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Tripled Fixed Point Methodologies for Finding a Solution to a System of Nonlinear Integral Equations on Orthogonal Metric Spaces

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Abstract:

This manuscript is devoted to obtaining tripled fixed point results in the setting of orthogonal metric spaces. Also, some corollaries are presented. Furthermore, we obtained some results of the existence and uniqueness for tripled fixed point for orthogonal complete metric spaces which are not satisfied for complete metric spaces and we discussed this in an example. Ultimately, to support the theoretical results, a system of nonlinear integral equations is given as an application.
keywords: Complete metric spaces, orthogonal metric space, tripled fixed points, nonlinear integral equations.

1 Introduction

Fixed point theorems in complete metric spaces (CMSs) became widely used in 1922 after Banach's theorem [1]. This technique has particular resonance in many important disciplines, such as topology, dynamical systems, differential and integral equations, economics, game theory, and biological sciences [2,3]. The applications and the technique of fixed point (FP) became major roles in several fields of mathematics, computer science, statistics, biology, economics, game theory, chemistry, theory of integral equations, differential equations, and mathematical economics, etc. (see, [2,3,4,5,6,7,8,9,10,11,12]). In 2011, the concept of triple fixed point (TFP) was initiated by Berinde and Borcut [13] as a generalization of coupled FPs which are presented by Bhaskar and Lakshmikantham [14]. Moreover, many authors presented some TFP theorems for contraction mappings in several spaces, (see [15,16,17,18,19,20]). Consequently, On the other hand, the concept of orthogonal sets (O-sets) was initiated by Gordji et al. [21] who extended the results of [1] in that study. Moreover, they discussed the existence of solutions for differential equations using their results. For more contributions in this field, see [22,23,24,25,26].

2 Preliminaries

We provide a few notations and definitions that are necessary to this manuscript in this part.

Definition 1.*[22] Assume that Ω is a nonempty set and $\perp \subseteq \Omega^2$ is a binary relation. We say that Ω is an orthogonal set and $\aleph$ is an orthogonal element, if the relation $\perp$ justifies*

$$\exists \aleph_0 \in \Omega : (\forall \zeta : \zeta \perp \aleph_0) \text{ or } (\forall \zeta : \aleph_0 \perp \zeta).$$

We denote this O-Set by $(\Omega, \perp)$.

Example 1.[22] Let $\Omega = [0,\infty)$ and consider for orthogonal elements $\aleph_0 = 0$ or $\aleph_0 = 1$ such that $\aleph \perp \zeta$ if $\aleph\zeta = \{\aleph, \zeta\}$, then $(\Omega, \perp)$ is an O-Set.

Definition 2.*[22] Suppose that $(\Omega, \perp)$ is an O-Set. A sequence $\{\aleph_\omega\}_{\omega\in\mathbb{N}}$ is said to be an orthogonal sequence (OS) if*

$$(\aleph_\omega \perp \aleph_{\omega+1}) \text{ or } (\aleph_{\omega+1} \perp \aleph_\omega), \text{ for all } \omega.$$

Definition 3.[22] *Let $(\Omega,\perp)$ be an O-Set and $(\Omega,\perp,\eth)$ be a metric space, then the trio $(\Omega,\perp,\eth)$ is called an orthogonal metric space (OMS).*

Definition 4.*Assume that $(\Omega,\perp,\eth)$ is an OMS and $\digamma:\Omega\to\Omega$ is a mapping, then*

1. Γ is orthogonally continuous in $\aleph\in\Omega$ if we have $\digamma(\aleph_\omega)\to\digamma(\mu)$, for all OS $\{\aleph_\omega\}_{\omega\in\mathbb{N}}$ in Ω where $\aleph_\omega\to\aleph$ as $\omega\to\infty$.
2. Γ is orthogonally continuous on Ω if $\digamma$ is orthogonally continuous for each $\aleph\in\Omega$.

Definition 5.[22] *Assume that $(\Omega,\perp,\eth)$ is an OMS. We say that $(\Omega,\perp,\eth)$ is an orthogonally complete metric space (OCMS) if each Cauchy OS is convergent.*

*Remark.*Every CMS is OCMS and the converse is not true in general.

Definition 6.[13] *Let $\Gamma:\Omega^3\to\Omega$ be a mapping. A point $(\aleph,\zeta,\delta)\in\Omega^3$ is called a TFP of Γ if*

$$\aleph=\Gamma(\aleph,\zeta,\delta),$$

$$\zeta=\Gamma(\zeta,\delta,\aleph),$$

and

$$\delta=\Gamma(\delta,\aleph,\zeta).$$

Definition 7.*Suppose that $(\Omega,\perp)$ is an O-Set. A mapping $\Gamma:\Omega^3\to\Omega$ is called orthogonally preserving (OP) if*

$$\aleph\perp\mu,\zeta\perp\upsilon,\delta\perp\rho \textit{ implies } \Gamma(\aleph,\zeta,\delta)\perp\Gamma(\mu,\upsilon,\rho).$$

3 Main results

This section is devoted to present some TFP results in OMSs.

Firstly, let's establish the following theorem:

Theorem 1.*Assume that $(\Omega,\perp,\eth)$ is an OCMS (not necessarily CMS) and the mapping $\Gamma:\Omega^3\to\Omega$ is OP. If for all $\aleph,\zeta,\delta,\mu,\upsilon,\rho\in\Omega$ with $\aleph\perp\mu$, $\zeta\perp\upsilon$ and $\delta\perp\rho$,*

$$\begin{aligned}&\eth(\Gamma(\aleph,\zeta,\delta),\Gamma(\mu,\upsilon,\rho))\\ &\leq\alpha\eth(\aleph,\mu)+\beta\eth(\zeta,\upsilon)+\gamma\eth(\delta,\rho)\end{aligned}\tag{1}$$

where $\alpha,\beta,\gamma\geq0$ and $\alpha+\beta+\gamma<1$, then Γ has a unique TFP.

*Proof.*Assume there are orthogonal elements $\aleph_0,\zeta_0,\delta_0\in\eth$ such that

$$(\aleph_0\perp\aleph) \text{ for all } \aleph\in\Omega \text{ or } (\aleph\perp\aleph_0) \text{ for all } \aleph\in\Omega,$$

$$(\zeta_0\perp\zeta) \text{ for all } \zeta\in\Omega \text{ or } (\zeta\perp\zeta_0) \text{ for all } \zeta\in\Omega$$

and

$$(\delta_0\perp\delta) \text{ for all } \delta\in\Omega \text{ or } (\delta\perp\delta_0) \text{ for all } \delta\in\Omega.$$

So, for $\aleph_0,\zeta_0,\delta_0\in\eth$, we obtain

$$\aleph_0\perp\Gamma(\aleph_0,\zeta_0,\delta_0) \text{ or } \Gamma(\aleph_0,\zeta_0,\delta_0)\perp\aleph_0,$$

$$\zeta_0\perp\Gamma(\zeta_0,\delta_0,\aleph_0) \text{ or } \Gamma(\zeta_0,\delta_0,\aleph_0)\perp\zeta_0$$

and

$$\delta_0\perp\Gamma(\delta_0,\aleph_0,\zeta_0) \text{ or } \Gamma(\delta_0,\aleph_0,\zeta_0)\perp\delta_0,$$

for $\omega\in\mathbb{N}$, we have

$$\begin{aligned}&\aleph_1=\Gamma(\aleph_0,\zeta_0,\delta_0),\ \zeta_1=\Gamma(\zeta_0,\delta_0,\aleph_0),\\ &\delta_1=\Gamma(\delta_1,\aleph_1,\zeta_1)\end{aligned}$$

and

$$\begin{aligned}&\aleph_2=\Gamma(\aleph_1,\zeta_1,\delta_1),\ \zeta_2=\Gamma(\zeta_1,\delta_1,\aleph_1),\\ &\delta_2=\Gamma(\delta_1,\aleph_1,\zeta_1),\end{aligned}$$

iteratively, we obtain

$$\begin{aligned}&\aleph_{\omega+1}=\Gamma(\aleph_\omega,\zeta_\omega,\delta_\omega),\\ &\zeta_{\omega+1}=\Gamma(\zeta_\omega,\delta_\omega,\aleph_\omega),\\ &\delta_{\omega+1}=\Gamma(\delta_\omega,\aleph_\omega,\zeta_\omega).\end{aligned}\tag{2}$$

