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UNIQUENESS OF GENERAL DIFFERENCE DIFFERENTIAL POLYNOMIALS AND MEROMORPHIC(ENTIRE) FUNCTIONS

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Abstract. This study explores the uniqueness of entire and meromorphic functions with equal weights $l \geq 0$ by investigating the general difference-differential polynomial $\Psi(z, f)$. We have extended the findings attributed to [3] and derived a new result. Additionally, we examine the implications when a polynomial of degree n shares a common value with the general difference-differential polynomial. We have also posed an open problem for future research work.

1. Background Information, Definitions and results

A meromorphic function is a non-constant function that exhibits poles as singularities throughout the complex plane. The Nevanlinna theory of meromorphic functions provides standard notations for the discussion, as referenced by [5], [9], and [10]. If $f(z)$ and $g(z)$ share $a(z)$ CM(IM), we refer to $a(z)$ as a small function concerning $f(z)$ if $T(r, a(z)) = S(r, f)$, where $S(r, f)$ is any small quantity satisfying $S(r, f) = o\{T(r, f)\}$, as $r \to \infty$, possibly outside a set of finite linear measure.

We use $N_{k)}\left(r, \frac{1}{f-a}\right)$ to represent the count of zeros of $f(z) - a$ with a multiplicity of up to k. We use $\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)$ to represent the corresponding count where the multiplicity is not considered. Similarly, $N_{(k}\left(r, \frac{1}{f-a}\right)$ represents the count of zeros of $f(z) - a$ with a multiplicity greater than or equal to k, and $\overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$ represents the corresponding count where the multiplicity is not considered.

Let's say we have a function f and a non-negative integer (or infinity) k. We can define $E_k(a; f)$ as the set of all points a where f equals a. If a appears as an a-point of

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f with multiplicity m, we count it m times if $m \leq k$ and $k+1$ times if $m > k$. When $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k.

If f and g share (a, k), they also share (a, p) for any $0 \leq p \leq k$. Furthermore, f and g share a value of a either in terms of identity (IM) or counting multiplicities (CM) only if they share $(a, 0)$ or (a, ∞) respectively.
We denote $N_L\left(r, \frac{1}{(f-1)}\right)$ as the counting function of zeros of $f-1$ where $p > q$, with $\overline{N}_L\left(r, \frac{1}{(f-1)}\right)$ representing the reduced counting function. Similarly, $N_E^{1)}\left(r, \frac{1}{(f-1)}\right)$ denotes the counting function of zeros of $f-1$ where $p = q = 1$. Suppose z_0 is a zero of $f-1$ with multiplicity p and a zero of $g-1$ with multiplicity q. We use $N_L\left(r, \frac{1}{(f-1)}\right)$ to count zeros of $f-1$ where $p \geq q$, and $N_E^{1)}\left(r, \frac{1}{(g-1)}\right)$ follows similarly. Additionally, $N_E^{(2}\left(r, \frac{1}{(f-1)}\right)$ counts those 1 points of f where $p = q \geq 2$, with $N_E^{(2}\left(r, \frac{1}{(g-1)}\right)$ defined in a parallel manner.

Definition 1.1. [12] The difference polynomial and its shifts in $f(z)$ is defined as

$$\Psi_0(z, f) = \sum_{\lambda \in I} a_\lambda(z) f(z)^{i_{\lambda,0}} f(z+c_1)^{i_{\lambda,1}} ... f(z+c_k)^{i_{\lambda,k}}, \tag{1}$$

where degree is denoted as $d(\Psi_0) = max_{\lambda \in I}\{d(\lambda)\}$ and $\lambda = \{i_{\lambda,0}, ..., i_{\lambda,k}\}$, I is a finite set of the index and meromorphic co-efficients $a_\lambda(z)$ are satisfying $T(r, a_\lambda(z)) = S(r, f)$, $\lambda \in I$. $f(z)^{i_{\lambda,0}} f(z+c_1)^{i_{\lambda,1}} ... f(z+c_k)^{i_{\lambda,k}}$ is monomial in $f(z)$ and $f(z+c_1), ..., f(z+c_k)$, where $c_1, ..., c_k$ are distinct non-zero complex constants and $d(\lambda) = i_{\lambda,0} + ... + i_{\lambda,k}$.

Definition 1.2. The definition of the general differential-difference polynomial of f(z) and its shifts, as provided in [1], is as follows.

$$\begin{aligned}
\Psi(z, f) &= \sum_{\lambda \in I} a_\lambda(z) f(z)^{\lambda_{0,0}} f^{(1)}(z)^{\lambda_{0,1}} ... f^{(m)}(z)^{\lambda_{0,m}} \\
&\times f(z+c_1)^{\lambda_{1,0}} f^{(1)}(z+c_1)^{\lambda_{1,1}} ... f^{(m)}(z+c_1)^{\lambda_{1,m}} \\
&... f(z+c_k)^{\lambda_{k,0}} f^{(1)}(z+c_k)^{\lambda_{k,1}} ... f^{(m)}(z+c_k)^{\lambda_{k,m}} \\
&= \sum_{\lambda \in I} a_\lambda(z) \prod_{i=0}^{k} \prod_{j=0}^{m} f^{(j)}(z+c_i)^{\lambda_{i,j}}
\end{aligned} \tag{2}$$

where I is a finite set of multi-indices $\lambda = (\lambda_{0,0}, ..., \lambda_{0,m}, \lambda_{1,0}, ..., \lambda_{1,m}, ..., \lambda_{k,0}, ..., \lambda_{k,m})$, $c_0(=0)$ and $c_1, c_2, ..., c_k$ are distinct complex constants. The growth of $a_\lambda(z), \lambda \in I$ is $S(r, f)$.
$d(\lambda) = \sum_{i=0}^{k} \sum_{j=0}^{m} \lambda_{i,j}$ denotes the degree of the monomial $\prod_{i=0}^{k} \prod_{j=0}^{m} f^{(j)}(z+c_i)^{\lambda_{i,j}}$ of $\Psi(z, f)$. Then $d(\Psi) = \max_{\lambda \in I}\{d(\lambda)\}, d^*(\Psi) = \min_{\lambda \in I}\{d(\lambda)\}$ denote the degree and the lower degree of $\Psi(z, f)$ respectively.

The differential-difference polynomial $\Psi(z, f)$ is called a homogeneous if $d(\Psi) = d^*(\Psi)$ otherwise, it is a non-homogeneous.

