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NON-OSCILLATORY BEHAVIOR OF HIGHER ORDER HILFER FRACTIONAL DIFFERENCE EQUATION

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ABSTRACT. In this paper, we look into the non-oscillatory behavior of the higher order Hilfer fractional difference equation. The approach we employ is based on certain fundamental concepts derived from discrete fractional calculus and mathematical inequalities. In order to aid in arriving at the important end result, a volterra-type summation equation is constructed as a similar representation of our problem. We were able to come up with new, easier to implement condition that satisfied by the non-oscillatory solutions to our analyzed equation. To demonstrate the empirical reliability of the theoretical finding, we lend a numerical example.

1. INTRODUCTION

Prior to the beginning of the 20th century, mathematical theory about fractional calculus had been put forth. The advent of new fractional operators broadens the range of definitions and clarifies their diverse uses. Its non-local feature opens up new paths for research and applications in important interdisciplinary issues in physical, biological and engineering sciences [12, 13, 2].

Discrete fractional calculus is a new discipline, similar to its continuous counterpart. The theory of delta fractional calculus has been expanded by the contributions of noted mathematicians such as Atici, Eloe, Abdeljawad, and Anastassiou, to point out a few [10, 11, 1, 8, 19, 22]. It has gained prominence in the last decade due to its intrinsic complexity and non-local features. The development of its theory is still in progress, which opens up new doors along with possibilities for exploration in this domain.

The most recent study has focused on the existence and uniqueness, stability, and oscillation of the solutions owing to figuring out the behavior of solutions

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is of the utmost importance for understanding equations [14, 23, 15]. Indeed, the oscillation theory offers important insights into the dynamics of solutions to equation-modeled problems in many scientific and technical domains. Many scientists have directed their research efforts toward the remarkably constructive and fast-advancing field of oscillation theory of fractional order difference equations in recent years [9, 6].

Motivated by the concept of Hilfer fractional derivative [24], Haider et.al [19] introduced the Hilfer discrete fractional operator, which is a generalization of Riemann-Liouville and Caputo operators. This operator interpolates both the Riemann-Liouville and Caputo operators, and T.Y.Uzun [25] studied the oscillatory behavior of some higher order fractional difference equation using this operator.

Although the oscillatory behavior was the focus of research, non-oscillatory behavior for nonlinear fractional difference equations still need work. Yet, a few mathematicians determined the non-oscillatory behavior for the solutions of the following sorts of nonlinear fractional difference equations:

Graef et.al [18] studied for the following forced fractional differential equation with positive and negative terms of the form

$$^{\mathfrak{c}}D^{\varpi}_{\mathfrak{b}}y(\xi)+\Upsilon_1(\xi,x(\xi))=\mathfrak{e}(\xi)+\mathfrak{g}(\xi)x^{\varrho}(\xi)+\Upsilon_2(\xi,x(\xi)),$$

where $\varpi\in(0,1)$ and $^{\mathfrak{c}}D^{\varpi}_{\mathfrak{b}}y$ denotes the Caputo fractional derivative of y of order ϖ. They examined for $y(\xi)=(\mathfrak{d}(\xi)(x'(\xi))^{\varrho})'$ and $y(\xi)=\mathfrak{d}(\xi)(x'(\xi))^{\varrho}$.

Alzabut et.al [4] studied for the following forced nabla fractional difference equation with positive and negative terms of the form

$$\nabla^{\varpi}_{\mathfrak{b}*}y(\xi)+\Upsilon_1(\xi,x(\xi))=\mathfrak{e}(\xi)+\mathfrak{g}(\xi)x(\xi)+\Upsilon_2(\xi,x(\xi)),$$

where $\varpi\in(0,1)$ and $\nabla^{\varpi}_{b*}y$ denotes the Caputo nabla fractional difference of y of order ϖ. They examined for $y(\xi)=\nabla(\mathfrak{d}(\xi)\nabla x(\xi))$ and $y(\xi)=\mathfrak{d}(\xi)\nabla x(\xi)$.

Alzabut et.al [3] studied for the following higher order forced nabla fractional difference equation with positive and negative terms of the form

$$\nabla^{\varpi}_{\mathfrak{b}*}y(\xi)+\Upsilon_1(\xi,x(\xi))=\mathfrak{e}(\xi)+\mathfrak{g}(\xi)x^{\varrho}(\xi)+\Upsilon_2(\xi,x(\xi)),$$

where $\varpi\in(0,1)$ and $\nabla^{\varpi}_{b*}y$ denotes the Caputo nabla fractional difference of y of order ϖ. They examined for $y(\xi)=\nabla^{\mathfrak{m}-1}\big(\mathfrak{d}(\xi)(\nabla x(\xi))^{\varrho}\big),\mathfrak{m}\in\mathbb{N}_1$.