Hence, one can write

$$\begin{aligned}&\aleph_0\perp\Gamma(\aleph_0,\zeta_0,\delta_0)=\aleph_1\\ &\qquad\text{or } \aleph_1=\Gamma(\aleph_0,\zeta_0,\delta_0)\perp\aleph_0,\end{aligned}$$

$$\begin{aligned}&\zeta_0\perp\Gamma(\zeta_0,\delta_0,\aleph_0)=\zeta_1\\ &\qquad\text{or } \zeta_1=\Gamma(\zeta_0,\delta_0,\aleph_0)\perp\zeta_0\end{aligned}$$

and

$$\begin{aligned}&\delta_0\perp\Gamma(\delta_0,\aleph_0,\zeta_0)=\delta_1\\ &\qquad\text{or } \delta_1=\Gamma(\delta_0,\aleph_0,\zeta_0)\perp\delta_0.\end{aligned}$$

As the mappinge Γ is OP, we get

$$\begin{aligned}&\aleph_1=\Gamma(\aleph_0,\zeta_0,\delta_0)\perp\Gamma(\aleph_1,\zeta_1,\delta_1)=\aleph_2\\ &\text{or } \aleph_2=\Gamma(\aleph_1,\zeta_1,\delta_1)\perp\Gamma(\aleph_0,\zeta_0,\delta_0)=\aleph_1,\end{aligned}$$

$$\begin{aligned}&\zeta_1=\Gamma(\zeta_0,\delta_0,\aleph_0)\perp\Gamma(\zeta_1,\delta_1,\aleph_1)=\zeta_2\\ &\text{or } \zeta_2=\Gamma(\zeta_1,\delta_1,\aleph_1)\perp\Gamma(\zeta_0,\delta_0,\aleph_0)=\zeta_1\end{aligned}$$

and

$$\begin{aligned}&\delta_1=\Gamma(\delta_0,\aleph_0,\zeta_0)\perp\Gamma(\delta_1,\aleph_1,\zeta_1)=\delta_2\\ &\text{or } \delta_2=\Gamma(\delta_1,\aleph_1,\zeta_1)\perp\Gamma(\delta_0,\aleph_0,\zeta_0)=\delta_1.\end{aligned}$$

With the samw manner, we have for all $\omega\in\mathbb{N}$

$$\begin{aligned}&\aleph_\omega\perp\aleph_{\omega+1} \text{ or } \aleph_{\omega+1}\perp\aleph_\omega,\\ &\zeta_\omega\perp\zeta_{\omega+1} \text{ or } \zeta_{\omega+1}\perp\zeta_\omega,\\ &\delta_\omega\perp\delta_{\omega+1} \text{ or } \delta_{\omega+1}\perp\delta_\omega.\end{aligned}\tag{3}$$

Thus, $\{\aleph_\omega\}_{\omega\in\mathbb{N}},\{\zeta_\omega\}_{\omega\in\mathbb{N}}$ and $\{\delta_\omega\}_{\omega\in\mathbb{N}}$ are OSs. Now, we show that $\{\aleph_\omega\}_{\omega\in\mathbb{N}},\{\zeta_\omega\}_{\omega\in\mathbb{N}}$ and $\{\delta_\omega\}_{\omega\in\mathbb{N}}$ are Cauchy OSs. By using (1), we can write for all $\omega\in\mathbb{N}$ and $\alpha+\beta+\gamma<1$,

$$\begin{aligned}&\eth(\aleph_\omega,\aleph_{\omega+1})\\ &=\eth(\Gamma(\aleph_{\omega-1},\zeta_{\omega-1},\delta_{\omega-1}),\Gamma(\aleph_\omega,\zeta_\omega,\delta_\omega))\\ &\leq\alpha\eth(\aleph_{\omega-1},\aleph_\omega)+\beta\eth(\zeta_{\omega-1},\zeta_\omega)+\gamma\eth(\delta_{\omega-1},\delta_\omega).\end{aligned}\tag{4}$$

Similarly, we have

$$\begin{aligned}&\partial(\zeta_\omega,\zeta_{\omega+1})\\&=\partial(\Gamma(\zeta_{\omega-1},\delta_{\omega-1},\aleph_{\omega-1}),\Gamma(\zeta_\omega,\delta_\omega,\aleph_\omega))\\&\leq\alpha\partial(\zeta_{\omega-1},\zeta_\omega)+\beta\partial(\delta_{\omega-1},\delta_\omega)+\gamma\partial(\aleph_{\omega-1},\aleph_\omega)\end{aligned}\tag{5}$$

and

$$\begin{aligned}&\partial(\delta_\omega,\delta_{\omega+1})\\&=\partial(\Gamma(\delta_{\omega-1},\aleph_{\omega-1},\zeta_{\omega-1}),\Gamma(\delta_\omega,\aleph_\omega,\zeta_\omega))\\&\leq\alpha\partial(\delta_{\omega-1},\delta_\omega)+\beta\partial(\aleph_{\omega-1},\aleph_\omega)+\gamma\partial(\zeta_{\omega-1},\zeta_\omega).\end{aligned}\tag{6}$$

Suppose that

$$\partial_\omega=\partial(\aleph_\omega,\aleph_{\omega+1})+\partial(\zeta_\omega,\zeta_{\omega+1})+\partial(\delta_\omega,\delta_{\omega+1}).$$

From (4), (5) and (6), we have for all $\omega\in\mathbb{N}$,

$$\begin{aligned}&\partial_\omega\\&=\partial(\aleph_\omega,\aleph_{\omega+1})+\partial(\zeta_\omega,\zeta_{\omega+1})+\partial(\delta_\omega,\delta_{\omega+1})\\&\leq\alpha\partial(\aleph_{\omega-1},\aleph_\omega)+\beta\partial(\zeta_{\omega-1},\zeta_\omega)+\gamma\partial(\delta_{\omega-1},\delta_\omega)\\&\quad+\alpha\partial(\zeta_{\omega-1},\zeta_\omega)+\beta\partial(\delta_{\omega-1},\delta_\omega)+\gamma\partial(\aleph_{\omega-1},\aleph_\omega)\\&\quad+\alpha\partial(\delta_{\omega-1},\delta_\omega)+\beta\partial(\aleph_{\omega-1},\aleph_\omega)+\gamma\partial(\zeta_{\omega-1},\zeta_\omega)\\&=(\alpha+\beta+\gamma)\begin{bmatrix}\partial(\aleph_{\omega-1},\aleph_\omega)+\partial(\zeta_{\omega-1},\zeta_\omega)\\+\partial(\delta_{\omega-1},\delta_\omega)\end{bmatrix}\\&=(\alpha+\beta+\gamma)\partial_{\omega-1}.\end{aligned}$$

