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SOME PROPERTIES OF ANALYTIC FUNCTIONS DEFINED BY POLYLOGARITHM FUNCTIONS

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Abstract. The main purpose of this paper, is to introduce a new subclass of analytic functions involving Polylogarithm functions and obtain coefficient inequalities, distortion properties, extreme points, radii of starlikeness and convexity, Hadamard product, and convolution and integral operators for the class.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of $\mathcal{A}$ consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of $\mathcal{A}$ consisting of functions $u(z)$ which are all univalent in $\mathbb{U}$. A function $u \in \mathcal{A}$ is a starlike function of the order $\xi,\ 0 \leq \xi < 1$, if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \xi,\ z \in \mathbb{U}. \tag{2}$$

We denote this class with $S^*(\xi)$. A function $u \in \mathcal{A}$ is a convex function of the order $\xi,\ 0 \leq \xi < 1$, if it satisfies

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \xi,\ z \in \mathbb{U}. \tag{3}$$

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We denote this class with $K(\xi)$. Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in $\mathbb{U}$ respectively. For $u \in \mathcal{A}$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{4}$$

their convolution (or Hadamard product), denoted by $(u * g)$, is defined as

$$(u * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z), \ (z \in \mathbb{U}). \tag{5}$$

Note that $u * g \in \mathcal{A}$.
Let T denotes the class of functions analytic in $\mathbb{U}$ that are of the form

$$u(z) = z - \sum_{k=2}^{\infty} a_k z^k, \ a_k \geq 0 \ (z \in \mathbb{U}) \tag{6}$$

and let $T^*(\xi) = T \cap S^*(\xi), \ C(\xi) = T \cap K(\xi)$. The class $T^*(\xi)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [30].
Let $u \in \mathcal{A}$. Denote by $D^\lambda : \mathcal{A} \to \mathcal{A}$ the operator defined by

$$D^\lambda = \frac{z}{(1-z)^{\lambda+1}} * u(z) \ (\lambda > -1)$$

It is obvious that $D^0 u(z) = u(z), D^1 u(z) = zu'(z)$ and

$$D^\delta u(z) = \frac{z(z^{\delta-1}u(z))^\delta}{\delta!}, \ (\delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

Note that $D^\delta u(z) = z + \sum_{k=2}^{\infty} C(\delta, k) a_k z^k$ where $C(\delta, k) = \binom{k+\delta-1}{\delta}$ and $\delta \in \mathbb{N}_0$. The operator $D^\delta u(z)$ is called the Ruscheweyh derivative operator (see [25]).
The evolution of polylogarithm function, also known as Jonquiere's function, was started in 1696 by two eminent mathematicians, Leibniz and Bernoulli [12]. In their work, the polylogarithm function was defined using an absolute convergent series. The development of this function was so significant that it was utilized in the research work of other prominent mathematicians such as Euler, Spence, Abel, Lobachevsky, Rogers, Ramanujan, etc., allowing them to discover various functional identities of great importance as a result [17]. It should come as no surprise that the increased utilization of the polylogarithm function appears to be related to its importance in a number of key areas of mathematics and physics such as topology, algebra, geometry, complex analysis quantum field theory, and mathematical physics [18, 13, 23].
Recntly, Al-Shaqsi and Darus [37], Danyal Soybas Santosh B. Joshi and Haridas Pawar [33], Al-Shaqsi and Darus [8], Stalin et al. [36] and Thirucheran et al. [37] generalized Ruscheweyh and Salagean operators using polylogarithm functions on class $\mathcal{A}$ of analytic functions (see also [1, 3, 4, 5, 6, 7, 18, 20, 24, 29, 35, 38]).