Alzabut et.al [5] studied for the following forced delta fractional difference equation with positive and negative terms of the form

$$\Delta^{\varpi}y(\xi)+\Upsilon_1(\xi+\varpi,x(\xi+\varpi))=\mathfrak{e}(\xi+\varpi)+\mathfrak{g}(\xi+\varpi)x^{\varrho}(\xi+\varpi)+\Upsilon_2(\xi+\varpi,x(\xi+\varpi)),$$

where $\varpi\in(0,1]$ and $\Delta^{\varpi}y$ is the Caputo delta fractional difference of y of order ϖ. They examined for $y(\xi)=\Delta(\mathfrak{d}(\xi)(\Delta x(\xi))^{\varrho})$, $y(\xi)=\mathfrak{d}(\xi)(\Delta x(\xi))^{\varrho}$ and $y(\xi)=x(\xi)$.

Urged by the aforementioned works, to provide an affirming response regarding the behavior of non-oscillatory solutions, we look into the higher-order forced fractional difference problem with the Hilfer difference operator of the form

$$\begin{cases}\Delta^{\varpi,\nu}_{\mathfrak{b}}y(\xi)+\Upsilon_1(\xi,x(\xi+\varpi-1))=\mathfrak{e}(\xi)+\mathfrak{g}(\xi)x^{\varrho}(\xi+\varpi-1)+\Upsilon_2(\xi,x(\xi+\varpi-1))\\ \Delta^{-(1-\tau)}_{\mathfrak{b}}y(\xi)|_{\xi=\mathfrak{b}+1-\tau}=y_0,\end{cases}\tag{1}$$

where $y(\xi) = \Delta^{\mathfrak{m}-1}[\mathfrak{d}(\xi)(\Delta x(\xi))^{\varrho}]$, $\mathfrak{m} \in \mathbb{N} \cup \{0\}$, $0 < \varpi \leq 1$, $0 \leq \nu \leq 1$, $\xi \in \mathbb{N}_{\mathfrak{b}+1-\varpi}$, $\tau = \varpi + \nu - \varpi\nu$ and $\Delta_{\mathfrak{b}}^{\varpi,\nu}$ is the Hilfer Type fractional difference operator of order ϖ and type ν. Also, $\Upsilon_1, \Upsilon_2 : \mathbb{N}_{\mathfrak{b}+1-\varpi} \times \mathbb{R} \to \mathbb{R}$, $\mathfrak{e}, \mathfrak{g}, \mathfrak{d} : \mathbb{N}_{\mathfrak{b}+1-\varpi} \to \mathbb{R}$, are continuous functions. Here $\mathbb{N}_{\mathfrak{a}} = \{\mathfrak{a}, \mathfrak{a}+1, \mathfrak{a}+2, ...\}$.

The layout of the article is as follows: A few fundamental yet significant definitions and findings from discrete calculus are presented in section 2. In section 3, we use discrete fractional calculus features and mathematical inequalities to offer sufficient conditions for a solution to be non-oscillatory. Section 4 presents appropriate illustrations to support the theoretical conclusions.

2. Preliminaries

In the subsequent section, we present some preliminary discrete fractional calculus observations that will be applied to the main findings.

Definition 1. [10] *Let $x : \mathbb{N}_{\mathfrak{b}} \to \mathbb{R}$ and $\varpi > 0$. Then the ϖ-th fractional sum of x is defined by*

$$\Delta_{\mathfrak{b}}^{-\varpi} x(\xi) := \sum_{\vartheta=\mathfrak{b}}^{\xi-\varpi} \mathfrak{h}_{\varpi-1}(\xi, \sigma(\vartheta)) x(\vartheta)$$

for $\xi \in \mathbb{N}_{\mathfrak{b}+\varpi}$, where $\vartheta^{(\varpi)} = \frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta+1-\varpi)}$ and $\mathfrak{h}_{\varpi}(\vartheta, \theta) = \frac{(\vartheta-\theta)^{(\varpi)}}{\Upsilon(\varpi+1)}$.

Definition 2. [10] *Let $x : \mathbb{N}_{\mathfrak{b}} \to \mathbb{R}$ and $\lceil \varpi \rceil = \mathfrak{n}$. Then the ϖ-th Riemann-Liouville fractional difference of x is defined by*

$$\Delta_{\mathfrak{b}}^{\varpi} x(\xi) := \Delta^{\mathfrak{n}} \Delta_{\mathfrak{b}}^{-(\mathfrak{n}-\varpi)} x(\xi), \qquad \xi \in \mathbb{N}_{\mathfrak{b}+\mathfrak{n}-\varpi}.$$

Definition 3. [7] *Let $x : \mathbb{N}_{\mathfrak{b}} \to \mathbb{R}$ and $\lceil \varpi \rceil = \mathfrak{n}$. Then the ϖ-th Caputo fractional difference of x is defined by*

$$\Delta_{\mathfrak{b}}^{\varpi} x(\xi) := \Delta_{\mathfrak{b}}^{-(\mathfrak{n}-\varpi)} \Delta^{\mathfrak{n}} x(\xi), \qquad \xi \in \mathbb{N}_{\mathfrak{b}+\mathfrak{n}-\varpi}.$$