Iteratively, one can obtain for all $\omega\in\mathbb{N}$,

$$\begin{aligned}&0\leq\partial_\omega\leq(\alpha+\beta+\gamma)\partial_{\omega-1}\leq(\alpha+\beta+\gamma)^2\partial_{\omega-2}\\&\quad\leq\cdots\leq(\alpha+\beta+\gamma)^\omega\partial_0.\end{aligned}\tag{7}$$

Let $\partial_0=0$, this implies that

$$\partial_0=\partial(\aleph_0,\aleph_1)+\partial(\zeta_0,\zeta_1)+\partial(\delta_0,\delta_1)=0.$$

Hence, we have

$$\partial(\aleph_0,\aleph_1)=0\Rightarrow\aleph_0=\aleph_1=\Gamma(\aleph_0,\zeta_0,\delta_0),$$

$$\partial(\zeta_0,\zeta_1)=0\Rightarrow\zeta_0=\zeta_1=\Gamma(\zeta_0,\delta_0,\aleph_0)$$

and

$$\partial(\delta_0,\delta_1)=0\Rightarrow\delta_0=\delta_1=\Gamma(\delta_0,\aleph_0,\zeta_0).$$

Thus, Γ has a unique TFP $(\aleph,\zeta,\delta)$. Assume that $\partial_0>0$, then we get for each positive integer ω and ψ with $\omega\leq\psi$,

$$\begin{aligned}\partial(\aleph_\omega,\aleph_\psi)&\leq\partial(\aleph_\omega,\aleph_{\omega+1})+\partial(\aleph_{\omega+1},\aleph_{\omega+2})\\&\quad+\cdots+\partial(\aleph_{\psi-1},\aleph_\psi),\end{aligned}\tag{8}$$

$$\begin{aligned}\partial(\zeta_\omega,\zeta_\psi)&\leq\partial(\zeta_\omega,\zeta_{\omega+1})+\partial(\zeta_{\omega+1},\zeta_{\omega+2})\\&\quad+\cdots+\partial(\zeta_{\psi-1},\zeta_\psi)\end{aligned}\tag{9}$$

and

$$\begin{aligned}\partial(\delta_\omega,\delta_\psi)&\leq\partial(\delta_\omega,\delta_{\omega+1})+\partial(\delta_{\omega+1},\delta_{\omega+2})\\&\quad+\cdots+\partial(\delta_{\psi-1},\delta_\psi).\end{aligned}\tag{10}$$

Adding (8), (9) and (10), we have for $\omega\leq\psi$,

$$\begin{aligned}&\partial(\aleph_\omega,\aleph_\psi)+\partial(\zeta_\omega,\zeta_\psi)+\partial(\delta_\omega,\delta_\psi)\\&\leq\partial(\aleph_\omega,\aleph_{\omega+1})+\partial(\zeta_\omega,\zeta_{\omega+1})+\partial(\delta_\omega,\delta_{\omega+1})+\\&\quad\partial(\aleph_{\omega+1},\aleph_{\omega+2})+\partial(\zeta_{\omega+1},\zeta_{\omega+2})+\partial(\delta_{\omega+1},\delta_{\omega+2})\\&\quad+\cdots+\\&\quad\partial(\aleph_{\psi-1},\aleph_\psi)+\partial(\zeta_{\psi-1},\zeta_\psi)+\partial(\delta_{\psi-1},\delta_\psi)\\&=\partial_\omega+\partial_{\omega+1}+\cdots+\partial_{\psi-1}\\&\leq\begin{bmatrix}(\alpha+\beta+\gamma)^\omega+(\alpha+\beta+\gamma)^{\omega+1}\\+\cdots+(\alpha+\beta+\gamma)^{\psi-1}\end{bmatrix}\partial_0\\&\leq\frac{(\alpha+\beta+\gamma)^\omega}{1-(\alpha+\beta+\gamma)}\partial_0.\end{aligned}$$

Taking limit as $\omega,\psi\to\infty$, since $\frac{(\alpha+\beta+\gamma)}{1-(\alpha+\beta+\gamma)}<1$,we obtain that $\{\aleph_\omega\}_{\omega\in\mathbb{N}},\{\zeta_\omega\}_{\omega\in\mathbb{N}}$ and $\{\delta_\omega\}_{\omega\in\mathbb{N}}$ are Cauchy OSs in Ω. Let $(\Omega,\perp,\partial)$ be an OCMS, there is $\varkappa,\eta,\theta\in\Omega$ so that

$$\aleph_\omega\to\varkappa,\ \zeta_\omega\to\eta,\ \delta_\omega\to\theta.$$

Hence, there is $\omega_0\in\mathbb{N}$ with

$$\begin{aligned}&\partial(\aleph_\omega,\varkappa)\leq\frac{\varepsilon}{2},\ \partial(\zeta_\omega,\eta)\leq\frac{\varepsilon}{2},\\&\partial(\delta_\omega,\theta)\leq\frac{\varepsilon}{2},\end{aligned}\tag{11}$$

for all $\omega\geq\omega_0$ and each $\varepsilon>0$. By choice of $\varkappa,\eta$ and θ, one can write

$$(\varkappa\perp\aleph_\omega)\text{ or }(\aleph_\omega\perp\varkappa),$$
$$(\eta\perp\zeta_\omega)\text{ or }(\zeta_\omega\perp\eta)$$

and

$$(\theta\perp\delta_\omega)\text{ or }(\delta_\omega\perp\theta).$$

Applying (1) and (11), we have for $\alpha+\beta+\gamma<1$,

$$\begin{aligned}&\partial(\Gamma(\varkappa,\eta,\theta),\varkappa)\\&\leq\partial(\Gamma(\varkappa,\eta,\theta),\aleph_{\omega+1})+\partial(\aleph_{\omega+1},\varkappa)\\&=\partial(\Gamma(\varkappa,\eta,\theta),\Gamma(\aleph_\omega,\zeta_\omega,\delta_\omega))+\partial(\aleph_{\omega+1},\varkappa)\\&\leq\alpha\partial(\aleph_\omega,\varkappa)+\beta\partial(\zeta_\omega,\eta)+\gamma\partial(\delta_\omega,\theta)\\&\quad+\partial(\aleph_{\omega+1},\varkappa)\\&<(\alpha+\beta+\gamma)\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<\varepsilon.\end{aligned}$$

This leads to $\partial(\Gamma(\varkappa,\eta,\theta),\varkappa)=0$ implies $\Gamma(\varkappa,\eta,\theta)=\varkappa$. Similarly, we obtain $\Gamma(\eta,\theta,\varkappa)=\eta$ and $\Gamma(\theta,\varkappa,\eta)=\theta$. Then $(\varkappa,\eta,\theta)$ is a TFP of Γ.

For uniqueness, suppose that $\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right)\in\partial^3$ is another TFP of Γ such that $\Gamma\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right)=\widehat{\varkappa}$, $\Gamma\left(\widehat{\eta},\widehat{\theta},\widehat{\varkappa}\right)=\widehat{\eta}$ and $\Gamma\left(\widehat{\theta},\widehat{\varkappa},\widehat{\eta}\right)=\widehat{\theta}$.