We recall here the definition of the well-known generalization of the Riemann Zeta and polylogarithm function, or simply the nth order polylogarithm function $G(n; z)$ given by

$$\Phi_n(b; z) = \sum_{k=1}^{\infty} \frac{z^k}{(k+b)^n} \ (n, b \in \mathbb{C}, \ z \in \mathbb{U}) \tag{7}$$

where any term with $k + b$ =0 is excluded (see Lerch [16] and also [[9], Section 1.10 and 1.12]). Using the the definition of Gamma function [[9], p.27] a simply transformation produces the integral formula

$$\Phi_n(b;z) = \frac{1}{\Gamma(n)}\int_0^1 z(log\frac{1}{t})^{n-1}\frac{t^b}{1-tz}dt,\ Re\ b > -1 \text{ and } Re\ n > 1.$$

we note that $\Phi_{-1}(0;z) = \frac{z}{(1-z)^2}$ is Koebe function. For more about polylogarithm in the theory of univalent functions see [23].
Now, for $u \in \mathcal{A}, n \in \mathbb{C},\ b \in \mathbb{C} \setminus \mathbb{Z}^-$ and $z \in \mathbb{U}$, we define the function $G(n, b; z)$ by

$$G(n,b;z) = (1+b)^n \Phi_n(b;z) = \sum_{k=1}^{\infty}\left(\frac{1+b}{k+b}\right)^n z^k. \tag{8}$$

Also we introduce a function $(G(n, b; z))^{-1}$ given by

$$G(n,b;z) * (G(n,b;z))^{-1} = \frac{z}{(1-z)^{\lambda+1}}, (n \in \mathbb{C},\ b \in \mathbb{C}\ \setminus \mathbb{Z}^-,\ \lambda > -1;\ z \in \mathbb{U}). \tag{9}$$

and obtain the following linear operator

$$\mathfrak{D}^n_{b,\lambda}u(z) = (G(n,b;z))^{-1} * u(z) \tag{10}$$

Now we find the explicit form of the function $(G(n, b; z))^{(-1)}$. It is well known that $\lambda > -1$

$$\frac{z}{(1-z)^{\lambda+1}} = \sum_{k=0}^{\infty}\frac{(\lambda+1)_k}{k!}z^{k+1}\ (z \in \mathbb{U}). \tag{11}$$

Putting (8) and (11) in (9), we get

$$\sum_{k=1}^{\infty}\left(\frac{1+b}{k+b}\right)^n z^k * (G(n,b;z))^{(-1)} = \sum_{k=1}^{\infty}\frac{(k+\lambda-1)!}{\lambda!(k-1)!}z^k.$$

Therefore the function $(G(n, b; z))^{(-1)}$ has the following form

$$(G(n,b;z))^{(-1)} = \sum_{k=1}^{\infty}\left(\frac{k+b}{1+b}\right)^n \frac{(k+\lambda-1)!}{\lambda!(k-1)!}z^k\ (z \in \mathbb{U}).$$

Now we note that

$$\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z) = z + \sum_{k=2}^{\infty}\Theta(k,b,\lambda,n)a_k z^k \tag{12}$$

$$(n \in \mathbb{C},\ b \in \mathbb{C}\ \setminus \mathbb{Z}^-,\ \lambda > -1;\ z \in \mathbb{U}).$$

where

$$\Theta(k,b,\lambda,n) = \left(\frac{k+b}{1+b}\right)^n \frac{(k+\lambda-1)!}{\lambda!(k-1)!} \tag{13}$$

It is clear that $\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}$ are multiplier transformations. For $n \in \mathbb{Z}, b = 1$ and $\lambda = 0$ the operators $\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}$ were studied by Uralegaddi and Somanatha [39], and for $n \in \mathbb{Z}, \lambda = 0$ the operators $\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}$ are closely related to the multiplier transformations studied by Flett [11], also, for $n = -1, \lambda = 0$, the operators $\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}$ is the integral operator studied by Owa and Srivastava [21]. And for any negative real number n and $b = 1, \lambda = 0$ the operators $\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}$ is the multiplier transformation studied by Jung et al.[15], and for any nonnegative integer n and $b = \lambda = 0$, the operators $\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}$ is the differential operator defined by Salagean [27]. Furthermore, for $n = 0$ and $\lambda \in \mathbb{N}_0$, the operators $\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}$ is the differential operator $\mathfrak{D}^{\mathfrak{n}}_{\lambda}$ defined by Ruscheweyh

[25]. For $n, \lambda \in \mathbb{N}_0$ and $b = 0$ the operators $\mathfrak{D}^{\mathfrak{n}}_{\lambda}$ is the differential operator defined [8]. Finally, for different choices of n, b and λ we obtain several operator investigated earlier by other author see, for example [2, 10] and [19].