A study on uniqueness under different conditions was conducted for $f(z)$ and $f^{(k)}(z)$ sharing a small function [2, 4, 6, 10, see]. In 2008, Zhang and Lu [11] concluded.

Theorem A. [11] *Suppose $k(\geq 1)$ and $n(\geq 1)$ are integers, and f is a non-constant meromorphic function. Moreover, consider a small meromorphic function $a(z)$ concerning f, where $a(z)$ is distinct from 0 and ∞. If f^n and $f^{(k)}$ share the value $a(z)$ IM and*

$$4\Theta(0, f) + (2k+6)\Theta(\infty, f) + 2\delta_{k+2}(0, f) > 12 + 2k - n,$$

or f^n *and* $f^{(k)}$ *share the value* $a(z)$ *CM and*

$$2\Theta(0,f)+(k+3)\Theta(\infty,f)+\delta_{k+2}(0,f)>6+k-n,$$

then $f\equiv f^{(k)}$.

In 2013, Bhoosnurmath and Kabbur extended the above result to a general differential polynomial and obtained the following results.

Theorem B. [1] *Consider a non-constant meromorphic function* f *and a small meromorphic function* $a(z)$ *such that* $a(z)$ *is not identically equal to* 0 *or* ∞. *Let* $\Psi[f]$ *represent a non-constant differential polynomial in* f. *If* f *and* $\Psi[f]$ *share the value* a *IM and*

$$(2Q+6)\Theta(\infty,f)+(2+3\underline{d}(\Psi))\delta(0,f)>2Q+2\underline{d}(\Psi)+\overline{d}(\Psi)+7,$$

then $f\equiv\Psi[f]$.

Theorem C. [1]*Given a non-constant meromorphic function* f *and a small meromorphic function* $a(z)$ *such that* $a(z)$ *is not identically equal to* 0 *or* ∞, *along with* $\Psi[f]$ *denoting a non-constant differential polynomial in* f, *if* f *and* $\Psi[f]$ *share the value* a *CM and*

$$3\Theta(\infty,f)+(\underline{d}(\Psi)+1)\delta(0,f)>4,$$

then $f\equiv\Psi[f]$.

Theorem D. [1] *Suppose* f *is a non-constant entire function and* $a(z)$ *is a small meromorphic function such that* $a(z)$ *is not identically equal to* 0 *or* ∞. *Let* $\Psi[f]$ *denote a non-constant differential polynomial in* f. *If* f *and* $\Psi[f]$ *share the value* a *IM and*

$$(3\underline{d}(\Psi)+2)\delta(0,f)>2\overline{d}(\Psi)+2,$$

then $f\equiv\Psi[f]$.

Theorem E. [1] *Consider* $f(z)$ *as a non-constant entire function and* $a(z)$ *as a small meromorphic function such that* $a(z)$ *is not identically equal to* 0 *or* ∞. *Let* $\Psi[f]$ *represent a non-constant differential polynomial in* f. *If* f *and* $\Psi[f]$ *share the value* a *CM and*

$$(\underline{d}(\Psi)+1)\,\delta(0,f)>1,$$

then $f\equiv\Psi[f]$.

In 2020, [3] studied $\Psi(z,f)$ instead of a differential polynomial in f and proved some results:

Theorem F. [3] *Given a non-constant meromorphic function* $f(z)$ *and a small meromorphic function* $a(z)$, *where* $a(z)$ *is not identically equal to* 0 *or* ∞, *let* $\Psi(z,f)$ *denote a non-constant differential-difference polynomial as defined in* (2). *If* $f(z)$ *and* $\Psi(z,f)$ *share the value* a *IM and*

$$\Theta(\infty,f)(2Q^*+6)+\delta(0,f)(3d^*(\Psi)+2)>2Q^*+2d(\Psi)+8, \tag{3}$$

then $f(z)\equiv\Psi(z,f)$.

Theorem G. [3] *Assume* $f(z)$ *is a non-constant meromorphic function and* $a(z)$ *is a small meromorphic function such that* $a(z)\not\equiv 0,\infty$. *Let* $\Psi(z,f)$ *be a non-constant differential-difference polynomial as defined in* (2). *If* $f(z)$ *and* $\Psi(z,f)$ *share the value* a *CM and*

$$3\Theta(\infty,f)+\delta(0,f)(d^*(\Psi)+1)>4, \tag{4}$$

then $f(z)\equiv\Psi(z,f)$.

Theorem H. [3] *Consider $f(z)$ as a non-constant entire function and $a(z)$ as a small meromorphic function such that $a(z)$ is not identically equal to 0 or ∞. Let $\Psi(z,f)$ denote a non-constant differential-difference polynomial as defined in (2). If $f(z)$ and $\Psi(z,f)$ share the value a IM and*

$$\delta(0,f)(3d^*(\Psi)+2) > 2d(\Psi)+2, \tag{5}$$

then $f(z) \equiv \Psi(z,f)$.

Theorem I. [3] *Given $f(z)$ as a non-constant entire function and $a(z)$ as a small meromorphic function, where $a(z)$ is not identically equal to 0 or ∞, let $\Psi(z,f)$ represent a non-constant differential-difference polynomial as defined in Definition 1. If $f(z)$ and $\Psi(z,f)$ share the value a CM and*

$$(d^*(\Psi)+1)\delta(0,f) > 1, \tag{6}$$

then $f(z) \equiv \Psi(z,f)$.

Question 1. What happens if the non-constant meromorphic function $f(z)$ and the differential-difference polynomial $\Psi(z,f)$ share a value a with finite weight?

Question 2. When examining a meromorphic function f within a polynomial $p(f)$ and a differential-difference polynomial $\Psi(z,f)$, what conclusions can be drawn regarding the uniqueness of $p(f)$ and $\Psi(z,f)$ when they share a value a CM(IM)?

In this paper, we try to answer these two questions. Indeed, the following theorems are the main results of the paper.