Definition 4. [19] *Let $x : \mathbb{N}_{\mathfrak{b}} \to \mathbb{R}$ and $\lceil \varpi \rceil = \mathfrak{n}$. Then the Hilfer like fractional difference of order ϖ and type $0 \leq \nu \leq 1$ of function x is defined by*

$$\Delta_{\mathfrak{b}}^{\varpi,\nu} x(\xi) := \Delta_{\mathfrak{b}+(1-\nu)(\mathfrak{n}-\varpi)}^{-\nu(\mathfrak{n}-\varpi)} \Delta^{\mathfrak{n}} \Delta_{\mathfrak{b}}^{-(1-\nu)(\mathfrak{n}-\varpi)} x(\xi), \qquad \xi \in \mathbb{N}_{\mathfrak{b}+\mathfrak{n}-\varpi}.$$

The special cases Riemann-Liouville fractional difference and Caputo fractional difference will be obtained by putting $\nu = 0$ and $\nu = 1$, respectively.

Lemma 1. *(Young's inequality* [20]*)*

(i) If χ and ζ are non-negative, $\mathfrak{u} > 1$ and $\frac{1}{\mathfrak{u}} + \frac{1}{\mathfrak{v}} = 1$, then $\chi\zeta \leq \frac{1}{\mathfrak{u}}\chi^{\mathfrak{u}} + \frac{1}{\mathfrak{v}}\zeta^{\mathfrak{v}}$, where equality holds if and only if $\zeta = \chi^{\mathfrak{u}-1}$.

(ii) If χ and ζ are non-negative, $0 < \mathfrak{u} < 1$ and $\frac{1}{\mathfrak{u}} + \frac{1}{\mathfrak{v}} = 1$, then $\chi\zeta \geq \frac{1}{\mathfrak{u}}\chi^{\mathfrak{u}} + \frac{1}{\mathfrak{v}}\zeta^{\mathfrak{v}}$, where equality holds if and only if $\zeta = \chi^{\mathfrak{u}-1}$.

Lemma 2. *If $\mathbb{q} > \mathbb{r}$ and $0 < \mathbb{p} < 1$, then*

$$\frac{(\mathbb{q}-\mathbb{r})^{\mathbb{p}}}{\Gamma(\mathbb{p}+1)} < \mathfrak{h}_{\mathbb{p}}(\mathbb{q}, \mathbb{r}) < \frac{(\mathbb{q}-\mathbb{r}+1)^{\mathbb{p}}}{\Gamma(\mathbb{p}+1)}.$$

Proof. Using Gaustchis inequality [16], we can easily get the proof. □

Lemma 3. [17] *Let* $x : \mathbb{N}_{\mathfrak{b}} \to \mathbb{R}$, $\mathfrak{l} \in \mathbb{N}_0$, $\mathfrak{p} - 1 < \varpi_1 < \mathfrak{p}$ *and* $\mathfrak{q} - 1 < \varpi_2 \leq \mathfrak{q}$. *Then*

(i) For $\xi \in \mathbb{N}_{\mathfrak{b}+\varpi_1+\varpi_2}$, $\Delta^{-\varpi_1}_{\mathfrak{b}+\varpi_2}\Delta^{-\varpi_2}_{\mathfrak{b}}x(\xi) = \Delta^{-\varpi_1-\varpi_2}_{\mathfrak{b}}x(\xi) = \Delta^{-\varpi_2}_{\mathfrak{b}+\varpi_1}\Delta^{-\varpi_2}_{\mathfrak{b}}x(\xi)$.

(ii) For $\xi \in \mathbb{N}_{\mathfrak{b}+\varpi_1}$, $\Delta^{\mathfrak{l}}\Delta^{-\varpi_1}_{\mathfrak{b}}x(\xi) = \Delta^{\mathfrak{l}-\varpi_1}_{\mathfrak{b}}x(\xi)$.

(iii) For $\xi \in \mathbb{N}_{\mathfrak{b}+\mathfrak{q}-\varpi_2+\varpi_1}$, $\Delta^{-\varpi_1}_{\mathfrak{b}+\mathfrak{q}-\varpi_2}\Delta^{\varpi_2}_{\mathfrak{b}}x(\xi) = \Delta^{\varpi_2-\varpi_1}_{\mathfrak{b}}x(\xi) - \sum_{\mathfrak{k}=0}^{\mathfrak{q}-1}\mathfrak{h}_{\mathfrak{b}-\mathfrak{q}+\mathfrak{k}}(\xi, \mathfrak{b} + \mathfrak{q} - \varpi_1)$

$$\Delta^{\mathfrak{k}-(\mathfrak{b}-\varpi_2)}_{\alpha}x(\mathfrak{b} + \mathfrak{q} - \varpi_2).$$

Lemma 4. *If* $\mathbb{f} : \mathbb{N}_{\mathbb{a}} \to \mathbb{R}$, $\mathbb{k} \in \mathbb{N}_0$, *then*

$$\Delta^{-\mathbb{k}}\Delta^{\mathbb{k}}\mathbb{f}(\xi) = \mathbb{f}(\xi) - \sum_{\mathbb{j}=0}^{\mathbb{k}-1}\Delta^{\mathbb{j}}\mathbb{f}(\mathbb{a})\mathfrak{h}_{\mathbb{j}}(\xi, \mathbb{a}),$$

for $\xi \in \mathbb{N}_{\mathbb{a}+\mathbb{k}}$.

