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Discrete Alpha Power Transformed Weibull -G Family of distributions

Dr. Mai Amed Ibrahim Hegazy

Abstract

In this paper, a new flexible discrete family of distributions called discrete alpha power transformed Weibull -G family is introduced. Some of its distributional and reliability properties including quantiles, mean time to failure, Rényi entropy, moments and order statistics are obtained. The maximum likelihood method is used for estimating the family parameters. The proposed distributions which are members of this family are very flexible and can be used to model various types of data with increasing, decreasing or bathtub-shaped hazard rates. Discrete alpha power transformed Weibull- exponential distribution, as a member from this family, is studied in detail. A simulation study is conducted to investigate the precision of the theoretical results based on simulated and real data through some measurements of accuracy. two real data sets are analyzed to illustrate the suitability and applicability of the proposed model.

Keywords: *Alpha power transformation; Weibull-G family; Discrete distributions; Discrete alpha power transformed Weibull -G family of distributions; Weibull distribution; Maximum likelihood estimation*.

1. Introduction

Statistical literature is rich in many continuous distributions and their successful applications. Hower, in many applied areas such as lifetime analysis, finance and insurance, most of these distributions are not suitable to model some of real data sets. As a result, for application purposes, it is required to get the extended forms of these distributions for various fields. So, several attempts have been introduced by many researchers to generate new families of probability distributions that extend well-known families of distributions and at the same time provide great flexibility in modeling data in practice. So, several methods for generating new families of distributions have been studied. Some of prominent families are, Marshall and Olkin (1997), Eugene *et al.* (2002), Cordeiro and Castro (2011), Alzaatreh *et al.* (2013), Lee *et al.* (2013), Bourguignon *et al.* (2014) and Jones (2015).

Mahdavi and Kundu (2017) presented a method to add an extra parameter to a family of distributions, such an addition of parameters makes the resulting distribution richer and more flexible for modeling data. The suggested method is called *alpha power transformation* (APT) and it is useful to incorporate skewness to a family of distributions. The APT method was applied to many distributions by many researchers, such as Nassar *et al.* (2017), Dey *et al.* (2017), Nadarajah and Okorie (2018), Mead *et al.* (2019) and Nassar *et al.* (2020).

Let $G(x;\delta)$ and $g(x;\delta)$are the *cumulative distribution function* (cdf) and the *probability density function* (pdf) of any baseline distribution depending on a vector of parameter δ and the cdf of APT family is $F_{APT}(x;\alpha,\delta)$ for $x \in \mathbb{R}$ is

$$F_{APT}(x;\alpha,\delta) = \begin{cases} \frac{\alpha^{G(x;\delta)}-1}{\alpha-1}, & \alpha > 0,\ \alpha \neq 1, \\ G(x;\delta), & \alpha = 1\ , \end{cases} \tag{1}$$

and the corresponding pdf is

$$f_{APT}(x;\alpha,\delta) = \begin{cases} \frac{\log\alpha}{\alpha-1} g(x;\delta)\alpha^{g(x;\delta)}, & \alpha > 0,\ \alpha \neq 1, \\ g(x;\delta), & \alpha = 1, \end{cases} \tag{2}$$

where α is the shape parameter.

The Weibull distribution is a very popular lifetime distribution and it has been extensively used for modeling data in reliability, engineering, medical, quality control and biological studies. It is generally suitable for modeling monotone hazard rates. However, when the hazard rates are bathtub, upside down bathtub or bimodal shapes, it does not work well. As a result, researchers are motivated to develop several generalizations and extensions of Weibull distribution to model different kind of data sets.