Theorem 1.1. *Let $f(z)$ be a non-constant meromorphic function and l be a non-negative integer. Suppose $a(\neq 0,\infty)$ is a meromorphic function satisfying $T(r,a)=o(T(r,f))$ as $r\to\infty$ such that $f(z)$ and $\Psi(z,f)$ share (a,l). If $l\geq 2$ and*

$$\Theta(\infty,f)\left(Q^*+3\right)+2\Theta(0,f)+\delta(0,f)d(\Psi) \geq Q^*+2d(\Psi)-2d^*(\Psi)+5, \tag{7}$$

or $l=1$ and

$$\Theta(\infty,f)\left(Q^*+\frac{7}{2}\right)+\Theta(0,f)\frac{5}{2}+\delta(0,f)d(\Psi) \geq 2d(\Psi)+Q^*-d^*(\Psi)+6, \tag{8}$$

or $l=0$ and

$$\Theta(\infty,f)(2Q^*+6)+4\Theta(0,f)+\delta(0,f)2d(\Psi) \geq 4d(\Psi)+2Q^*-2d^*(\Psi)+10, \tag{9}$$

then $f(z) \equiv \Psi(z,f)$.

Example 1.1. *Let $\Psi(z,f) = -f(z)f^{(1)}$, where $f(z)=e^z$. Then $\Psi(z,f)$ and f share $(0,\infty)$ all the conditions (7) - (9) of Theorem 1.1 are satisified but $\Psi(z,f)\not\equiv f(z)$.*

This example shows that the condition $a\not\equiv 0$ is necessary for Theorem 1.1.

Theorem 1.2. *Suppose $f(z)$ is a non-constant meromorphic function and $a(z)$ is a small function where $a(z)\neq 0,\infty$. Let $p(z)$ be a non-zero polynomial of degree $n\geq 1$, and $\Psi(z,f)$ be a non-constant differential-difference polynomial. If $p(f)$ and $\Psi(z,f)$ share the value a IM and*

$$\Theta(\infty,f)(2Q^*+6)+\delta(0,f)(3d^*(\Psi)+2n) > 2Q^*+2d(\Psi)+2n+6, \tag{10}$$

then $p(f)\equiv \Psi(z,f)$.

Theorem 1.3. *Given a non-constant meromorphic function $f(z)$ and a small function $a(z)$ with $a(z) \neq 0, \infty$, let $p(z)$ denote a non-zero polynomial of degree $n \geq 1$. Additionally, consider $\Psi(z,f)$ as a non-constant differential-difference polynomial. If $p(f)$ and $\Psi(z,f)$ share the value a CM and*

$$3\Theta(\infty,f) + (d^*(\Psi)+n)\delta(0,f) > 3+n, \tag{11}$$

then $p(f) \equiv \Psi(z,f)$.

Theorem 1.4. *Considering $f(z)$ as a non-constant entire function and $a(z)$ as a small function with $a(z) \neq 0, \infty$, let $p(z)$ represent a non-zero polynomial of degree $n \geq 1$. Furthermore, let $\Psi(z,f)$ be a non-constant differential-difference polynomial. If $p(f)$ and $\Psi(z,f)$ share the value a CM and*

$$\delta(0,f)(d^*(\Psi)+n) > n, \tag{12}$$

then $p(f) \equiv \Psi(z,f)$.

Theorem 1.5. *Given $f(z)$, a non-constant entire function, and $a(z)$, a small function with $a(z) \neq 0, \infty$, along with $p(z)$, a non-zero polynomial of degree $n \geq 1$, and $\Psi(z,f)$, a non-constant differential-difference polynomial, suppose $p(f)$ and $\Psi(z,f)$ share the value a IM and*

$$(3d^*(\Psi)+2n)\delta(0,f) > 2d(\Psi)+2n, \tag{13}$$

then $p(f) \equiv \Psi(z,f)$.

Example 1.2. *Let p be a polynomial of degree one and $f = e^z$, $\Psi(z,f) = f^{(2)}(z)^{\frac{1}{2}} f(z+2\pi i)^{\frac{1}{2}}$. Here, by definition of (1.1) and by $\Psi(z,f)$ we observe that $d(\Psi) = \lambda_{0,1} + \lambda_{1,0} = \frac{1}{2} + \frac{1}{2} = 1$, i.e., $d(\Psi) = 1$, $d^*(\Psi) = \lambda_{0,1} + \lambda_{1,0} = \frac{1}{2} + \frac{1}{2} = 1$, i.e., $d^*(\Psi) = 1$ and $Q^* = 3\lambda_{0,1} + \lambda_{1,0} = 2$, i.e., $Q^* = 2$. Also $\overline{N}(r,f) = S(r,f)$ and $\overline{N}(r,0;f) = \overline{N}(r,0;e^z) \sim T(r,f)$. Then $\Theta(\infty,f) = 1$ and $\delta(0,f) = 0$. The deficiency conditions in* (10), (11), (12), *and* (13) *are not satisfied, but $p(f) \equiv \Psi(z,f)$.*

Hence, this example demonstrates that the conditions we have obtained are sufficient but not necessary for ensuring $p(f) \equiv P(z,f)$, in Theorems 1.1, 1.2, 1.3 and 1.4

Remark 1. *Let's examine the cases where $i = 0$ or $i = 1$. Assuming $c_1 = 0$, according to the definition of $\Psi(z,f)$, we obtain*

$$\begin{aligned}\Psi(z,f) &= \sum_{\lambda \in I} a_\lambda(z) f(z)^{\lambda_{0,0}+\lambda_{1,0}} f^{(1)}(z)^{\lambda_{0,1}+\lambda_{1,1}} \ldots f^{(m)}(z)^{\lambda_{0,m}+\lambda_{1,m}} \\ &= \sum_{\lambda \in I} a_\lambda(z) f(z)^{n_{i0}} f^{(1)}(z)^{n_{i1}} \ldots f^{(m)}(z)^{n_{im}} = \Psi[f],\end{aligned}$$

where $n_{i0} = \lambda_{0,0} + \lambda_{1,0}, n_{i1} = \lambda_{0,1} + \lambda_{1,1}, \ldots, n_{im} = \lambda_{0,m} + \lambda_{1,m}, i = 0,1$. Then taking $d(\Psi) = \overline{d}(\Psi)$, and $d^*(\Psi) = \underline{d}(\Psi)$, we get

(1) In theorem 1.2, we get

$$\Theta(\infty,f)(2Q+6) + \delta(0,f)(3\underline{d}(\Psi)+2) > 2Q + 2\overline{d}(\Psi) + 8,$$

this signifies an advancement upon the outcome presented in Theorem. B.

(2) In Theorem 1.3, we get

$$3\Theta(\infty,f) + (\underline{d}(\Psi)+1)\delta(0,f) > 4,$$

which aligns with Theorem C.