Proof. Using Lemma 3(iii), we get the result. □

Lemma 5. [21] *Let* $(\mathbb{y}_{\mathbb{n}})$ *and* $(\mathbb{g}_{\mathbb{n}})$ *be non-negative sequences and* $\mathbb{c}$ *and* $\mathbb{n}$ *be non-negative constants. If*

$$\mathbb{y}_{\mathbb{n}} \leq \mathbb{c} + \sum_{\mathbb{t}=0}^{\mathbb{n}}\mathbb{y}_{\mathbb{n}}\mathbb{g}_{\mathbb{n}},$$

then

$$\mathbb{y}_{\mathbb{n}} \leq \mathbb{c}\exp\sum_{\mathbb{t}=0}^{\mathbb{n}}\mathbb{g}_{\mathbb{n}}.$$

Lemma 6. *The Hilfer fractional difference problem* (1) *has an unique solution*

$$y(\xi) = \mathfrak{h}_{\mathfrak{b}-1}(\xi, \mathfrak{b} + 1 - \tau)y_0 + \sum_{\mathfrak{s}=\mathfrak{b}+1-\varpi}^{\xi-\varpi}\mathfrak{h}_{\varpi-1}(\xi, \sigma(\mathfrak{s}))[\mathfrak{e}(\mathfrak{s}) + \mathfrak{g}(\mathfrak{s})x^{\rho}(\mathfrak{s} + \varpi - 1) + \Upsilon_2(\mathfrak{s}, x(\mathfrak{s} + \varpi - 1) - \Upsilon_1(\xi, x(\xi + \varpi - 1)))].$$

Proof. Applying $\Delta^{-\varpi}_{\mathfrak{b}+1-\varpi}$ on both sides of (1), we obtain

$$\Delta^{-\varpi}_{\mathfrak{b}+1-\varpi}\Delta^{\varpi,\nu}_{b}y(\xi) = \Delta^{-\varpi}_{\mathfrak{b}+1-\varpi}[\mathfrak{e}(\xi) + \mathfrak{g}(\xi)x^{\rho}(\xi + \varpi - 1) + \Upsilon_2(\xi, x(\mathfrak{s} + \varpi - 1) - \Upsilon_1(\xi, x(\xi + \varpi - 1)))].$$

Now,

$$\begin{aligned}\Delta^{-\varpi}_{\mathfrak{b}+1-\varpi}\Delta^{\varpi,\nu}_{b}y(\xi) &= \Delta^{-\varpi}_{\mathfrak{b}+1-\varpi}[\Delta^{-\nu(1-\varpi)}_{\mathfrak{b}+(1-\nu)(1-\varpi)}\Delta^{1}\Delta^{-(1-\nu)(1-\varpi)}_{\mathfrak{b}}y(\xi)] \\ &= \Delta^{-\varpi-\nu(1-\varpi)}_{\mathfrak{b}+1-\varpi-\nu(1-\varpi)}\Delta^{1}\Delta^{-(1-\nu)(1-\varpi)}_{\mathfrak{b}}y(\xi) \\ &= \Delta^{-\varpi-\nu(1-\varpi)}_{\mathfrak{b}+1-\varpi}\Delta^{\varpi+\nu(1-\varpi)}_{\mathfrak{b}}y(\xi) \\ &= y(\xi) - \mathfrak{h}_{\tau-1}(\xi, \mathfrak{b} + 1 - \tau)y_0.\end{aligned}$$

This completes the proof. □

3. Behavior of Non-Oscillatory solutions

In this section, we figure out the behavior of non-oscillatory solutions for the Hilfer fractional difference problem (1). To get the result, we take into account the following considerations:

($\mathscr{C}$**.1**) $\frac{\Upsilon_{\mathfrak{k}}(\xi,x)}{x} > 0, \quad (\mathfrak{k}=1,2), \quad x \neq 0, \quad \xi \in \mathbb{N}_{\mathfrak{b}+1-\varpi},$

($\mathscr{C}$**.2**) $|\Upsilon_1(\xi,x)| \geq \Omega_1(\xi)|x|^{\alpha_1}, \quad |\Upsilon_2(\xi,x)| \leq \Omega_2(\xi)|x|^{\alpha_2}, \quad x \neq 0, \quad \xi \in \mathbb{N}_{\mathfrak{b}+1-\varpi}$ for some continuous functions $\Omega_{\mathfrak{k}} : \mathbb{N}_{\mathfrak{b}+1-\varpi} \to \mathbb{R}^+$, $\mathfrak{k}=1,2$ and for some positive real numbers $\alpha_1 > \alpha_2$.