We can now describe a new subclass of functions belonging to the class $\mathcal{A}$ by using the linear operator $\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}$.

Definition 1 For $-1 \leq \upsilon < 1$ and $\varrho \geq 0$, we let $TS^{n}_{b,\lambda}(\upsilon, \varrho)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (6) and fulfiling the analytic condition

$$\Re\left\{\frac{z(\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z))'}{\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z)} - \upsilon\right\} \geq \varrho\left|\frac{z(\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z))'}{\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z)} - 1\right|, \tag{14}$$

for $z \in E$. The class $TS^{n}_{b,\lambda}(\upsilon, \varrho)$ can be reduced to the class studied earlier by Ronning [19, 21] by suitably specialising the values of υ and ϱ. The primary aim of this paper is to examine some common geometric function theory properties such as coefficient bounds, distortion properties, extreme points, radii of starlikeness and convexity, Hadamard product, and convolution and integral operators for the class.

2. Coefficient bounds

We get a required and adequate condition for function $u(z)$ in the class $TS^{n}_{b,\lambda}(\upsilon, \varrho)$ in this section. To find the coefficient estimates for our class, we use the approach proposed by Aqlan et al. [2].

Theorem 1 The function u defined by (6) is in the class $TS^{n}_{b,\lambda}(\upsilon, \varrho)$ if and only if

$$\sum_{k=2}^{\infty}[k(1+\varrho) - (\upsilon+\varrho)]\Theta(k, b, \lambda, n)|a_k| \leq 1 - \upsilon, \tag{15}$$

where $-1 \leq \upsilon < 1, \varrho \geq 0$. The result is sharp.

Proof. We have $f \in TS^{n}_{b,\lambda}(\upsilon, \varrho)$ if and only if the condition (14) satisfied. Upon the fact that

$$\Re(w) > \varrho|w-1| + \upsilon \Leftrightarrow \Re\{w(1+\varrho e^{i\theta}) - \varrho e^{i\theta}\} > \upsilon, \ -\pi \leq \theta \leq \pi.$$

Equation (14) may be written as

$$\Re\left\{\frac{z(\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z))'}{\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z)}(1+\varrho e^{i\theta}) - \varrho e^{i\theta}\right\} \tag{16}$$

$$= \Re\left\{\frac{z(\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z))'(1+\varrho e^{i\theta}) - \varrho e^{i\theta}\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z)}{\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z)}\right\} > \upsilon. \tag{17}$$

Now, we let

$$E(z) = z(\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z))'(1+\varrho e^{i\theta}) - \varrho e^{i\theta}\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z)$$
$$F(z) = \mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z).$$

Then (16) is equivalent to

$$|E(z) + (1-\upsilon)F(z)| > |E(z) - (1+\upsilon)F(z)|, \text{ for } 0 \leq \upsilon < 1.$$

For $E(z)$ and $F(z)$ as above, we have

$$\begin{aligned}&|E(z)+(1-\upsilon)F(z)|\\ &\geq (2-\upsilon)|z|-\sum_{k=2}^{\infty}[k+1-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)|a_k||z^k|\end{aligned}$$

and similarly

$$\begin{aligned}&|E(z)-(1+\upsilon)F(z)|\\ &\leq \upsilon|z|-\sum_{k=2}^{\infty}[k-1-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)|a_k||z^k|.\end{aligned}$$