(3) *In Theorem 1.4, we get*
$$(\underline{d}(\Psi)+1)\delta(0,f) > 1,$$
which aligns with Theorem E.

(4) *In Theorem 1.5, we get*
$$(3\underline{d}(\Psi)+2)\delta(0,f) > 2d(\Psi)+2,$$
which aligns with Theorem D.

2. **Lemmas**

Lemma 2.1. [8] *Suppose $f(z)$ is a non-constant meromorphic function.*
$$N\left(r,\frac{1}{f^{(k)}}\right) = N\left(r,\frac{1}{f}\right) + T(r,f^{(k)}) - T(r,f) + S(r,f), \tag{14}$$

$$N\left(r,\frac{1}{f^{(k)}}\right) \leq k\overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + S(r,f). \tag{15}$$

Lemma 2.2. [9] *Consider the expression $\varphi = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$, where F and G are two non-constant meromorphic functions. If F and G share 1 IM and $\varphi \not\equiv 0$, then*
$$N_E^{1)}\left(r,\frac{1}{F-1}\right) \leq N(r,\varphi) + S(r,F) + S(r,G). \tag{16}$$

Lemma 2.3. [7] *Suppose $f(z)$ is a transcendental meromorphic function of zero order, and let q and η be two non-zero complex constants. Then*
$$\begin{aligned}
T(r,f(qz+\eta)) &= T(r,f(z)) + S(r,f),\\
N(r,\infty;f(qz+\eta)) &\leq N(r,\infty;f(z)) + S(r,f),\\
N(r,0;f(qz+\eta)) &\leq N(r,0;f(z)) + S(r,f),\\
\overline{N}(r,\infty;f(qz+\eta)) &\leq \overline{N}(r,\infty;f(z)) + S(r,f),\\
\overline{N}(r,0;f(qz+\eta)) &\leq \overline{N}(r,0;f(z)) + S(r,f).
\end{aligned}$$

Lemma 2.4. [3] *Suppose $f(z)$ is a meromorphic function and $\Psi(z,f)$ is a differential-difference polynomial in f. Then*
$$m\left(r,\frac{\Psi(z,f)}{f^{d^*(\Psi)}}\right) \leq (d(\Psi)-d^*(\Psi))m(r,f) + S(r,f). \tag{17}$$

Lemma 2.5. [3] *Consider $f(z)$ as a meromorphic function and $\Psi(z,f)$ as a differential-difference polynomial in f. Then*
$$m\left(r,\frac{\Psi(z,f)}{f^{d(\Psi)}}\right) \leq (d(\Psi)-d^*(\Psi))m\left(r,\frac{1}{f}\right) + S(r,f). \tag{18}$$

Lemma 2.6. [3]*Consider $f(z)$ as a meromorphic function and $\Psi(z,f)$ as a differential-difference polynomial in f. Then*
$$N(r,\Psi(z,f)) \leq d(\Psi)N(r,f) + Q^*\overline{N}(r,f) + S(r,f). \tag{19}$$

Lemma 2.7. [3] *Consider $f(z)$ as a meromorphic function and a differential-difference polynomial $\Psi(z,f)$ in f. Then*
$$N\left(r,\frac{\Psi(z,f)}{f^{d(\Psi)}}\right) \leq Q^*\left(\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right)\right) + (d(\Psi)-d^*(\Psi))N\left(r,\frac{1}{f}\right) + S(r,f) \tag{20}$$

Lemma 2.8. [3]*Consider $f(z)$ as a meromorphic function and a differential-difference polynomial $\Psi(z,f)$ in f. Then*

$$N\left(r,\frac{\Psi(z,f)}{f^{d^*(\Psi)}}\right)\leq (d(\Psi)-d^*(\Psi))N\left(r,f\right)+Q^*\left(\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right)+S(r,f). \quad (21)$$

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Lemma 2.9. [3] *Consider $f(z)$ as a meromorphic function and a differential-difference polynomial $\Psi(z,f)$ in f. Then*

$$T(r,\Psi(z,f))\leq d(\Psi)T(r,f)+Q^*\overline{N}(r,f)+S(r,f), \quad (22)$$

where $Q^*=\max\limits_{0\leq i\leq k,\lambda\in I}\{\lambda_{i,1}+2\lambda_{i,2}+...+m\lambda_{i,m}\}$.

Lemma 2.10. [3] *Consider $f(z)$ as a meromorphic function and a differential-difference polynomial $\Psi(z,f)$ in f. If $\Psi(z,f)\not\equiv 0$, then we have*

$$\begin{aligned}N\left(r,\frac{1}{\Psi(z,f)}\right)\leq\ & T(r,\Psi(z,f))-T(r,f^{d(\Psi)})+(d(\Psi)-d^*(\Psi))m\left(r,\frac{1}{f}\right)\\ &+N\left(r,\frac{1}{f^{d(\Psi)}}\right)+S(r,f),\end{aligned} \quad (23)$$

$$N\left(r,\frac{1}{\Psi(z,f)}\right)\leq Q^*\overline{N}(r,f)+(d(\Psi)-d^*(\Psi))m\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{d(\Psi)}}\right)+S(r,f), \quad (24)$$

where $Q^*=\max_{0\leq i\leq k,\lambda\in I}\{\lambda_{i,1}+2\lambda_{i,2}+...+m\lambda_{i,m}\}$.

Lemma 2.11. [3] *Consider $f(z)$ as a meromorphic function and a differential-difference polynomial $\Psi(z,f)$ in f of degree d and let $Q^*=\lambda_{0,1}+2\lambda_{0,2}+...+m\lambda_{0,m}$. Then*

$$T(r,\Psi(z,f))=O(T(r,f)),S(r,\Psi(z,f))=S(r,f).$$

Lemma 2.12. [3] *Consider f and g a non constant meromorphic functions*

i) if f and g share $(0,1)$, then

$$N_L\left(r,\frac{1}{f-1}\right)\leq\overline{N}\left(r,f\right)+\overline{N}\left(r,\frac{1}{f}\right)+S(r), \quad (25)$$

Here, as r approaches infinity, $S(r)=o(T(r))$, where $T(r)=\max T(r,f),T(r,g)$.

ii) if f and g share $(1,1)$, then

$$\begin{aligned}2\overline{N}_L\left(r,\frac{1}{g-1}\right)+2\overline{N}_L\left(r,\frac{1}{f-1}\right)-\overline{N}_{f\geq 2}\left(r,\frac{1}{g-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{f-1}\right)\\ \leq N\left(r,\frac{1}{g-1}\right)-\overline{N}\left(r,\frac{1}{g-1}\right).\end{aligned} \quad (26)$$

3. **Proof of Main Results**

Proof of Theorem 1.1. Consider $F=\frac{f}{a}$ and $G=\frac{\Psi(z,f)}{a}$. Then $F-1=\frac{f-a}{a}$ and $G-1=\frac{\Psi(z,f)-a}{a}$.