For $\mathbb{s} \in \mathbb{N}_{\mathfrak{b}+1-\varpi}$, denote

$$\mathcal{G}(\mathbb{s}) = \frac{\alpha_1-\alpha_2}{\alpha_2}\Omega_1(\mathbb{s})\left[\frac{\alpha_2}{\alpha_1}\frac{\Omega_2(\mathbb{s})}{\Omega_1(\mathbb{s})}\right]^{\frac{\alpha_1}{\alpha_1-\alpha_2}}$$

and

$$\mathcal{H}(\mathbb{s}) = \sum_{\mathbb{u}=\mathfrak{b}+1}^{\mathbb{s}+\varpi}\left(\frac{1}{\mathfrak{d}(\mathbb{u})}\right)^{1/\varrho}.$$

Theorem 1. *Suppose ($\mathscr{C}$.1) and ($\mathscr{C}$.2) agree and assume that there exists a positive real number β such that*

$$\lim_{\xi\to\infty}\sum_{\mathfrak{s}=\mathfrak{b}+1-\varpi}^{\xi-\varpi}\mathfrak{h}_{\varpi-1}(\xi,\sigma(\mathfrak{s}))\mathfrak{e}(\mathfrak{s}) < \infty, \tag{2}$$

$$\lim_{\xi\to\infty}\sum_{\mathfrak{s}=\mathfrak{b}+1-\varpi}^{\xi-\varpi}\mathfrak{h}_{\varpi-1}(\xi,\sigma(\mathfrak{s}))\mathcal{G}(\mathfrak{s}) < \infty \tag{3}$$

and

$$\lim_{\xi\to\infty}\sum_{\mathfrak{s}=\mathfrak{b}+1-\varpi}^{\xi-\mathfrak{m}-1}\mathfrak{g}^{\beta}(\mathfrak{s}+1-\varpi)\mathfrak{s}^{\mathfrak{m}\beta}\mathcal{H}^{\varrho\beta}(\mathfrak{s}) < \infty. \tag{4}$$

Then every non-oscillatory solution $x(\xi)$ of (1) *satisfies*

$$\limsup_{\xi\to\infty}\frac{|x(\xi)|}{\xi^{\mathfrak{m}/\varrho}\mathcal{H}(\xi)} < \infty. \tag{5}$$

Proof.

Suppose that $x(\xi)$ represents a non-oscillatory solution to (1). Assuming that $x(\xi)$ is eventually a positive solution of (1) on $\mathbb{N}_{\mathfrak{b}}$, we can get a sufficiently large $\xi_1 > \mathfrak{b}+1$ such that $x(\xi) > 0$ for $\xi \in \mathbb{N}_{\xi_1}$. By using Lemma 6, we can say

$$\begin{aligned} y(\xi) = {} & \mathfrak{h}_{\tau-1}(\xi,\mathfrak{b}+1-\tau)y_0 + \sum_{\mathfrak{s}=\mathfrak{b}+1-\varpi}^{\xi-\varpi}\mathfrak{h}_{\varpi-1}(\xi,\sigma(\mathfrak{s}))\mathfrak{e}(\mathfrak{s}) + \\ & \sum_{\mathfrak{s}=\mathfrak{b}+1-\varpi}^{\xi_1-\varpi-1}\mathfrak{h}_{\varpi-1}(\xi,\sigma(\mathfrak{s}))\mathfrak{g}(\mathfrak{s})x^{\rho}(\mathfrak{s}+\varpi-1) + \sum_{\mathfrak{s}=\xi_1-\varpi}^{\xi-\varpi}\mathfrak{h}_{\varpi-1}(\xi,\sigma(\mathfrak{s}))\mathfrak{g}(\mathfrak{s})x^{\rho}(\mathfrak{s}+\varpi-1) + \\ & \sum_{\mathfrak{s}=\mathfrak{b}+1-\varpi}^{\xi_1-\varpi-1}\mathfrak{h}_{\varpi-1}(\xi,\sigma(\mathfrak{s}))[\Upsilon_2(\mathfrak{s},x(\mathfrak{s}+\varpi-1)) - \Upsilon_1(\xi,x(\xi+\varpi-1))] + \end{aligned}$$

$$\sum_{\mathfrak{s}=\xi_1-\varpi}^{\xi-\varpi} \mathfrak{h}_{\varpi-1}(\xi,\sigma(\mathfrak{s}))[\Upsilon_2(\mathfrak{s},x(\mathfrak{s}+\varpi-1)-\Upsilon_1(\xi,x(\xi+\varpi-1)))].$$