Therefore

$$\begin{aligned}&|E(z)+(1-\upsilon)F(z)|-|E(z)-(1+\upsilon)F(z)|\\ &\geq 2(1-\upsilon)-2\sum_{k=2}^{\infty}[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)|a_k|\end{aligned}$$

$$\text{or}\quad \sum_{k=2}^{\infty}[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)|a_k|\leq(1-\upsilon),$$

which yields (15).
On the other hand, we must have

$$\Re\left\{\frac{z(\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z))'}{\mathfrak{D}^{\mathfrak{n}}_{\mathfrak{b},\lambda}u(z)}(1+\varrho e^{i\theta})-\varrho e^{i\theta}\right\}\geq\upsilon.$$

Upon choosing the values of z on the positive real axis where $0\leq|z|=r<1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r-\sum_{k=2}^{\infty}[k-\upsilon+\varrho e^{i\theta}(k-1)]\Theta(k,b,\lambda,n)|a_k|\ r^k}{z-\sum_{k=2}^{\infty}\Theta(k,b,\lambda,n)|a_k|\ r^k}\right\}\geq 0.$$

Since $\Re(-e^{i\theta})\geq-|e^{i\theta}|=-1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r-\sum_{k=2}^{\infty}[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)|a_k|\ r^k}{z-\sum_{k=2}^{\infty}\Theta(k,b,\lambda,n)|a_k|\ r^k}\right\}\geq 0.$$

Letting $r\to 1^-$, we get the desired result. Finally the result is sharp with the extremal function u given by

$$u(z)=z-\frac{1-\upsilon}{[k(1+\varrho)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)}z^k. \tag{18}$$

3. Growth and Distortion Theorems

Theorem 2 Let the function u defined by (6) be in the class $TS^n_{b,\lambda}(\upsilon,\varrho)$. Then for $|z|=r$

$$r-\frac{1-\upsilon}{\Theta(2,b,\lambda,n)(2-\upsilon+\varrho)}r^2\leq|u(z)|\leq r+\frac{1-\upsilon}{\Theta(2,b,\lambda,n)(2-\upsilon+\varrho)}r^2. \tag{19}$$

Equality holds for the function

$$u(z) = z - \frac{1-\upsilon}{\Theta(2,b,\lambda,n)(2-\upsilon+\varrho)} z^2. \tag{20}$$

Proof. Since the other inequality can be explained using identical reasoning, we just prove the right hand side inequality in (6). In view of Theorem 2, we have

$$\sum_{k=2}^{\infty} |a_k| \leq \frac{1-\upsilon}{\Theta(2,b,\lambda,n)(2-\upsilon+\varrho)}. \tag{21}$$

Since,

$$\begin{aligned} u(z) &= z - \sum_{k=2}^{\infty} a_k z^k \\ |u(z)| &= \left| z - \sum_{k=2}^{\infty} a_k z^k \right| \leq r + \sum_{k=2}^{\infty} |a_k| r^k \leq r + r^2 \sum_{k=2}^{\infty} |a_k| \\ &\leq r + \sum_{k=2}^{\infty} \frac{1-\upsilon}{\Theta(2,b,\lambda,n)(2-\upsilon+\varrho)} r^2 \end{aligned}$$

which yields the right hand side inequality of (19).

Next, by using the same technique as in proof of Theorem 3, we give the distortion result.

Theorem 3 Let the function u defined by (6) be in the class $TS_{b,\lambda}^n(\upsilon,\varrho)$. Then for $|z| = r$

$$1 - \frac{(1-\upsilon)}{\Theta(2,b,\lambda,n)(2-\upsilon+\varrho)} r \leq |u'(z)| \leq 1 + \frac{(1-\upsilon)}{\Theta(2,b,\lambda,n)(2-\upsilon+\varrho)} r.$$

Equality holds for the function given by (20).