(i)If $(\varkappa\perp\widehat{\varkappa})$ or $(\widehat{\varkappa}\perp\varkappa)$, $(\eta\perp\widehat{\eta})$ or $(\widehat{\eta}\perp\eta)$ and $\left(\theta\perp\widehat{\theta}\right)$ or $\left(\widehat{\theta}\perp\theta\right)$, by (1), *we can write*

$$\begin{aligned}\partial(\varkappa,\widehat{\varkappa})&=\partial\left(\Gamma(\varkappa,\eta,\theta),\Gamma\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right)\right)\\&\leq\alpha\partial(\varkappa,\widehat{\varkappa})+\beta\partial(\eta,\widehat{\eta})+\gamma\partial\left(\theta,\widehat{\theta}\right),\end{aligned}$$

$$\begin{aligned}\partial(\eta,\widehat{\eta}) &= \partial\left(\Gamma(\eta,\theta,\varkappa),\Gamma\left(\widehat{\eta},\widehat{\theta},\widehat{\varkappa}\right)\right)\\ &\le \alpha\partial(\eta,\widehat{\eta})+\beta\partial\left(\theta,\widehat{\theta}\right)+\gamma\partial(\varkappa,\widehat{\varkappa}),\end{aligned}$$

and

$$\begin{aligned}\partial\left(\theta,\widehat{\theta}\right) &= \partial\left(\Gamma(\theta,\varkappa,\eta),\Gamma\left(\widehat{\theta},\widehat{\varkappa},\widehat{\eta}\right)\right)\\ &\le \alpha\partial\left(\theta,\widehat{\theta}\right)+\beta\partial(\varkappa,\widehat{\varkappa})+\gamma\partial(\eta,\widehat{\eta}),\end{aligned}$$

thus,

$$\begin{aligned}&\partial(\varkappa,\widehat{\varkappa})+\partial(\eta,\widehat{\eta})+\partial\left(\theta,\widehat{\theta}\right)\\ &\le(\alpha+\beta+\gamma)\left[\partial(\varkappa,\widehat{\varkappa})+\partial(\eta,\widehat{\eta})+\partial\left(\theta,\widehat{\theta}\right)\right].\end{aligned}$$

As $\alpha+\beta+\gamma<1$, we have

$$\partial(\varkappa,\widehat{\varkappa})+\partial(\eta,\widehat{\eta})+\partial\left(\theta,\widehat{\theta}\right)=0.$$

this implies

$$\varkappa=\widehat{\varkappa},\ \eta=\widehat{\eta} \text{ and } \theta=\widehat{\theta}.$$

(ii)If not, from the assumption in the beginning of the proof, the are orthogonal elements $\aleph_0,\zeta_0,\delta_0\in\partial$, we get

$$(\aleph_0\perp\varkappa,\ \aleph_0\perp\widehat{\varkappa}) \text{ or } (\varkappa\perp\aleph_0,\ \widehat{\varkappa}\perp\aleph_0),$$

$$(\zeta_0\perp\eta,\ \zeta_0\perp\widehat{\eta}) \text{ or } (\eta\perp\zeta_0,\ \widehat{\eta}\perp\zeta_0)$$

and

$$\left(\delta_0\perp\theta,\ \delta_0\perp\widehat{\theta}\right) \text{ or } \left(\theta\perp\delta_0,\ \widehat{\theta}\perp\delta_0\right).$$

Therefore, using (*1*), *we have*

$$\begin{aligned}\partial(\varkappa,\widehat{\varkappa}) &= \partial\left(\Gamma(\varkappa,\eta,\theta),\Gamma\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right)\right)\\ &\le \partial(\Gamma(\varkappa,\eta,\theta),\Gamma(\aleph_0,\zeta_0,\delta_0))\\ &\quad+\partial\left(\Gamma(\aleph_0,\zeta_0,\delta_0),\Gamma\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right)\right)\\ &\le \alpha\partial(\aleph_0,\varkappa)+\beta\partial(\zeta_0,\eta)+\gamma\partial(\delta_0,\theta)\\ &\quad+\alpha\partial(\aleph_0,\widehat{\varkappa})+\beta\partial(\zeta_0,\widehat{\eta})+\gamma\partial\left(\delta_0,\widehat{\theta}\right).\end{aligned}$$

Passing $\omega\to\infty$, we get $\partial(\varkappa,\widehat{\varkappa})=0$. Thus, $\varkappa=\widehat{\varkappa}$. Similarly, we have

$$\begin{aligned}\partial(\eta,\widehat{\eta}) &= \partial\left(\Gamma(\eta,\theta,\varkappa),\Gamma\left(\widehat{\eta},\widehat{\theta},\widehat{\varkappa}\right)\right)\\ &\le \partial(\Gamma(\eta,\theta,\varkappa),\Gamma(\zeta_0,\delta_0,\aleph_0))\\ &\quad+\partial\left(\Gamma(\zeta_0,\delta_0,\aleph_0),\Gamma\left(\widehat{\eta},\widehat{\theta},\widehat{\varkappa}\right)\right)\\ &\le \alpha\partial(\zeta_0,\eta)+\beta\partial(\delta_0,\theta)+\gamma\partial(\aleph_0,\varkappa)\\ &\quad+\alpha\partial(\zeta_0,\widehat{\eta})+\beta\partial\left(\delta_0,\widehat{\theta}\right)+\gamma\partial(\aleph_0,\widehat{\varkappa}).\end{aligned}$$

Passing $\omega\to\infty$, we get $\partial(\eta,\widehat{\eta})=0$. Thus, $\eta=\widehat{\eta}$. Again similarly, we have

$$\begin{aligned}\partial\left(\theta,\widehat{\theta}\right) &= \partial\left(\Gamma(\theta,\varkappa,\eta),\Gamma\left(\widehat{\theta},\widehat{\varkappa},\widehat{\eta}\right)\right)\\ &\le \partial(\Gamma(\theta,\varkappa,\eta),\Gamma(\delta_0,\aleph_0,\zeta_0))\\ &\quad+\partial\left(\Gamma(\delta_0,\aleph_0,\zeta_0),\Gamma\left(\widehat{\theta},\widehat{\varkappa},\widehat{\eta}\right)\right)\\ &\le \alpha\partial(\delta_0,\theta)+\beta\partial(\aleph_0,\varkappa)+\gamma\partial(\zeta_0,\eta)\\ &\quad+\alpha\partial\left(\delta_0,\widehat{\theta}\right)+\beta\partial(\aleph_0,\widehat{\varkappa})+\gamma\partial(\zeta_0,\widehat{\eta}).\end{aligned}$$

Passing $\omega\to\infty$, we get $\partial\left(\theta,\widehat{\theta}\right)=0$. Thus, $\theta=\widehat{\theta}$.

This leads to $(\varkappa,\eta,\theta)=\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right)$. Then, Γ has a unique TFP in Ω.

If we put $\alpha=\beta=\gamma$ in Theorem *1*, we obtain the following corollary.

Corollary 1.*Suppose that $(\Omega,\perp,\partial)$ is an OCMS (not necessarily CMS) and the mapping $\Gamma:\Omega^3\to\Omega$ is OP. If for all $\aleph,\zeta,\mu,\delta,\upsilon,\rho\in\Omega$ with $\aleph\perp\mu$, $\zeta\perp\upsilon$ and $\delta\perp\rho$,*

$$\begin{aligned}&\partial(\Gamma(\aleph,\zeta,\delta),\Gamma(\mu,\upsilon,\rho))\\ &\le\frac{\alpha}{3}[\partial(\aleph,\mu)+\partial(\zeta,\upsilon)+\partial(\delta,\rho)]\end{aligned}\qquad(12)$$

where $0\le\alpha<1$, then Γ has a unique TFP.

Now, we will give the next theorem which TFP theorem of generalized Kannan type mapping in OMSs.