Given that $f(z)$ and $\Psi(z,f)$ share (a,l), we can conclude that F and G share $(1,l)$ except at the zeros and poles of a. Additionally, observe that

$$\overline{N}(r,F)=\overline{N}(r,f),$$

$$\overline{N}(r,G)=\overline{N}(r,\Psi(z,f)=\overline{N}(r,f)+s(r,f).$$

Define,

$$\varphi=\left(\frac{F^{''}}{F^{'}}-\frac{2F^{'}}{F-1}\right)-\left(\frac{G^{''}}{G^{'}}-\frac{2G^{'}}{G-1}\right), \tag{27}$$

Claim $\varphi=0$,
suppose on the contrary that $\varphi\neq 0$. Theerefore from (27), we have

$$m(r,f)=S(r,f).$$

By the Nevanlinna Second fundemental theoerm of, we have

$$\begin{aligned}T(r,G)+T(r,F)\leq \overline{N}\left(r,\frac{1}{F}\right)+2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)\\ +\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f).\end{aligned} \tag{28}$$

$N_0\left(r,\frac{1}{F'}\right)$ represents the counting function of zeros of $F^{'}$ that are distinct from the zeros of $F\left(F-1\right)$. Similarly $N_0\left(r,\frac{1}{G'}\right)$ is defined.

Case 1. From (28), when $l\geq 1$, we have

$$\begin{aligned}\overline{N}_E^{1)}\left(r,\frac{1}{F-1}\right)&\leq N\left(r,\frac{1}{\varphi}\right)+S(r,f),\leq N\left(r,\varphi\right)+S(r,f)\\ &\leq \overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,F\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\ &\quad+\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f),\end{aligned}$$

and so,

$$\begin{aligned}\overline{N}\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)&=N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\ &\quad+\overline{N}\left(r,\frac{1}{G-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+S\left(r,f\right),\\ &\leq \overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,f\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\ &\quad+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)\\ &\quad+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,f).\end{aligned} \tag{29}$$

Subcase 1.1. When $l=1$, we have,

$$\overline{N}_L\left(r,\frac{1}{F-1}\right)\leq\frac{1}{2}N\left(r,1/F^{'}|F\neq 0\right)\leq\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+\frac{1}{2}\overline{N}\left(r,F\right), \tag{30}$$

Here, $N\left(r,1/F^{'}|F\neq 0\right)$ represents the zeros of $F^{'}$ excluding those of F. Combining (26) and (30), we obtain

$$\begin{aligned}2\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\ \leq N\left(r,\frac{1}{G-1}\right)+\frac{1}{2}\left(\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,F\right)\right)+S(r,f).\end{aligned} \tag{31}$$

Thus, from (31) and (30), we have

$$\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\leq \overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,f\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\frac{1}{2}\overline{N}\left(r,f\right)$$
$$+T(r,G)+N_0\left(r,\frac{1}{F'}\right)+\frac{1}{2}\overline{N}\left(r,\frac{1}{f}\right)+N_0\left(r,\frac{1}{G'}\right)$$
$$+S(r,f). \tag{32}$$

From (28), (32), and using (24), we have

$$T(r,F)\leq \frac{7}{2}\overline{N}\left(r,f\right)+N\left(r,\frac{1}{G}\right)+\frac{5}{2}\overline{N}\left(r,\frac{1}{f}\right)+S(r,f),$$

$$T(r,f)\leq\left[(1-\Theta\left(\infty,f\right))\left(Q^*+\frac{7}{2}\right)+(1-\Theta\left(0,f\right))\frac{5}{2}+(1-\delta\left(0,f\right))d(\Psi)+(d(\Psi)-d^*(\Psi))\right]$$
$$T(r,f)+S(r,f),$$

$$\Theta\left(\infty,f\right)\left(Q^*+\frac{7}{2}\right)+\Theta\left(0,f\right)\frac{5}{2}+\delta\left(0,f\right)d(\Psi)\leq Q^*+2d(\Psi)-d^*(\Psi)+5,$$

This contradicts the assertion in (8).

Subcase 1.2. For $l\geq 2$, under these circumstances, we have

$$2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)$$
$$\leq N\left(r,\frac{1}{G-1}\right)+S(r,f).$$

Derived from (29), we acquire,

$$\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\leq \overline{N}\left(r,f\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+N\left(r,\frac{1}{G-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)$$
$$+N_0\left(r,\frac{1}{G'}\right)+N_0\left(r,\frac{1}{G'}\right)$$
$$\leq \overline{N}\left(r,f\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+T(r,G)$$
$$+N_0\left(r,\frac{1}{G'}\right)+N_0\left(r,\frac{1}{F'}\right)+S(r,f). \tag{33}$$

Now from (28), (24) and (33), we obtain

$$T(r,F)\leq 3\overline{N}\left(r,f\right)+2\overline{N}\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+S(r,f)$$
$$\leq (Q^*+3)\,\overline{N}\left(r,f\right)+2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right)d(\Psi)+m\left(r,\frac{1}{f}\right)(d(\Psi)-d^*(\Psi))$$
$$+S(r,f),$$

$$T(r,f)\leq[(1-\Theta\left(\infty,f\right))\left(Q^*+3\right)+2(1-\Theta\left(0,f\right))+(1-\delta\left(0,f\right))d(\Psi)+(d(\Psi)-d^*(\Psi))]$$
$$T(r,f)+S(r,f),$$
$$\Theta\left(\infty,f\right)\left(Q^*+3\right)+2\Theta\left(0,f\right)+\delta\left(0,f\right)d(\Psi)\leq 2d(\Psi)+Q^*-d^*(\Psi)+4,$$

This contradicts the assertion in (7).