Proof. Since $f \in TS_{b,\lambda}^n(\upsilon,\varrho)$ by Theorem 2, we have that

$$\Theta(2,b,\lambda,n)\left[2(1+\varrho) - (\upsilon+\varrho)\right] \sum_{k=2}^{\infty} k a_k \leq \left[k(1+\varrho) - (\upsilon+\varrho)\right] \Theta(k,b,\lambda,n)|a_k| \leq 1-\upsilon$$

or

$$\sum_{k=2}^{\infty} k|a_k| \leq \frac{(1-\upsilon)}{\Theta(2,b,\lambda,n)(2-\upsilon+\varrho)} \ .$$

Thus from (21), we obtain

$$\begin{aligned} |u'(z)| &\leq 1 + r \sum_{k=2}^{\infty} k|a_k| \\ &\leq 1 + \frac{(1-\upsilon)}{\Theta(2,b,\lambda,n)(2-\upsilon+\varrho)} r \end{aligned}$$

which is right hand inequality of Theorem 3.

On the other hand, similarly

$$|u'(z)| \geq 1 - \frac{(1-\upsilon)}{\Theta(2,b,\lambda,n)(2-\upsilon+\varrho)} r$$

and thus proof is completed.

Theorem 4 If $u \in TS^n_{b,\lambda}(\upsilon, \varrho)$ then $u \in TS^n_{b,\lambda}(\gamma)$, where

$$\gamma = 1 - \frac{(k-1)(1-\upsilon)}{[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)-(1-\upsilon)}.$$

Equality holds for the function given by (20).

Proof. It is sufficient to show that (15) implies

$$\sum_{k=2}^{\infty}(k-\gamma)|a_k| \le 1-\gamma,$$

that is

$$\frac{k-\gamma}{1-\gamma} \le \frac{[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)}{(1-\upsilon)}$$

then

$$\gamma \le 1 - \frac{(k-1)(1-\upsilon)}{[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)-(1-\upsilon)}.$$

The above inequality holds true for $k \in \mathbb{N}_0, k \ge 2, \varrho \ge 0$ and $0 \le \upsilon < 1$.

4. Extreme points

Theorem 5 Let $u_1(z) = z$ and

$$u_k(z) = z - \frac{1-\upsilon}{[k(\varrho+1)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)}z^k, \tag{22}$$

for $k = 2, 3, \cdots$. Then $u(z) \in TS^n_{b,\lambda}(\upsilon, \varrho)$ if and only if $u(z)$ can be expressed in the form $u(z) = \sum\limits_{k=1}^{\infty} \zeta_k u_k(z)$, where $\zeta_k \ge 0$ and $\sum\limits_{k=1}^{\infty} \zeta_k = 1$.

Proof. Suppose $u(z)$ can be expressed as in (22). Then

$$\begin{aligned}
u(z) &= \sum_{k=1}^{\infty}\zeta_k u_k(z) = \zeta_1 u_1(z) + \sum_{k=2}^{\infty}\zeta_k u_k(z) \\
&= \zeta_1 u_1(z) + \sum_{k=2}^{\infty}\zeta_k\left\{z - \frac{1-\upsilon}{[k(\varrho+1)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)}z^k\right\} \\
&= \zeta_1 z + \sum_{k=2}^{\infty}\zeta_k z - \sum_{k=2}^{\infty}\zeta_k\left\{\frac{1-\upsilon}{[k(\varrho+1)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)}z^k\right\} \\
&= z - \sum_{k=2}^{\infty}\zeta_k\left\{\frac{1-\upsilon}{[k(\varrho+1)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)}z^k\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{k=2}^{\infty}\zeta_k\left(\frac{1-\upsilon}{[k(\varrho+1)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)}\right)\left(\frac{[k(\varrho+1)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)}{1-\upsilon}\right) \\
&= \sum_{k=2}^{\infty}\zeta_k = \sum_{k=1}^{\infty}\zeta_k - \zeta_1 = 1-\zeta_1 \le 1.
\end{aligned}$$

So, by Theorem 2, $u \in TS^n_{b,\lambda}(\upsilon, \varrho)$.

