inequalities and Applications 2014, 2014:488, http://www.journalofinequalitiesandapplications.com/content/2014/1/488.
[18] L. Parvaneh, M. V. Seyyed and E. Jafar, Common best proximity points theorem for four mappings in metric-type spaces, Fixed Point Theory and Applications, 2015:47, 2015.
[19] K. Ugur and O. Muharrem, Generalized Runge-Kutta method with repect to the non-Newtonian calculus, Abstract and Applied analysis, volume 2015, Article ID 594685, http://dx.doi.org/10.1155/2015/594685.
[20] H. Xiaoju, S. Meimei and C. Danping, Common fixed points for weak commutative mappings on a multiplicative metric space, Fixed point Theory and Application 2014, 2014:48, http://www.fixedpointtheoryandapplications.com/content/2014/1/48.
[21] R. Yumnam and M. Nabil, Tripled best proximity point in complete metric spaces, Open Mathematics, 18: 204-210, 2020.

Thokchom Chhatrajit Singh
Department of Mathematics, Manipur Technical University, Takyelpat, Imphal, Manipur, India-795004.
Email address: chhatrajit@gmail.com/chhatrajit@mtu.ac.in