Case 2. In the case where $l=0$, we then have:

$$N_E^{1)}\left(r,\frac{1}{F-1}\right)=N_E^{1)}\left(r,\frac{1}{G-1}\right)+S(r,f),\qquad \overline{N}_E^{(2}\left(r,\frac{1}{G-1}\right)=\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+S(r,f).$$

And also, from (3.2), we have

$$\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\leq N_E^{1)}\left(r,\frac{1}{F-1}\right)+N_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)$$
$$+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,f),$$

$$\begin{aligned}\overline{N}\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)\leq{}&\overline{N}\left(r,F\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)+N_E^{(2}\left(r,\frac{1}{F-1}\right)\\&+\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\&+S(r,f).\end{aligned}\tag{34}$$

From (25), (26), (28), and (8), we get

$$T(r,G)+T(r,F)\leq 2\overline{N}\left(r,f\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)$$
$$-\left(N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)\right)+S(r,f),$$

$$T(r,F)\leq 4\overline{N}\left(r,\frac{1}{f}\right)+6\overline{N}\left(r,f\right)+2N\left(r,\frac{1}{G}\right)+S(r,f),$$

$$T(r,f)\leq\left[\left(1-\Theta(\infty,f)\right)\left(2Q^*+6\right)+\left(1-\Theta(0,f)\right)4+\left(1-\delta(0,f)\right)2d(\Psi)+2\left(d(\Psi)-d^*(\Psi)\right)\right]$$
$$T(r,f)+S(r,f).$$

We obtain,

$$\Theta(\infty,f)\left(2Q^*+6\right)+\Theta(0,f)4+\delta(0,f)2d(\Psi)\leq 4d(\Psi)+2Q^*-2d^*(\Psi)+9.$$

This contradicts the assertion in (9).

This confirms the assertion, demonstrating that $\varphi\equiv 0$. Thus, according to (27) , we deduce that

$$\frac{G^{''}}{G'}-\frac{2G^{'}}{G-1}=\frac{F^{''}}{F'}-\frac{2F^{'}}{F-1},$$

so on integrating twice, we obtain

$$\frac{1}{F-1}=\frac{\mathcal{A}}{G-1}+\mathcal{B}.\tag{35}$$

$\mathcal{A}\neq 0$ and $\mathcal{B}$ are constant.

In this context, three possible cases can emerge:

Subcase 1.1. When $\mathcal{B}\neq 0,-1$, from (35), we get

$$\frac{F-1}{\mathcal{B}+1-\mathcal{B}F}=\frac{G-1}{\mathcal{A}},\quad \overline{N}\left(r,\frac{1}{F-\frac{\mathcal{B}+1}{\mathcal{B}}}\right)=\overline{N}\left(r,G\right).$$

Under these conditions, the Nevanlinna Second fundamental theorem provides:

$$\begin{aligned}T(r,f)&=T(r,F)+S(r,f),\\&\leq\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,F\right)+\overline{N}\left(r,\frac{1}{F-\frac{D+1}{D}}\right)+S(r,f),\\&\leq\left[\left(1-\Theta(0,f)\right)+2\left(1-\Theta(\infty,f)\right)\right]T(r,f)+S(r,f),\end{aligned}$$

$$\Theta(0,f)+2\Theta(\infty,f)\leq 2.$$

This contradicts the assertion in (7), (8) and (9).

Subcase 1.2. Assuming $\mathcal{B}=0$, according to (35), we get:

$$G=\mathcal{A}F-(\mathcal{A}-1). \tag{36}$$

Our assertion is that $\mathcal{A}=1$. Suppose $\mathcal{A}\neq 1$. Then, based on (36), we obtain:

$$\overline{N}(r,G)=\overline{N}\left(r,\frac{1}{F-\frac{\mathcal{A}-1}{\mathcal{A}}}\right).$$

Using the Nevanlinna second fundamental theorem and (24), we obtain

$$\begin{aligned}T(r,f)&=T(r,F)+S(r,f)\\&\leq\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F-\frac{D+1}{D}}\right)+S(r,f),\\&\leq[(1-\Theta(\infty,f))(Q^*+1)-\Theta(0,f)+(1-\delta(0,f))\,d(\Psi)+d(\Psi)\\&\quad-d^*(\Psi)+1]T(r,f)+S(r,f),\end{aligned}$$

$$\Theta(\infty,f)(Q^*+1)+\Theta(0,f)+\delta(0,f)d(\Psi)\leq 2d(\Psi)+Q^*-d^*(\Psi)+1.$$

Thus $\mathcal{A}=1$, and in this case, from (3.11)

$$F=G,$$

and so $f(z)=\Psi(z,f)$.

Subcase 1.3. Suppose $\mathcal{B}=-1$ from (35),

$$\frac{1}{F-1}=\frac{\mathcal{A}}{G-1}-1, \tag{37}$$

$$\Longrightarrow\ F=\frac{\mathcal{A}}{\mathcal{A}-G+1}. \tag{38}$$

If $\mathcal{A}\neq-1$

$$\overline{N}\left(r,\frac{1}{F-\frac{\mathcal{A}}{\mathcal{A}+1}}\right)=\overline{N}\left(r,\frac{1}{G}\right).$$

Applying the same reasoning as in subcase 1.2 leads to a contradiction. Hence, $\mathcal{A}=-1$.

From (38), we have:

$$GF\equiv 1,$$

$$i.e., f(z).\,[\Psi(z,f)]\equiv a^2. \tag{39}$$

Therefore, under these conditions, we have $\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)=S(r,f)$,

Based on (38) and (39), along with the first fundamental theorem,

$$\begin{aligned}(1+d(\Psi))\,T(r,f)&=T\left(r,\frac{1}{f^{d(\Psi)+1}}\right),\\&\leq m\left(r,\frac{\Psi(z,f)}{f^{d(\Psi)}}\right)+N\left(r,\frac{\Psi(z,f)}{f^{d(\Psi)}}\right)+S(r,f),\\&\leq T(r,f)\,(d(\Psi)-d^*(\Psi))+S(r,f),\end{aligned}$$

$$(1+d^*(\Psi))\,T(r,f)\leq S(r,f).$$

Which is a contradiction.