Conversely, we suppose $u \in TS^n_{b,\lambda}(\upsilon, \varrho)$. Since

$$|a_k| \le \frac{1-\upsilon}{[k(\varrho+1)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)},\ k \ge 2.$$

We may set

$$\zeta_k = \frac{[k(\varrho+1)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)}{1-\upsilon}|a_k|,\ \ k\geq 2$$

and $\zeta_1 = 1-\sum\limits_{k=2}^{\infty}\zeta_k$. Then

$$\begin{aligned} u(z) &= z-\sum_{k=2}^{\infty}a_k z^k = z-\sum_{k=2}^{\infty}\zeta_k\frac{1-\upsilon}{[k(\varrho+1)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)}z^k \\ &= z-\sum_{k=2}^{\infty}\zeta_k[z-u_k(z)] = z-\sum_{k=2}^{\infty}\zeta_k z+\sum_{k=2}^{\infty}\zeta_k u_k(z) \\ &= \zeta_1 u_1(z)+\sum_{k=2}^{\infty}\zeta_k u_k(z) = \sum_{k=1}^{\infty}\zeta_k u_k(z). \end{aligned}$$

Corollary 1 The extreme points of $TS_{b,\lambda}^{n}(\upsilon,\varrho)$ are the functions $u_1(z)=z$ and

$$u_k(z) = z-\frac{1-\upsilon}{[k(\varrho+1)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)}z^k,\ \ k\geq 2.$$

5. Radii of Close-to-convexity, Starlikeness and Convexity

A function $u\in TS_{b,\lambda}^{n}(\upsilon,\varrho)$ is said to be close-to-convex of order δ if it satisfies

$$\Re\{u'(z)\}>\delta,\ (0\leq\delta<1;\ z\in E).$$

Also A function $u\in TS_{b,\lambda}^{n}(\upsilon,\varrho)$ is said to be starlike of order δ if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\}>\delta,\ (0\leq\delta<1;\ z\in E).$$

Further a function $u\in TS_{b,\lambda}^{n}(\upsilon,\varrho)$ is said to be convex of order δ if and only if $zu'(z)$ is starlike of order δ that is if

$$\Re\left\{1+\frac{zu'(z)}{u(z)}\right\}>\delta,\ (0\leq\delta<1;\ z\in E).$$

Theorem 6 Let $u\in TS_{b,\lambda}^{n}(\upsilon,\varrho)$. Then u is close-to-convex of order δ in $|z|<R_1$, where

$$R_1 = \inf_{k\geq 2}\left[\frac{(1-\delta)[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)}{k(1-\upsilon)}\right]^{\frac{1}{k-1}}.$$

The result is sharp with the extremal function u is given by (18).

Proof. It is sufficient to show that $|u'(z)-1|\leq 1-\delta$, for $|z|<R_1$. We have

$$|u'(z)-1| = \left|-\sum_{k=2}^{\infty}ka_k z^{k-1}\right|\leq\sum_{k=2}^{\infty}ka_k|z|^{k-1}.$$

Thus $|u'(z)-1|\leq 1-\delta$ if

$$\sum_{k=2}^{\infty}\frac{k}{1-\delta}|a_k||z|^{k-1}\leq 1. \tag{23}$$

But Theorem 2 confirms that

$$\sum_{k=2}^{\infty}\frac{[k(\varrho+1)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)}{1-\upsilon}|a_k|\leq 1. \tag{24}$$

Hence (23) will be true if

$$\frac{k|z|^{k-1}}{1-\delta} \leq \frac{[k(\varrho+1)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)}{1-\upsilon}.$$

We obtain

$$|z| \leq \left[\frac{(1-\delta)[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)}{k(1-\upsilon)}\right]^{\frac{1}{k-1}}, k \geq 2$$

as required.

Theorem 7 Let $u \in TS^n_{b,\lambda}(\upsilon,\varrho)$. Then u is starlike of order δ in $|z| < R_2$, where

$$R_2 = \inf_{k\geq 2}\left[\frac{(1-\delta)[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)}{(k-\delta)(1-\upsilon)}\right]^{\frac{1}{k-1}}.$$

The result is sharp with the extremal function u is given by (18).