Simplifying using Lemma 2, (2), ($\mathscr{C}$.**1**) and ($\mathscr{C}$.**2**) , we get

$$y(\xi) < \mathcal{C}_0 + \sum_{\mathfrak{s}=\xi_1-\varpi}^{\xi-\varpi} \mathfrak{h}_{\varpi-1}(\xi,\sigma(\mathfrak{s}))\mathfrak{g}(\mathfrak{s})x^{\rho}(\mathfrak{s}+\varpi-1)+$$

$$\sum_{\mathfrak{s}=\xi_1-\varpi}^{\xi-\varpi} \mathfrak{h}_{\varpi-1}(\xi,\sigma(\mathfrak{s}))[\Omega_2(\mathfrak{s})x^{\alpha_2}(\mathfrak{s}+\varpi-1)-\Omega_1(\mathfrak{s})x^{\alpha_1}(\mathfrak{s}+\varpi-1)],$$

for some constant $\mathcal{C}_0 > 0$. By choosing $\chi = x^{\alpha_2}(\mathfrak{s}+\varpi-1)$, $\zeta = \frac{\alpha_2}{\alpha_1}[\frac{\Omega_2(\mathfrak{s})}{\Omega_1(\mathfrak{s})}]$, $\mathfrak{u} = \alpha_1/\alpha_2$ and $\mathfrak{v} = \alpha_1/(\alpha_1-\alpha_2)$ in Lemma 1(i), we have

$$\Omega_2(\mathfrak{s})x^{\alpha_2}(\mathfrak{s}+\varpi-1)-\Omega_1(\mathfrak{s})x^{\alpha_1}(\mathfrak{s}+\varpi-1) \leq \mathcal{G}(\mathfrak{s}). \tag{6}$$

Using (6) and (3), we obtain

$$\Delta^{\mathfrak{m}-1}[\mathfrak{d}(\xi)(\Delta x(\xi))^{\varrho}] < \mathcal{C}_1 + \sum_{\mathfrak{s}=\xi_1-\varpi}^{\xi-\varpi} \mathfrak{h}_{\varpi-1}(\xi,\sigma(\mathfrak{s}))\mathfrak{g}(\mathfrak{s})x^{\rho}(\mathfrak{s}+\varpi-1),$$

for some constant $\mathcal{C}_1 > 0$. Using Lemma 4, we obtain

$$\mathfrak{d}(\xi)(\Delta x(\xi))^{\varrho} < \sum_{\mathfrak{j}=0}^{\mathfrak{m}-2} \Delta^{\mathfrak{j}}[\mathfrak{d}(\xi_1)(\Delta x(\xi_1))^{\varrho}]\mathfrak{h}_{\mathfrak{j}}(\xi,\xi_1)+$$

$$\sum_{\mathfrak{t}=\xi_1}^{\xi-\mathfrak{m}+1} \mathfrak{h}_{\mathfrak{m}-2}(\xi,\sigma(\mathfrak{t}))\left[\mathcal{C}_1 + \sum_{\mathfrak{s}=\xi_1-\varpi}^{\mathfrak{t}-\varpi} \mathfrak{h}_{\varpi-1}(\mathfrak{t},\sigma(\mathfrak{s}))\mathfrak{g}(\mathfrak{s})x^{\rho}(\mathfrak{s}+\varpi-1)\right].$$

Simplifying the aforementioned inequality using Lemma 2, we have,

$$\mathfrak{d}(\xi)(\Delta x(\xi))^{\varrho} < \mathcal{C}_2\xi^{\mathfrak{m}} + \sum_{\mathfrak{s}=\xi_1}^{\xi-\mathfrak{m}-1} (\xi-\mathfrak{s}+1+\varpi)^{(\varpi-1)}\mathfrak{g}(\mathfrak{s}-\varpi)x^{\rho}(\mathfrak{s}-1), \tag{7}$$

for some constant $\mathcal{C}_2 > 0$. Now,

$$\sum_{\mathfrak{s}=\xi_1}^{\xi-\mathfrak{m}-1} (\xi-\mathfrak{s}+1+\varpi)^{(\varpi-1)}\mathfrak{g}(\mathfrak{s}-\varpi)x^{\rho}(\mathfrak{s}-1)$$

$$\leq \left[\sum_{\mathfrak{s}=\xi_1}^{\xi-\mathfrak{m}-1} \left((\xi-\mathfrak{s}+1+\varpi)^{(\varpi-1)}\right)^{\alpha}\right]^{1/\alpha} \left[\sum_{\mathfrak{s}=\xi_1}^{\xi-\mathfrak{m}-1} \mathfrak{g}^{\beta}(\mathfrak{s}-\varpi)x^{\rho\beta}(\mathfrak{s}-1)\right]^{1/\beta}$$

$$< \mathcal{C}_3\xi^{\mathfrak{m}}\left[\sum_{\mathfrak{s}=\xi_1}^{\xi-\mathfrak{m}-1} \mathfrak{g}^{\beta}(\mathfrak{s}-\varpi)x^{\rho\beta}(\mathfrak{s}-1)\right]^{1/\beta},$$

for some constant $\mathcal{C}_3 > 0$. Therefore, (7) becomes

$$\mathfrak{d}(\xi)(\Delta x(\xi))^{\varrho} < \mathcal{C}_2\xi^{\mathfrak{m}} + \mathcal{C}_3\xi^{\mathfrak{m}}\left[\sum_{\mathfrak{s}=\xi_1}^{\xi-\mathfrak{m}-1} \mathfrak{g}^{\beta}(\mathfrak{s}-\varpi)x^{\rho\beta}(\mathfrak{s}-1)\right]^{1/\beta}.$$