Theorem 2.*Assume that $(\Omega,\perp,\partial)$ is an OCMS (not necessarily CMS) and the mapping $\Gamma:\Omega^3\to\Omega$ is OP. If for all $\aleph,\zeta,\delta,\mu,\upsilon,\rho\in\Omega$ with $\aleph\perp\mu$, $\zeta\perp\upsilon$ and $\delta\perp\rho$,*

$$\begin{aligned}&\partial(\Gamma(\aleph,\zeta,\delta),\Gamma(\mu,\upsilon,\rho))\\ &\le\alpha\partial(\Gamma(\aleph,\zeta,\delta),\aleph)+\beta\partial(\Gamma(\mu,\upsilon,\rho),\mu),\end{aligned}\qquad(13)$$

where $\alpha,\beta\ge0$ and $\alpha+\beta<1$, then Γ has a unique TFP.

*Proof.*Based on Proof of Theorem *1* $\{\aleph_\omega\}_{\omega\in\mathbb{N}},\{\zeta_\omega\}_{\omega\in\mathbb{N}}$ and $\{\delta_\omega\}_{\omega\in\mathbb{N}}$ are OSs which have the same properties, so that (*2*) *and* (*3*) are satisfied for all $\omega\in\mathbb{N}$. Suppose that $\frac{\alpha}{1-\beta}=a$ and $\frac{1}{1-\alpha}=b$. Therefore by (*13*), one can get

$$\begin{aligned}&\partial(\aleph_\omega,\aleph_{\omega+1})\\ &=\partial(\Gamma(\aleph_{\omega-1},\zeta_{\omega-1},\delta_{\omega-1}),\Gamma(\aleph_\omega,\zeta_\omega,\delta_\omega))\\ &\le\alpha\partial(\Gamma(\aleph_{\omega-1},\zeta_{\omega-1},\delta_{\omega-1}),\aleph_{\omega-1})\\ &\quad+\beta\partial(\Gamma(\aleph_\omega,\zeta_\omega,\delta_\omega),\aleph_\omega)\\ &=\alpha\partial(\aleph_\omega,\aleph_{\omega-1})+\beta\partial(\aleph_{\omega+1},\aleph_\omega).\end{aligned}$$

Hence, we have

$$\partial(\aleph_\omega,\aleph_{\omega+1})\le a\partial(\aleph_\omega,\aleph_{\omega-1}),$$

with $a<1$. By continuing this process, we obtain for all $\omega\in\mathbb{N}$,

$$\partial(\aleph_\omega,\aleph_{\omega+1})\le a^\omega\partial(\aleph_0,\aleph_1).$$

Therefore, we can write for each positive integer ω and ψ with $\omega \leq \psi$,

$$\begin{aligned}\eth\left(\aleph_{\omega},\aleph_{\psi}\right) &\leq \eth\left(\aleph_{\omega},\aleph_{\omega+1}\right)+\eth\left(\aleph_{\omega+1},\aleph_{\omega+2}\right)\\ &\quad+\cdots+\eth\left(\aleph_{\psi-1},\aleph_{\psi}\right)\\ &\leq \left(a^{\omega}+a^{\omega+1}+\cdots+a^{\psi}\right)\eth\left(\aleph_0,\aleph_1\right)\\ &\leq \frac{a^{\omega}}{1-a}\eth\left(\aleph_0,\aleph_1\right).\end{aligned}$$

Taking limit as $\omega,\psi\to\infty$, since $a<1$,we obtain that $\{\aleph_{\omega}\}_{\omega\in\mathbb{N}}$ is a Cauchy OS. Similarly, we can prove that $\{\zeta_{\omega}\}_{\omega\in\mathbb{N}}$ and $\{\delta_{\omega}\}_{\omega\in\mathbb{N}}$ are Cauchy OSs in Ω. Let $(\Omega,\perp,\eth)$ be an OCMS, there is $\varkappa,\eta,\theta\in\Omega$ so that

$$\aleph_{\omega}\to\varkappa,\ \zeta_{\omega}\to\eta,\ \delta_{\omega}\to\theta.$$

By choice of $\varkappa,\eta$ and θ, one can write

$$(\varkappa\perp\aleph_{\omega})\ \text{or}\ (\aleph_{\omega}\perp\varkappa),$$

$$(\eta\perp\zeta_{\omega})\ \text{or}\ (\zeta_{\omega}\perp\eta)$$

and

$$(\theta\perp\delta_{\omega})\ \text{or}\ (\delta_{\omega}\perp\theta).$$

Then by (*13*), one can write for $a<1$ and $b<1$,

$$\begin{aligned}&\eth\left(\Gamma(\varkappa,\eta,\theta),\varkappa\right)\\ &\leq \eth\left(\Gamma(\varkappa,\eta,\theta),\aleph_{\omega+1}\right)+\eth\left(\aleph_{\omega+1},\varkappa\right)\\ &= \eth\left(\Gamma(\varkappa,\eta,\theta),\Gamma(\aleph_{\omega},\zeta_{\omega},\delta_{\omega})\right)+\eth\left(\aleph_{\omega+1},\varkappa\right)\\ &\leq \alpha\eth\left(\Gamma(\varkappa,\eta,\theta),\varkappa\right)+\beta\eth\left(\Gamma(\aleph_{\omega},\zeta_{\omega},\delta_{\omega}),\aleph_{\omega}\right)\\ &\quad+\eth\left(\aleph_{\omega+1},\varkappa\right)\end{aligned}$$

this implies that

$$\begin{aligned}&\eth\left(\Gamma(\varkappa,\eta,\theta),\varkappa\right)\\ &\leq a\eth\left(\aleph_{\omega+1},\aleph_{\omega}\right)+b\eth\left(\aleph_{\omega+1},\varkappa\right)\\ &\leq a\left[\eth\left(\aleph_{\omega+1},\varkappa\right)+\eth\left(\varkappa,\aleph_{\omega}\right)\right]\\ &\quad+b\eth\left(\aleph_{\omega+1},\varkappa\right).\end{aligned}$$

This leads to $\eth(\Gamma(\varkappa,\eta,\theta),\varkappa)=0$ implies $\Gamma(\varkappa,\eta,\theta)=\varkappa$. Similarly, we obtain $\Gamma(\eta,\theta,\varkappa)=\eta$ and $\Gamma(\theta,\varkappa,\eta)=\theta$. Then $(\varkappa,\eta,\theta)$ is a TFP of Γ.

Now, for uniqueness, suppose that $\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right)\in\eth^3$ is another TFP of Γ such that $\Gamma\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right)=\widehat{\varkappa},\Gamma\left(\widehat{\eta},\widehat{\theta},\widehat{\varkappa}\right)=\widehat{\eta}$ and $\Gamma\left(\widehat{\theta},\widehat{\varkappa},\widehat{\eta}\right)=\widehat{\theta}$.

(i)If $(\varkappa\perp\widehat{\varkappa})$ or $(\widehat{\varkappa}\perp\varkappa)$, $(\eta\perp\widehat{\eta})$ or $(\widehat{\eta}\perp\eta)$ and $\left(\theta\perp\widehat{\theta}\right)$ or $\left(\widehat{\theta}\perp\theta\right)$, by (*13*) *, we can write*

$$\begin{aligned}&\eth(\varkappa,\widehat{\varkappa})\\ &=\eth\left(\Gamma(\varkappa,\eta,\theta),\Gamma\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right)\right)\\ &\leq \alpha\eth\left(\Gamma(\varkappa,\eta,\theta),\varkappa\right)+\beta\eth\left(\Gamma\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right),\widehat{\varkappa}\right)\\ &=\alpha\eth(\varkappa,\varkappa)+\beta\eth(\widehat{\varkappa},\widehat{\varkappa})=0.\end{aligned}$$

Therefore, we get $\varkappa=\widehat{\varkappa}$, similiarly, $\eta=\widehat{\eta}$ and $\theta=\widehat{\theta}$.