Proof. We must show that $\left|\frac{zu'(z)}{u(z)}-1\right| \leq 1-\delta$, for $|z| < R_2$.

We have

$$\begin{aligned}\left|\frac{zu'(z)}{u(z)}-1\right| &= \left|\frac{-\sum\limits_{k=2}^{\infty}(k-1)a_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_k z^{k-1}}\right| \\ &\leq \frac{\sum\limits_{k=2}^{\infty}(k-1)|a_k||z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}|a_k||z|^{k-1}} \\ &\leq 1-\delta. \end{aligned} \tag{25}$$

Hence (25) holds true if

$$\sum_{k=2}^{\infty}(k-1)|a_k||z|^{k-1} \leq (1-\delta)\left(1-\sum_{k=2}^{\infty}|a_k||z|^{k-1}\right)$$

or equivalently,

$$\sum_{k=2}^{\infty}\frac{k-\delta}{1-\delta}|a_k||z|^{k-1} \leq 1. \tag{26}$$

Hence, by using (24) and (26) will be true if

$$\frac{k-\delta}{1-\delta}|z|^{k-1} \leq \frac{[k(\varrho+1)-(\upsilon+\varrho)]\Theta(k,b,\lambda,n)}{1-\upsilon}$$

$$\Rightarrow |z| \leq \left[\frac{(1-\delta)[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)}{(k-\delta)(1-\upsilon)}\right]^{\frac{1}{k-1}}, k \geq 2$$

which completes the proof.

By using the same approach in the proof of Theorem 5, we can show that $\left|\frac{zu''(z)}{u'(z)}-1\right| \leq 1-\delta$, for $|z| < R_3$, with the aid of Theorem 2.

Thus we have the assertion of the following Theorem 5.

Theorem 8 Let $u \in TS^n_{b,\lambda}(\upsilon, \varrho)$. Then u is convex of order δ in $|z| < R_3$, where

$$R_3 = \inf_{k\geq 2}\left[\frac{(1-\delta)[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)}{k(k-\delta)(1-\upsilon)}\right]^{\frac{1}{k-1}}.$$

The result is sharp with the extremal function u is given by (18).

6. Inclusion theorem involving modified Hadamard products

For functions

$$u_j(z) = z - \sum_{k=2}^{\infty}|a_{k,j}|z^k, \;\; j = 1, 2 \tag{27}$$

in the class A, we define the modified Hadamard product $(u_1 * u_2)(z)$ of $u_1(z)$ and $u_2(z)$ given by

$$(u_1 * u_2)(z) = z - \sum_{k=2}^{\infty}|a_{k,1}||a_{k,2}|z^k.$$

We can prove the following.

Theorem 9 Let the function $u_j,\; j = 1, 2$, given by (27) be in the class $TS^n_{b,\lambda}(\upsilon, \varrho)$ respectively. Then $(u_1 * u_2)(z) \in TS(\upsilon, \varrho, \lambda, t, \xi)$, where

$$\xi = 1 - \frac{(1-\upsilon)^2}{(k+1)(2-\upsilon)(2-\upsilon+\varrho)(1+\lambda)-(1-\upsilon)^2}.$$

Proof. Employing the approach used earlier by Schild and Silverman [28], we need to find the biggest ξ such that

$$\sum_{k=2}^{\infty}\frac{[k-\xi+\varrho(k-1)]\Theta(k,b,\lambda,n)}{1-\xi}|a_{k,1}||a_{k,2}| \leq 1.$$

Since $u_j \in TS^n_{b,\lambda}(\upsilon, \varrho),\; j = 1, 2$, then we have

$$\sum_{k=2}^{\infty}\frac{[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)}{1-\upsilon}|a_{k,1}| \leq 1$$
$$\text{and} \sum_{k=2}^{\infty}\frac{[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)}{1-\upsilon}|a_{k,2}| \leq 1,$$

by the Cauchy-Schwartz inequality, we have

$$\sum_{k=2}^{\infty}\frac{[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)}{1-\upsilon}\sqrt{|a_{k,1}||a_{k,2}|} \leq 1.$$