Taking $\mathbb{m}(\xi) = \mathcal{C}_2 + \mathcal{C}_3\left[\sum_{\mathfrak{s}=\xi_1}^{\xi-\mathfrak{m}-1} \mathfrak{g}^{\beta}(\mathfrak{s}-\varpi)x^{\rho\beta}(\mathfrak{s}-1)\right]^{1/\beta}$, we have

$$\mathfrak{d}(\xi)(\Delta x(\xi))^{\varrho} < \xi^{\mathfrak{m}}\mathbb{m}(\xi),$$

and hence

$$\Delta x(\xi) < \left(\frac{\xi^{\mathfrak{m}}\mathbb{m}(\xi)}{\mathfrak{d}(\xi)}\right)^{\frac{1}{\varrho}}.$$

Computing the aforementioned inequality's sum from ξ_1 to $\xi - 1$, we get

$$x(\xi) < x(\xi_1) + (\xi^{\mathfrak{m}}\mathbb{m}(\xi))^{\frac{1}{\varrho}} \sum_{\mathfrak{s}=\xi_1}^{\xi-1}\left(\frac{1}{\mathfrak{d}(\mathfrak{s})}\right)^{\frac{1}{\varrho}}$$
$$\implies \frac{x(\xi)}{\mathcal{H}(\xi)} < \frac{x(\xi_1)}{\mathcal{H}(\xi_1)} + (\xi^{\mathfrak{m}}\mathbb{m}(\xi))^{\frac{1}{\varrho}} < \mathcal{C}_4 + (\xi^{\mathfrak{m}}\mathbb{m}(\xi))^{\frac{1}{\varrho}}.$$

By employing the fundamental inequality, $(\mathcal{A}+\mathcal{B})^{\kappa} \leq 2^{\kappa-1}(\mathcal{A}^{\kappa}+\mathcal{B}^{\kappa})$, $\mathcal{A},\mathcal{B} \geq 0, \kappa \geq 1$, we get

$$\left(\frac{x(\xi)}{\mathcal{H}(\xi)}\right)^{\varrho} < \mathcal{C}_5 + \mathcal{C}_6\left[\sum_{\mathfrak{s}=\xi_1}^{\xi-\mathfrak{m}-1} \mathfrak{g}^{\beta}(\mathfrak{s}-\varpi)x^{\varrho\beta}(\mathfrak{s}-1)\right]^{1/\beta},$$

for some positive constants $\mathcal{C}_5$ and $\mathcal{C}_6$. Again employing the above fundamental inequality, we get

$$\left(\frac{x(\xi)}{\xi^{\mathfrak{m}/\varrho}\mathcal{H}(\xi)}\right)^{\varrho\beta} < \mathcal{C}_7 + \mathcal{C}_8\left[\sum_{\mathfrak{s}=\xi_1-1}^{\xi-\mathfrak{m}-1} \mathfrak{g}^{\beta}(\mathfrak{s}+1-\varpi)\mathfrak{s}^{\mathfrak{m}\beta}\mathcal{H}^{\varrho\beta}(\mathfrak{s})\left(\frac{x(\mathfrak{s})}{\mathfrak{s}^{\mathfrak{m}/\varrho}\mathcal{H}(\mathfrak{s})}\right)^{\varrho\beta}\right],$$

for some positive constants $\mathcal{C}_7$ and $\mathcal{C}_8$. Using Lemma 5, we obtain

$$\left(\frac{x(\xi)}{\xi^{\mathfrak{m}/\varrho}\mathcal{H}(\xi)}\right)^{\varrho\beta} < \mathcal{C}_9 \exp\left[\sum_{\mathfrak{s}=\xi_1-1}^{\xi-\mathfrak{m}-1} \mathfrak{g}^{\beta}(\mathfrak{s}+1-\varpi)\mathfrak{s}^{\mathfrak{m}\beta}\mathcal{H}^{\varrho\beta}(\mathfrak{s})\right],$$

where $\mathcal{C}_9$ is a constant. In accordance with the hypothesis (4), we have

$$\limsup_{\xi\to\infty} \frac{x(\xi)}{\xi^{\mathfrak{m}/\varrho}\mathcal{H}(\xi)} < \infty.$$

The procedure is identical for an eventually negative solution, and so we have left it out here. Thus, the theorem is verified. □