(ii)If not, from the assumption in the beginning of the proof, the are orthogonal elements $\aleph_0,\zeta_0,\delta_0\in\eth$, we get

$$(\aleph_0\perp\varkappa,\ \aleph_0\perp\widehat{\varkappa})\ \text{or}\ (\varkappa\perp\aleph_0,\ \widehat{\varkappa}\perp\aleph_0),$$

$$(\zeta_0\perp\eta,\ \zeta_0\perp\widehat{\eta})\ \text{or}\ (\eta\perp\zeta_0,\ \widehat{\eta}\perp\zeta_0)$$

and

$$\left(\delta_0\perp\theta,\ \delta_0\perp\widehat{\theta}\right)\ \text{or}\ \left(\theta\perp\delta_0,\ \widehat{\theta}\perp\delta_0\right).$$

Therefore, using (*13*)*, we have*

$$\begin{aligned}&\eth(\varkappa,\widehat{\varkappa})\\ &=\eth\left(\Gamma(\varkappa,\eta,\theta),\Gamma\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right)\right)\\ &\leq \eth\left(\Gamma(\varkappa,\eta,\theta),\Gamma(\aleph_0,\zeta_0,\delta_0)\right)\\ &\quad+\eth\left(\Gamma(\aleph_0,\zeta_0,\delta_0),\Gamma\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right)\right)\\ &\leq \alpha\eth\left(\Gamma(\varkappa,\eta,\theta),\varkappa\right)+\beta\eth\left(\Gamma(\aleph_0,\zeta_0,\delta_0),\aleph_0\right)\\ &\quad+\alpha\eth\left(\Gamma(\aleph_0,\zeta_0,\delta_0),\aleph_0\right)+\beta\eth\left(\Gamma\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right),\widehat{\varkappa}\right).\end{aligned}$$

Passing $\omega\to\infty$, we get $\eth(\varkappa,\widehat{\varkappa})=0$. Thus, $\varkappa=\widehat{\varkappa}$. Similarly, we have $\eta=\widehat{\eta}$ and $\theta=\widehat{\theta}$.

This leads to $(\varkappa,\eta,\theta)=\left(\widehat{\varkappa},\widehat{\eta},\widehat{\theta}\right)$. Then, Γ has a unique TFP in Ω.

If we put $\alpha=\beta$ in Theorem *2*, we obtain the following corollary.

Corollary 2.*Suppose that $(\Omega,\perp,\eth)$ is an OCMS (not necessarily CMS) and the mapping $\Gamma:\Omega^3\to\Omega$ is OP. If for all $\aleph,\zeta,\delta,\mu,\upsilon,\rho\in\Omega$ with $\aleph\perp\mu$, $\zeta\perp\upsilon$ and $\delta\perp\rho$,*

$$\begin{aligned}&\eth\left(\Gamma(\aleph,\zeta,\delta),\Gamma(\mu,\upsilon,\rho)\right)\\ &\leq \frac{\alpha}{2}\left[\eth\left(\Gamma(\aleph,\zeta,\delta),\aleph\right)+\eth\left(\Gamma(\mu,\upsilon,\rho),\mu\right)\right]\end{aligned}$$

where $0\leq\alpha<1$, then Γ has a unique TFP.

Now, we will give the next theorem which TFP theorem of generalized Chatterjea type mapping in OMSs.

Theorem 3.*Assume that $(\Omega,\perp,\eth)$ is an OCMS (not necessarily CMS) and the mapping $\Gamma:\Omega^3\to\Omega$ is OP. If for all $\aleph,\zeta,\mu,\delta,\upsilon,\rho\in\Omega$ with $\aleph\perp\mu$, $\zeta\perp\upsilon$ and $\delta\perp\rho$,*

$$\begin{aligned}&\eth\left(\Gamma(\aleph,\zeta,\delta),\Gamma(\mu,\upsilon,\rho)\right)\\ &\leq \alpha\eth\left(\Gamma(\aleph,\zeta,\delta),\mu\right)+\beta\eth\left(\Gamma(\mu,\upsilon,\rho),\aleph\right),\end{aligned}\qquad(14)$$

where $\alpha,\beta\geq0$ and $\alpha+\beta<1$, then Γ has a unique TFP.

*Proof.*Based on Proof of Theorem *1* $\{\aleph_{\omega}\}_{\omega\in\mathbb{N}},\{\zeta_{\omega}\}_{\omega\in\mathbb{N}}$ and $\{\delta_{\omega}\}_{\omega\in\mathbb{N}}$ are OSs which have the same properties, so that (2) and (3) are satisfied for all $\omega \in \mathbb{N}$. Therefore by (14), one can get

$$\begin{aligned} &\eth(\aleph_\omega, \aleph_{\omega+1}) \\ &= \eth(\Gamma(\aleph_{\omega-1}, \zeta_{\omega-1}, \delta_{\omega-1}), \Gamma(\aleph_\omega, \zeta_\omega, \delta_\omega)) \\ &\leq \alpha\eth(\Gamma(\aleph_{\omega-1}, \zeta_{\omega-1}, \delta_{\omega-1}), \aleph_\omega) \\ &\quad +\beta\eth(\Gamma(\aleph_\omega, \zeta_\omega, \delta_\omega), \aleph_{\omega-1}) \\ &= \alpha\eth(\aleph_\omega, \aleph_\omega) + \beta\eth(\aleph_{\omega+1}, \aleph_{\omega-1}) \\ &\leq \beta\eth(\aleph_{\omega+1}, \aleph_\omega) + \beta\eth(\aleph_\omega, \aleph_{\omega-1}). \end{aligned}$$

It follows that

$$\eth(\aleph_\omega, \aleph_{\omega+1}) \leq \frac{1}{1-\beta}\eth(\aleph_\omega, \aleph_{\omega-1}),$$

with $\frac{1}{1-\beta} < 1$. By continuing like as the proof of Theorem 2. Then, we obtain that $\{\aleph_\omega\}_{\omega\in\mathbb{N}}$ is a Cauchy OS. Let $(\Omega, \perp, \eth)$ be an OCMS, there is $\varkappa, \eta, \theta \in \Omega$ so that

$$\aleph_\omega \to \varkappa,\ \zeta_\omega \to \eta,\ \delta_\omega \to \theta.$$

By choice of $\varkappa, \eta$ and θ, one can write

$$(\varkappa \perp \aleph_\omega) \text{ or } (\aleph_\omega \perp \varkappa),$$

$$(\eta \perp \zeta_\omega) \text{ or } (\zeta_\omega \perp \eta)$$

and

$$(\theta \perp \delta_\omega) \text{ or } (\delta_\omega \perp \theta).$$

Then by (14), one can write for $\alpha < 1$,

$$\begin{aligned} &\eth(\Gamma(\varkappa, \eta, \theta), \varkappa) \\ &\leq \eth(\Gamma(\varkappa, \eta, \theta), \aleph_{\omega+1}) + \eth(\aleph_{\omega+1}, \varkappa) \\ &= \eth(\Gamma(\varkappa, \eta, \theta), \Gamma(\aleph_\omega, \zeta_\omega, \delta_\omega)) + \eth(\aleph_{\omega+1}, \varkappa) \\ &\leq \alpha\eth(\Gamma(\varkappa, \eta, \theta), \aleph_\omega) + \beta\eth(\Gamma(\aleph_\omega, \zeta_\omega, \delta_\omega), \varkappa) \\ &\quad +\eth(\aleph_{\omega+1}, \varkappa) \\ &\leq \alpha[\eth(\Gamma(\varkappa, \eta, \theta), \varkappa) + \eth(\varkappa, \aleph_\omega)] \\ &\quad +\beta\eth(\Gamma(\aleph_\omega, \zeta_\omega, \delta_\omega), \varkappa) + (\beta+1)\eth(\aleph_{\omega+1}, \varkappa). \end{aligned}$$