Thus it is sufficient to show that

$$\frac{[k-\xi+\varrho(k-1)]\Theta(k,b,\lambda,n)}{1-\xi}|a_{k,1}||a_{k,2}|$$
$$\leq \frac{[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)}{1-\upsilon}\sqrt{|a_{k,1}||a_{k,2}|},\ k\geq 2,$$

that is

$$\sqrt{|a_{k,1}||a_{k,2}|}\leq \frac{(1-\xi)[k-\upsilon+\varrho(k-1)]}{1-\upsilon)[k-\xi+\varrho(k-1)]}.$$

Note that

$$\sqrt{|a_{k,1}||a_{k,2}|}\leq \frac{(1-\upsilon)}{[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)}.$$

Consequently, we need only to prove that

$$\frac{(1-\upsilon)}{[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)}\leq \frac{(1-\xi)[k-\upsilon+\varrho(k-1)]}{1-\upsilon)[k-\xi+\varrho(k-1)]},\ k\geq 2,$$

or equivalently

$$\xi\leq 1-\frac{(k-1)(1+\varrho)(1-\upsilon)^2}{[k-\upsilon+\varrho(k-1)]^2\Theta(k,b,\lambda,n)-(1-\upsilon)^2},\ k\geq 2.$$

Since

$$A(k)=1-\frac{(k-1)(1+\varrho)(1-\upsilon)^2}{[k-\upsilon+\varrho(k-1)]^2\Theta(k,b,\lambda,n)-(1-\upsilon)^2},\ k\geq 2$$

is an increasing function of $k, k\geq 2$, letting $k=2$ in last equation, we obtain

$$\xi\leq A(2)=1-\frac{(1+\varrho)(1-\upsilon)^2}{[2-\upsilon+\varrho]^2\Theta(k,b,\lambda,n)-(1-\upsilon)^2}.$$

Finally, by taking the function given by (20), we can see that the result is sharp.

7. Convolution and Integral Operators

Let $u(z)$ be defined by (6) and suppose that $g(z)=z-\sum\limits_{k=2}^{\infty}|b_k|z^k$. Then the Hadamard product (or convolution) of $u(z)$ and $g(z)$ defined here by

$$u(z)*g(z)=(u*g)(z)=z-\sum_{k=2}^{\infty}|a_k||b_k|z^k.$$

Theorem 10 Let $u\in TS^n_{b,\lambda}(\upsilon,\varrho)$ and $g(z)=z-\sum\limits_{k=2}^{\infty}|b_k|z^k, 0\leq|b_k|\leq 1$. Then $u*g\in TS^n_{b,\lambda}(\upsilon,\varrho)$.

Proof. In view of Theorem 2, we have

$$\sum_{k=2}^{\infty}[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)|a_k||b_k|$$
$$\leq\sum_{k=2}^{\infty}[k-\upsilon+\varrho(k-1)]\Theta(k,b,\lambda,n)|a_k|$$
$$\leq(1-\upsilon).$$

Theorem 11 Let $u \in TS^n_{b,\lambda}(\upsilon, \varrho)$ and $\hbar$ be real number such that $\hbar > -1$. Then the function $Q(z) = \frac{\hbar+1}{z^\hbar} \int\limits_0^z t^{\hbar-1} u(t) dt$ also belongs to the class $TS^n_{b,\lambda}(\upsilon, \varrho)$.

Proof. From the representation of $Q(z)$, it follows that

$$Q(z) = z - \sum_{k=2}^{\infty} |A_k| z^k, \text{ where } A_k = \left(\frac{\hbar+1}{\hbar+k}\right) |a_k|.$$

Since $\hbar > -1$, than $0 \leq A_k \leq |a_k|$. Which in view of Theorem 2, $Q \in TS^n_{b,\lambda}(\upsilon, \varrho)$.

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