Bourguignon *et al.* (2014) proposed the *Weibull-G* (W-G) family of distributions the cdf of W-G family is $F_{WG}(x;\theta,\delta)$ for $x \in \mathbb{R}$ is

$$F_{WG}(x;\theta,\delta) = \int_0^{\frac{G(x;\delta)}{\bar{G}(x;\delta)}} \theta\, t^{\theta-1} e^{-t^{\theta}}\, dt = 1 - \exp\left\{-\left[\frac{G(x;\delta)}{\bar{G}(x;\delta)}\right]^{\theta}\right\}, \tag{3}$$

and the corresponding (pdf) is

$$f_{WG}(x;\theta,\delta) = \theta\, g(x;\delta) \frac{(G(x;\delta))^{\theta-1}}{(\bar{G}(x;\delta))^{\theta+1}}\, exp\left\{-\left[\frac{G(x;\delta)}{\bar{G}(x;\delta)}\right]^{\theta}\right\}. \tag{4}$$

Elbatal *et al.* (2021) introduced a new extended generator called *alpha power transformed Weibull-G* (APTW-G) family based on combining the APT family with the Weibull-G family of distributions. They expected that the proposed distributions will be more flexible and will perform better than some existing probability distributions to model life testing data. They provided three sub models of this family by taking the baseline distributions as exponential, Rayleigh and

Lindley. They used the *maximum likelihood* (ML) method to estimate the parameters.

The cdf of APTW-G family of distributions can be stated by substituting (3) in (1) as follows:

$$F_{APTW-G}(x;\alpha,\delta,\theta)=\begin{cases}\frac{\alpha^{1-e^{-t_i^{\theta}}}-1}{\alpha-1}, & \alpha>0,\ \alpha\neq 1,\\ 1-e^{-t_i^{\theta}}, & \alpha=1\,,\end{cases} \qquad (5)$$

and the corresponding pdf is

$$f_{APTW-G}(x;\alpha,\delta,\theta)=$$
$$\begin{cases}log(\alpha)\,\theta g(x;\delta)\frac{(G(x;\delta))^{\theta-1}}{(\alpha-1)(\bar{G}(x;\delta))^{\theta+1}}e^{-t_i^{\theta}}\alpha^{1-e^{-t_i^{\theta}}}, & \alpha>0,\ \alpha\neq 1,\\ \theta\, g(x;\delta)\frac{(G(x;\delta))^{\theta-1}}{(\bar{G}(x;\delta))^{\theta+1}}\,exp\left\{-\left[\frac{G(x;\delta)}{\bar{G}(x;\delta)}\right]^{\theta}\right\}, & \alpha=1,\end{cases} \qquad (6)$$

where $t_i=\frac{G(x;\delta)}{\bar{G}(x;\delta)}$. (7)

The *survival function* (sf); $S_{APTW-G}(x;\alpha,,\delta,\theta)$, is given by

$$S_{APTW-G}(x;\alpha,\delta,\theta,)=\begin{cases}\frac{\alpha-\alpha^{1-e^{-t_i^{\theta}}}}{\alpha-1}, & \alpha>0,\ \alpha\neq 1,\\ e^{-t_i^{\theta}}, & \alpha=1\,,\end{cases} \qquad (8)$$

The discretization phenomenon is desirable when the existing continuous distributions aren't appropriate or don't provide sufficient adequacy in modeling lifetime data. Some well-known discrete distributions have limited applicability to model discrete failure times. Thus, there is a need to derive appropriate discrete distributions by discretizing the continuous distributions to fit various types of data. Therefore, the study of discretization of continuous is meaningful. Although there are several methods to construct discrete distributions from the continuous ones, the general approach of discretization (survival discretization method) of some known continuous distributions have been attracting great concern for use as lifetime distributions. One of the advantages of using this approach of discretizing is that the sf for discrete distributions has the same functional form of the sf for the continuous distributions; as a result, many reliability characteristics and properties remain unchanged [see, Roy (2003, 2004)]. Many researchers studied the general approach of discretization of some known continuous distributions for use as lifetime distributions. [See for example, Nakagawa and Osaki (1975), Khan *et al.*