Proof of theorem 1.2.

$$F=\frac{\Psi\left(z,f\right)}{a}\quad G=\frac{p(f)}{a}, \tag{40}$$

Given that $p(f)$ and $\Psi(z,f)$ share a IM, it implies that F and G also share 1 IM. Now, utilizing lemmaLemma 2.11 and from (1), we can deduce

$$T(r,G)\leq T(r,f)+S(r,f), \tag{41}$$

$$\begin{aligned}&\overline{N}(r,F)=\overline{N}(r,\Psi(z,f)=\overline{N}(r,f)+S(r,f),\\&\overline{N}(r,G)=\overline{N}(r,f)+S(r,f),\end{aligned} \tag{42}$$

$$\overline{N}_E^{1)}\left(r,\frac{1}{F-1}\right)=\overline{N}_E^{1)}\left(r,\frac{1}{G-1}\right)+S(r,f), \tag{43}$$

$$\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)=\overline{N}_E^{(2}\left(r,\frac{1}{G-1}\right)+S(r,f), \tag{44}$$

$$\overline{N}_L\left(r,\frac{1}{F-1}\right)\leq\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+S(r,f), \tag{45}$$

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)&=\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,f)\\&\leq N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{F-1}\right)\\&\quad+N_L\left(r,\frac{1}{G-1}\right)+S(r,f).\end{aligned} \tag{46}$$

Suppose that $\varphi\not\equiv 0$. Then we have,

$$\begin{aligned}N(r,\varphi)\leq{}&N_{(2}\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+N_{(2}\left(r,\frac{1}{G}\right)+N_L\left(r,\frac{1}{F-1}\right)\\&+N_L\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right),\end{aligned} \tag{47}$$

Here, $N_0\left(r,\frac{1}{F'}\right)$ represents the counting function for the zeros of $F^{'}$ excluding those shared with F and $F-1$. Similarly, $N_0\left(r,\frac{1}{G'}\right)$ is defined similarly.
Applying the second fundamental theorem yields

$$\begin{aligned}T(r,G)+T(r,F)\leq{}&\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)\\&+\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f).\end{aligned} \tag{48}$$

Given that F and G share 1 IM, we deduce from (46)

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)={}&2N_E^{1)}\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{F-1}\right)\\&+2N_L\left(r,\frac{1}{G-1}\right)+2\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right).\end{aligned} \tag{49}$$

From this, (16) and (47), we get

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\leq{}&N_{(2}\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+N_{(2}\left(r,\frac{1}{G}\right)+N_0\left(r,\frac{1}{F'}\right)\\&+3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)\\&+2N_E^{(2}\left(r,\frac{1}{F-1}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f).\end{aligned} \tag{50}$$

We now note that

$$N_L\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{G-1}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right) \leq N\left(r,\frac{1}{G-1}\right)\leq T(r,G)+O(1). \tag{51}$$

Combining (50) and (51) yields

$$\overline{N}\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)\leq N_{(2}\left(r,\frac{1}{F}\right)+N_{(2}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)+2N_L\left(r,\frac{1}{F-1}\right) + N_L\left(r,\frac{1}{G-1}\right)+T(r,G)+N_0\left(r,\frac{1}{F'}\right) + N_0\left(r,\frac{1}{G'}\right)+S(r,f). \tag{52}$$

Employing (52) within (48) and (42), results in

$$T(r,F)\leq 3\overline{N}(r,G)+N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+2N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right) + S(r,f). \tag{53}$$

Utilizing (53) and (23) yields

$$T(r,\Psi(z,f))\leq N\left(r,\frac{1}{\Psi(z,f)}\right)+3\overline{N}(r,G)+N\left(r,\frac{1}{f}\right)+2N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right) + S(r,f),$$

$$T(r,f)d(\Psi)\leq nN\left(r,\frac{1}{f}\right)+3\overline{N}(r,f)+\left(d(\Psi)-m\left(r,\frac{1}{f}\right)d^*(\Psi)\right)+N\left(r,\frac{1}{f^{d(\Psi)}}\right) + 2N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right)+S(r,f). \tag{54}$$

From (15), (23), and (40), we get

$$2N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right)\leq 2N\left(r,\frac{1}{F'}\right)+N\left(r,\frac{1}{G'}\right) \leq \overline{N}(r,f)(2Q^*+3)+(2d(\Psi)+n)N\left(r,\frac{1}{f}\right) + 2m\left(r,\frac{1}{f}\right)(d(\Psi)-d^*(\Psi))+S(r,f). \tag{55}$$

Again using (55) in (54), we get

$$T(r,f)d(\Psi)\leq nN\left(r,\frac{1}{f}\right)+3\overline{N}(r,G)+(d(\Psi)-d^*(\Psi))m\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{d(\Psi)}}\right) + \overline{N}(r,f)(2Q^*+3)+N\left(r,\frac{1}{f}\right)(2d(\Psi)+n) + 2(d(\Psi)-d^*(\Psi))m\left(r,\frac{1}{f}\right)+S(r,f),$$

$$T(r,f)(3d^*(\Psi)-2d(\Psi))\leq \overline{N}(r,f)(2Q^*+6)+N\left(r,\frac{1}{f}\right)(3d^*(\Psi)+2n)+S(r,f).$$

Therefore, we obtain

$$\Theta(\infty,f)(2Q^*+6)+\delta(0,f)(3d^*(\Psi)+2n)\leq 2(Q^*+n+d(\Psi))+6, \tag{56}$$

which contradicts (10).

Thus, $\varphi = 0$.
Integrating φ results in

$$\frac{1}{G-1} = \frac{\mathcal{A}}{F-1} + \mathcal{B}, \tag{57}$$

Here, ($\mathcal{A} \neq 0$) and $\mathcal{B}$ are constants. Consequently,

$$G = \frac{(\mathcal{B}+1)F + (\mathcal{A}-\mathcal{B}-1)}{\mathcal{B}F + (\mathcal{A}-\mathcal{B})}, \quad F = \frac{(\mathcal{B}-\mathcal{A})G + (\mathcal{A}-\mathcal{B}-1)}{\mathcal{B}G - (\mathcal{B}+1)}. \tag{58}$$

We examine the following three cases.
Case 1. Suppose $\mathcal{B} \neq 0, -1$. According to (58), we have

$$\overline{N}\left(r, \frac{1}{G - \frac{(\mathcal{B}+1)}{\mathcal{B}}}\right) = \overline{N}(r, F). \tag{59}$$

From this, along with the second fundamental theorem, we have

$$T(r,G) \leq \overline{N}\left(r, \frac{1}{G - \frac{(\mathcal{B}+1)}{\mathcal{B}}}\right) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G}\right) + S(r,f),$$

$$nT(r,f) \leq (2Q^* + 6)\overline{N}(r,f) + (3d^*(\Psi) + 2n)N\left(r, \frac{1}{f}\right) + S(r,f),$$