Corollary 1. *Suppose ($\mathscr{C}$.1), ($\mathscr{C}$.2),* (2), (3) *and* $y(\xi) = x(\xi)$ *agree and assume that there exists a positive real number* β *such that*

$$\lim_{\xi\to\infty} \sum_{\mathfrak{s}=\mathfrak{b}+1-\varpi}^{\xi-1} \mathfrak{g}^{\beta}(\mathfrak{s}+1-\varpi)(\mathfrak{s}+\varpi) < \infty. \tag{8}$$

Then every non-oscillatory solution $x(\xi)$ *of* (1) *satisfies*

$$\limsup_{\xi\to\infty} \frac{|x(\xi)|}{\xi} < \infty. \tag{9}$$

4. Example

Here, we highlight our main result by offering a numerical example.

Example 1. *Let us examine*

$$\begin{cases} \Delta_{.1}^{.3,.2} y(\xi) + (\xi+1)^4 x^2(\xi - .5) = \frac{2\pi}{\xi^2} + \frac{1}{\xi^{10}} x^3(\xi + \varpi - 1) + \xi x(\xi - .5) \\ \Delta_{\mathfrak{b}}^{-(1-\tau)} y(\xi)|_{\xi=\mathfrak{b}+1-\tau} = y_0, \end{cases} \tag{10}$$

where $y(\xi) = \Delta^2[\frac{1}{\xi^3}(\Delta x(\xi))^3]$.

Here $\varpi = .5, \nu = .2, \mathfrak{b} = .1$, $\mathfrak{g}(\xi) = 1/\xi^{10}$, $\mathfrak{e}(\xi) = \frac{2\sqrt{\pi}}{\xi^2}$, $\mathfrak{m} = \varrho = 3$ and $\mathfrak{d}(\xi) = 1/\xi^3$. Then,

$$\lim_{\xi\to\infty} \sum_{\mathfrak{s}=\mathfrak{b}+1-\varpi}^{\xi-\varpi} \mathfrak{h}_{\varpi-1}(\xi, \sigma(\mathfrak{s}))\mathfrak{e}(\mathfrak{s}) = \lim_{\xi\to\infty} \sum_{\mathfrak{s}=.6}^{\xi-.5} \mathfrak{h}_{-.5}(\xi, \sigma(\mathfrak{s})) \frac{2\sqrt{\pi}}{\mathfrak{s}^2} < \lim_{\xi\to\infty} \sum_{\mathfrak{s}=.6}^{\xi-.5} \frac{1}{\mathfrak{s}^2} < \infty.$$

Let us choose $\Omega_1(\xi) = \xi^4$, $\Omega_2(\xi) = \xi$, $\alpha_1 = 2$ and $\alpha_2 = 1$. Then, $(\mathscr{C}\mathbf{.1})$ and $(\mathscr{C}\mathbf{.2})$ are satisfied and $\mathcal{G}(\xi) = 1/4\xi^2$. So, we have

$$\lim_{\xi\to\infty} \sum_{\mathfrak{s}=\mathfrak{b}+1-\varpi}^{\xi-\varpi} \mathfrak{h}_{\varpi-1}(\xi, \sigma(\mathfrak{s}))\mathcal{G}(\mathfrak{s}) = \lim_{\xi\to\infty} \sum_{\mathfrak{s}=.6}^{\xi-.5} \mathfrak{h}_{-.5}(\xi, \sigma(\mathfrak{s})) \frac{1}{\mathfrak{s}^2} < \infty.$$

Next, by taking $\beta = 2$, we get

$$\lim_{\xi\to\infty} \sum_{\mathfrak{s}=\mathfrak{b}+1-\varpi}^{\xi-\mathfrak{m}-1} \mathfrak{g}^{\beta}(\mathfrak{s}+1-\varpi)\mathfrak{s}^{\mathfrak{m}\beta}\mathcal{H}^{\varrho\beta}(\mathfrak{s}) = \lim_{\xi\to\infty} = \sum_{\mathfrak{s}=.6}^{\xi-2} \frac{\mathfrak{s}^6(\mathfrak{s}-.6)^6(\mathfrak{s}+1.6)^6}{32(\mathfrak{s}+.5)^{20}} < \infty.$$

Therefore, by Theorem 1, every non-oscillatory solution $x(\xi)$ of (10) satisfies

$$\limsup_{\xi\to\infty} \frac{|x(\xi)|}{\xi(\xi-.6)(\xi+1.6)} < \infty.$$

Conclusion

In line with prior research that focuses on oscillation criteria, our work introduced few conditions for studying the behavior of non-oscillatory solutions to the Hilfer fractional difference equation. The main equation has a broad scope and can be applied to particular cases [5]. Our main result is proven by constructing an equivalent representation of the main equation and by using mathematical inequalities. The validity of the theoretical result is confirmed with a numerical example. Further, it strengthened some previous findings in the literature.

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