Passing $\omega \to \infty$, we obtain

$$\eth(\Gamma(\varkappa, \eta, \theta), \varkappa) \leq \alpha\eth(\Gamma(\varkappa, \eta, \theta), \varkappa)$$

This leads to $\eth(\Gamma(\varkappa, \eta, \theta), \varkappa) = 0$ implies $\Gamma(\varkappa, \eta, \theta) = \varkappa$. Similarly, we obtain $\Gamma(\eta, \theta, \varkappa) = \eta$ and $\Gamma(\theta, \varkappa, \eta) = \theta$. Then $(\varkappa, \eta, \theta)$ is a TFP of Γ. Also, we can prove that the uniqueness of TFP similar to the other results. Then, Γ has a unique TFP in Ω.

If we put $\alpha = \beta$ in Theorem 3, we obtain the following corollary.

Corollary 3. *Suppose that $(\Omega, \perp, \eth)$ is an OCMS (not necessarily CMS) and the mapping $\Gamma : \Omega^3 \to \Omega$ is OP. If for all $\aleph, \zeta, \delta, \mu, \upsilon, \rho \in \Omega$ with $\aleph \perp \mu$, $\zeta \perp \upsilon$ and $\delta \perp \rho$,*

$$\begin{aligned} &\eth(\Gamma(\aleph, \zeta, \delta), \Gamma(\mu, \upsilon, \rho)) \\ &\leq \frac{\alpha}{2}[\eth(\Gamma(\aleph, \zeta, \delta), \mu) + \eth(\Gamma(\mu, \upsilon, \rho), \aleph)] \end{aligned}$$

where $0 \leq \alpha < 1$, then Γ has a unique TFP.

Example 2. Suppose that $\Omega = \{0, 1, 2, \cdots\}$ and $0 < \delta - (\aleph + \zeta)$. Then, $(\Omega, \perp)$ is an O-set. Define Euclidian metric $\eth$ on Ω. $(\Omega, \perp, \eth)$ is an OCMS . Let the mapping $\Gamma : \Omega^3 \to \Omega$ described as

$$\Gamma(\aleph, \zeta, \delta) = \begin{cases} \frac{\delta - (\aleph + \zeta)}{8}, & \aleph + \zeta < \delta \\ 0, & \text{otherwise} \end{cases},$$

for all $\aleph, \zeta, \delta, \mu, \upsilon, \rho \in \Omega$. It is clear that Γ is OP on Ω. Assume that $\aleph \perp \mu$, $\zeta \perp \upsilon$ and $\delta \perp \rho$, we consider the following cases for all $\aleph, \zeta, \delta, \mu, \upsilon, \rho \in \Omega$:

(i)If $\aleph + \zeta < \delta$ and $\mu + \upsilon < \rho$, then $\Gamma(\aleph, \zeta, \delta) = \frac{\delta - (\aleph + \zeta)}{8}$ and $\Gamma(\mu, \upsilon, \rho) = \frac{\rho - (\mu + \upsilon)}{8}$.
(ii)If $\aleph + \zeta < \delta$ and $\mu + \upsilon \geq \rho$, then $\Gamma(\aleph, \zeta, \delta) = \frac{\delta - (\aleph + \zeta)}{8}$ and $\Gamma(\mu, \upsilon, \rho) = 0$.
(iii)If $\aleph + \zeta \geq \delta$ and $\mu + \upsilon < \rho$, then $\Gamma(\aleph, \zeta, \delta) = 0$ and $\Gamma(\mu, \upsilon, \rho) = \frac{\rho - (\mu + \upsilon)}{8}$.
(iv)If $\aleph + \zeta \geq \delta$ and $\mu + \upsilon \geq \rho$, then $\Gamma(\aleph, \zeta, \delta) = 0$ and $\Gamma(\mu, \upsilon, \rho) = 0$.

For these cases, then

$$\begin{aligned} &|\Gamma(\aleph, \zeta, \delta) - \Gamma(\mu, \upsilon, \rho)| \\ &\leq \frac{\alpha}{3}[|\aleph - \mu| + |\zeta - \upsilon| + |\delta - \rho|]. \end{aligned} \tag{15}$$

is fulfilled for $0 \leq \alpha < 1$ and for all $\aleph, \zeta, \delta, \mu, \upsilon, \rho \in \Omega$: According to Corollary 1, Γhas a unique TFP $(0,0,0)$. If $(\Omega, \perp)$ is not O-set, then (15) is not fulfilled. To prove this, take $\aleph =$, $\zeta =$, $\delta =$, $\mu =$, $\upsilon =$ and $\rho =$, for any $0 \leq \alpha < 1$, we have

$$\begin{aligned} &|\Gamma(1,2,5) - \Gamma(2,3,4)| = 1 \\ &> \frac{\alpha}{3}[|1-2| + |2-3| + |5-4|] = \alpha. \end{aligned}$$

On the other hand, in this example, if we defined the mapping $\Gamma : \Omega^3 \to \Omega$ by $\Gamma(\aleph, \zeta, \delta) = \frac{\aleph + \zeta + \delta}{3}$ for O-set X, then (15) fulfills for $\alpha = 1$. So, $(0,0,0)$and $(1,1,1)$ are two TFPs of Γ. This implies that the TFP of Γ is not unique. Hence, conditions $\alpha < 1$ and $\alpha + \beta + \gamma < 1$ in Corollary 1 and Theorem 1, respectively, are the most suitable conditions to give the uniqueness of the TFP.

4 Solve a system of nonlinear integral equations

In this part, we apply Theorem 1 to discuss the existence and uniqueness solution for the following system of the nonlinear integral equations

$$\begin{aligned} \aleph(\lambda) &= \textstyle\int_0^P F(\lambda, \aleph(\varphi), \zeta(\varphi), \delta(\varphi))\, d\varphi \\ \zeta(\lambda) &= \textstyle\int_0^P F(\lambda, \zeta(\varphi), \delta(\varphi), \aleph(\varphi))\, d\varphi\ , \\ \delta(\lambda) &= \textstyle\int_0^P F(\lambda, \delta(\varphi), \aleph(\varphi), \zeta(\varphi))\, d\varphi \end{aligned} \tag{16}$$

where $P>0$, $\lambda\in[0,P]$ and $\digamma:[0,P]\times\mathbb{R}^3\to\mathbb{R}$. Let $C([0,P],\mathbb{R})$ is the space of all real valued continuous functions defined on $[0,P]$.