Therefore, we have

$$\Theta(\infty, f)(2Q^* + 6) + \delta(0,f)(3d^*(\Psi) + 2n) \leq 3d^*(\Psi) + 2Q^* + 7,$$

which contradicts (10).
Case 2. If $\mathcal{B} = 0$, then according to (58), we have

$$G = \frac{F + (\mathcal{A}-1)}{\mathcal{A}}, \quad F = \mathcal{A}G - (\mathcal{A}-1). \tag{60}$$

Our assertion is that $\mathcal{A} = 1$. Assuming $\mathcal{A} \neq 1$, then from (60), we obtain

$$N\left(r, \frac{1}{F}\right) = N\left(r, \frac{1}{G - \frac{(\mathcal{A}-1)}{\mathcal{A}}}\right). \tag{61}$$

With this and the Nevanlinna second fundamental theorem, we obtain

$$T(r,G) \leq \overline{N}\left(r, \frac{1}{G}\right) + N\left(r, \frac{1}{G - \frac{(\mathcal{A}-1)}{\mathcal{A}}}\right) + \overline{N}(r,G) + S(r,f),$$

$$\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\Psi(z,f)}\right) + S(r,f),$$

$$[n - d(\Psi) + d^*(\Psi)]T(r,f) \leq (Q^*+1)\overline{N}(r,f) + (n + d(\Psi))N\left(r, \frac{1}{f}\right) + S(r,f).$$

So, we have

$$(Q^*+1)\Theta(\infty,f) + (n + d(\Psi))\delta(0,f) \leq Q^* + 2d(\Psi) - d^*(\Psi) + 1,$$

this contradicts (10).
Hence, $\mathcal{A} = 1$. According to (60), we have $G \equiv F$.
Thus, $p(f(z)) \equiv \Psi(z,f)$.
Case 3. If $\mathcal{B} = -1$, then according to (58), we have

$$G = \frac{\mathcal{A}}{-F + \mathcal{A} + 1}, \quad F = \frac{(1+\mathcal{A})G - \mathcal{A}}{G}. \tag{62}$$

In case $\mathcal{A} \neq -1$, we deduce from (62) that

$$N\left(r, \frac{1}{G - \frac{\mathcal{A}}{(\mathcal{A}+1)}}\right) = N\left(r, \frac{1}{F}\right). \tag{63}$$

Using the same reasoning as in case 2, we arrive at a contradiction.
Hence, $\mathcal{A} = -1$.
From (62), we obtain

$$GF = 1. \tag{64}$$

That is,

$$p(f).\Psi(z,f) = a^2. \tag{65}$$

From (65), we have

$$N\left(r, \frac{1}{f}\right) + N(r,f) = S(r,f). \tag{66}$$

Employing (62), (65), Lemma Lemma 2.10, and the Nevanlinna first fundamental theorem, we derive

$$\begin{aligned} T(r,f)(d(\Psi)+n) &= T\left(r, \frac{1}{f^{d(\Psi)+n}}\right) \\ &= T\left(r, \frac{\Psi(z,f)}{f^{d(\Psi)}.a^2}\right) + S(r,f) \\ &\leq T\left(r, \frac{1}{f}\right)(d(\Psi) - d^*(\Psi)) + S(r,f). \end{aligned}$$

We have,

$$(d^*(\Psi)+n)T(r,f) \leq S(r,f), \tag{67}$$

This leads to a contradiction.
Thus, the proof of Theorem 1.2 is complete.

Proof of Theorem 1.3. Consider the definitions of F and G as given in (40).
From the theorem's hypothesis, it follows that F and G share 1 CM. Hence,

$$\overline{N}_L\left(r, \frac{1}{F-1}\right) = \overline{N}_L\left(r, \frac{1}{G-1}\right) = 0. \tag{68}$$

Continuing similarly to the Proof of Theorem 1.1, we arrive at (54), which is:

$$\begin{aligned} T(r,f)d(\Psi) \leq & nN\left(r, \frac{1}{f}\right) + 3\overline{N}(r,G) + m\left(r, \frac{1}{f}\right)(d(\Psi) - d^*(\Psi)) + N\left(r, \frac{1}{f^{d(\Psi)}}\right) \\ & + 2N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) + S(r,f). \end{aligned}$$

Using (68) in (54), we get

$$\begin{aligned} T(r,f)d(\Psi) \leq & N\left(r, \frac{1}{f^{d(\Psi)}}\right) + 3\overline{N}(r,G) + m\left(r, \frac{1}{f}\right)(d(\Psi) - d^*(\Psi)) + nN\left(r, \frac{1}{f}\right) \\ & + S(r,f) \\ & \leq 3\overline{N}(r,f) + (d(\Psi) - d^*(\Psi))\left[T(r,f) - N\left(r, \frac{1}{f}\right)\right] + d(\Psi)N\left(r, \frac{1}{f}\right) \\ & + nN\left(r, \frac{1}{f}\right) + S(r,f), \end{aligned}$$

$$T(r,f)d^*(\Psi) \leq (d^*(\Psi)+n)N\left(r, \frac{1}{f}\right) + 3\overline{N}(r,f) + S(r,f).$$

Thus, we have

$$3\Theta(\infty,f) + \delta(0,f)(d^*(\Psi)+n) \leq 3+n,$$

This contradicts (11).
Therefore, $\varphi \equiv 0$. Following a similar approach to the Proof of theorem 1.2, we establish Theorem 1.3.
Thus, the proof of Theorem 1.3 is concluded.

3.1. **Proof of Theorem 1.4.** Given the hypothesis that $f(z)$ is a non-constant entire function, we can employ $N(r,f) = S(r,f)$ in the Proof of Theorem 1.2 to derive the proof of Theorem 1.4.

3.2. **Proof of Theorem 1.5.** Given the hypothesis that $f(z)$ is a non-constant entire function, we can utilize $N(r,f) = S(r,f)$ in the Proof of Theorem 1.3 to derive the proof of Theorem 1.5.

Open Question 1.1. Considering the non-constant meromorphic function $f_1^p p(f_1)$, where $f_1 = z - c$ for some $c \in \mathbb{C}$, along with the differential-difference polynomial $\Psi(z,f)$, what implications arise if they share a value a with finite weight?

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