Theorem 4. *Assume that $\digamma:[0,P]\times\mathbb{R}^3\to\mathbb{R}$ is a mapping and the following conditions fulfilled:*

1. $\digamma$ is continuous,
2. there is $\alpha,\beta,\gamma\geq 0$ with $\alpha+\beta+\gamma<1$ such that

$$\begin{aligned}0&\leq \digamma(\lambda,\mu,\upsilon,\rho)-\digamma(\lambda,\aleph,\zeta,\delta)\\&\leq\frac{1}{P}[\alpha(\mu-\aleph)+\beta(\upsilon-\zeta)+\gamma(\rho-\delta)],\end{aligned}$$

for all $\aleph,\zeta,\delta,\mu,\upsilon,\rho\in\mathbb{R}$, $\aleph,\zeta,\delta,\mu,\upsilon,\rho\geq 0$ with $\mu-\aleph\geq 0$, $\upsilon-\zeta\geq 0$, $\rho-\delta\geq 0$ and for each $\lambda\in[0,P]$.

Then, the system (16) has a unique solution.

Proof. Define $\Omega=\{\aleph\in C([0,P],\mathbb{R}):\aleph(\lambda)\geq 0,\ \forall\lambda\in[0,P]\}$. Describe the orthogonality relationship in Ω by

$$\aleph\perp\zeta\Leftrightarrow\zeta(\lambda)-\aleph(\lambda)\geq 0,\ \forall\lambda\in[0,P].$$

Taking an arbitrary λ and for all $\aleph,\zeta,\delta\in\Omega$, let

$$\eth(\aleph,\zeta)=\sup_{\lambda\in[0,P]}|\aleph(\lambda)-\zeta(\lambda)|.$$

Clearly, $(\Omega,\eth)$ is a metric space. We need to prove the orthogonally completeness of Ω. We consider a Cauchy OS $\{\aleph_\omega\}_{\omega\in\mathbb{N}}\subseteq\Omega$. It is clear to say that $\{\aleph_\omega\}_{\omega\in\mathbb{N}}$ is convergent to a point $\aleph\in C([0,P],\mathbb{R})$. Therefore, we prove that $\aleph\in\Omega$. Taking an arbitrary $\lambda\in[0,P]$. By applying definition of $\perp$, we can write that $(\aleph_\omega\perp\aleph_{\omega+1})$ for all $\omega\in\mathbb{N}$. As $\aleph_\omega(\lambda)\geq 0$ for all $\omega\in\mathbb{N}$, this sequence converges to $\varkappa(\lambda)$. It follows that $\varkappa(\lambda)\geq 0$. Since $\lambda\in[0,P]$, $\varkappa\geq 0$ and so $\varkappa\in\Omega$. Now, let $\Gamma:\Omega^3\to\Omega$ be a mapping defined by

$$\Gamma(\aleph,\zeta,\delta)(\lambda)=\int_0^P\digamma(\lambda,\aleph(\varphi),\zeta(\varphi),\delta(\varphi))\,d\varphi$$

for each $\lambda\in[0,P]$, $\aleph,\zeta,\delta\in\Omega$. Now, we get $\digamma$ is OP. For all $\aleph,\zeta,\delta,\mu,\upsilon,\rho\in\Omega$ with $\aleph\perp\mu$, $\zeta\perp\upsilon$, $\delta\perp\rho$, and $\lambda\in[0,P]$, from (ii), we have

$$\begin{aligned}0\leq\ &\digamma(\lambda,\mu(\varphi),\upsilon(\varphi),\rho(\varphi))\\&-\digamma(\lambda,\aleph(\varphi),\zeta(\varphi),\delta(\varphi)),\end{aligned}$$

this leads to

$$\begin{aligned}&\digamma(\lambda,\aleph(\varphi),\zeta(\varphi),\delta(\varphi))\\&\leq\digamma(\lambda,\mu(\varphi),\upsilon(\varphi),\rho(\varphi)).\end{aligned}$$

Hence, one can write

$$\begin{aligned}\Gamma(\aleph,\zeta,\delta)(\lambda)&=\int_0^P\digamma(\lambda,\aleph(\varphi),\zeta(\varphi),\delta(\varphi))\,d\varphi\\&\leq\int_0^P\digamma(\lambda,\mu(\varphi),\upsilon(\varphi),\rho(\varphi))\,d\varphi\\&=\Gamma(\mu,\upsilon,\rho)(\lambda).\end{aligned}$$

This satisfies that $\Gamma(\aleph,\zeta,\delta)(\lambda)-\Gamma(\mu,\upsilon,\rho)(\lambda)\geq 0$. Therefore, we obtain $(\aleph,\zeta,\delta)\perp\Gamma(\mu,\upsilon,\rho)$. By (ii), for all $\aleph,\zeta,\delta,\mu,\upsilon,\rho\in\Omega$ with $\aleph\perp\mu$, $\zeta\perp\upsilon$, $\delta\perp\rho$, and $\lambda\in[0,P]$, we have

$$\begin{aligned}&|\Gamma(\mu,\upsilon,\rho)(\lambda)-\Gamma(\aleph,\zeta,\delta)(\lambda)|\\&=\left|\int_0^P\digamma(\lambda,\mu(\varphi),\upsilon(\varphi),\rho(\varphi))\,d\varphi\right.\\&\qquad\left.-\int_0^P\digamma(\lambda,\aleph(\varphi),\zeta(\varphi),\delta(\varphi))\,d\varphi\right|\\&\quad\int_0^P|\digamma(\lambda,\mu(\varphi),\upsilon(\varphi),\rho(\varphi))\\&\quad-\digamma(\lambda,\aleph(\varphi),\zeta(\varphi),\delta(\varphi))|\,d\varphi\\&\leq\frac{1}{P}\int_0^P[\alpha|\mu(\varphi)-\aleph(\varphi)|+\beta|\upsilon(\varphi)-\zeta(\varphi)|\\&\quad+\gamma|\rho(\varphi)-\delta(\varphi)|]\,d\varphi\\&\leq\frac{1}{P}\int_0^P\left[\alpha\sup_{s\in[0,P]}|\mu(s)-\aleph(s)|\right.\\&\quad+\beta\sup_{s\in[0,P]}|\upsilon(s)-\zeta(s)|\\&\quad\left.+\gamma\sup_{s\in[0,P]}|\rho(s)-\delta(s)|\right]d\varphi\\&\leq\alpha\sup_{s\in[0,P]}|\mu(s)-\aleph(s)|+\beta\sup_{\lambda\in[0,P]}|\upsilon(s)-\zeta(s)|\\&\quad+\gamma\sup_{s\in[0,P]}|\rho(s)-\delta(s)|.\end{aligned}$$

It follows that

$$\begin{aligned}&\sup_{s\in[0,P]}|\Gamma(\mu,\upsilon,\rho)(\lambda)-\Gamma(\aleph,\zeta,\delta)(\lambda)|\\&\leq\alpha\sup_{s\in[0,P]}|\mu(s)-\aleph(s)|+\beta\sup_{\lambda\in[0,P]}|\upsilon(s)-\zeta(s)|\\&\quad+\gamma\sup_{s\in[0,P]}|\rho(s)-\delta(s)|.\end{aligned}$$

Thus, $\aleph\perp\mu$, $\zeta\perp\upsilon$, $\delta\perp\rho$ and $\alpha+\beta+\gamma<1$, we obtain

$$\begin{aligned}&\eth(\Gamma(\aleph,\zeta,\delta),\Gamma(\mu,\upsilon,\rho))\\&\leq\alpha\eth(\aleph,\mu)+\beta\eth(\zeta,\upsilon)+\gamma\eth(\delta,\rho)\end{aligned}$$

Hence, Theorem 1 is satisfied. Then the system (16) has a unique solution.

5 Conclusion

In this work, we introduced some novel findings for the existence and uniqueness of TFPs of OMSs. Also, we presented some corollaries. Furthermore, to support our work, we gave an example and discussed the solution of a system of nonlinear integral equations for OMSs.

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