(1989), Bracquemond and Gaudoin (2003), Inusah and Kozubowski (2006), Krishna and Pundir (2009), Jazi *et al.* (2010), Gomez-Deniz and Calderin-Ojeda (2011) and Nekoukhou *et al*. (2012) and Chakraborty (2015)]. Although there are several discrete distributions in statistical literature, there is still a lot of space left to develop new discretized distributions that are suitable under different conditions. This encourages us to provide a flexible family of discrete distributions for analyzing a variety of discrete real-world data sets. Therefore, this paper will introduce a flexible discrete generator family of distributions, in the so-called *discrete* APTW-G (DAPTW-G) family. Our reasons for introducing the DAPTW-G family are the following:

- To generate models for modeling both lifetime and counting data sets.
- To generate models with a negatively skewed, a positively skewed, or a symmetric shape.
- To provide consistently better fits than other generated discrete distributions with the same base line model and other popular discrete distributions in statistical literature.
- To define special models with diverse shapes of hazard rate function.

The rest of this paper is organized as follows: in Section 2, DAPTW-G family of distributions is introduced and some of its properties are studied. In Section 3, some members of DAPTW-G family of distributions are presented. DAPTW-E is discussed in detail in Section 4. In Section 5, numerical results based on simulation study and real data sets are analyzed to evaluate the performance of the maximum likelihood estimates and demonstrate how the results can be used in practice.

2. Discrete Alpha Power Transformed Weibull-G Family of Distributions

In this section, DAPTW-G family of distributions is constructed using the general approach of discretizing, by introducing a grouping on the time axis see Roy (2003, 2004). If the *continuous random variable* (crv) X has the sf, $S(x) = P\,(X \geq x)$ and times are grouped into unit intervals so that the *discrete* rv (drv) of X denoted by $dX = \lfloor x \rfloor$; which is the largest integer less than or equal to *x*, will have the *probability mass function* (pmf)

$$P(dx = x) = P(x) = P(x \leq X \leq x + 1) = S(x) - S(x + 1), \qquad x = 0,1,2, \dots . \quad (9)$$

The pmf of the drv, dX, can be viewed as discrete concentration of pdf of X. So, given any continuous distribution it is possible to construct corresponding discrete distribution using (9).

The drv X is said to have the DAPTW-G family if its cdf is given by

$$F_{DAPTW-G}(x;\alpha,\delta,\theta)=\begin{cases}\frac{\alpha^{1-e^{-t_{i*}^{\theta}}}-1}{\alpha-1}, & \alpha>0,\ \alpha\neq1,\\ 1-e^{-t_{i*}^{\theta}}, & \alpha=1,\end{cases}\qquad(10)$$

where $t_{i*}=\frac{G(x+1;\delta)}{\bar{G}(x+1;\delta)}$, $x=0,1,2,\dots$, (11)

the pmf of DAPTW-G family can be expressed as

$$P_{DAPTW-G}(x;\alpha,\delta,\theta)=\begin{cases}\frac{\alpha^{1-e^{-t_{i*}^{\theta}}}-\alpha^{1-e^{-t_{i}^{\theta}}}}{\alpha-1}, & \alpha>0,\ \alpha\neq1,\\ e^{-t_{i}^{\theta}}-e^{-t_{i*}^{\theta}}, & \alpha=1.\end{cases}\qquad(12)$$

One can note that when $\alpha=1$, the family reduce to the *discrete odd Weibull-G* (DOW-G) family of distributions introduced by El-Morshedy *et al.* (2021).

The *hazard rate* (hrf)*, alternative hazard rate* (ahrf) *and reversed hazard rate functions* (rhrf) can be formulated as

$$h_{DAPTW-G}(x;\alpha,\delta,\theta)=\frac{P_{DAPTW-G}(x)}{S_{DAAPTW-G}(x)}=\frac{\alpha^{1-e^{-t_{i*}^{\theta}}}-\alpha^{1-e^{-t_{i}^{\theta}}}}{\alpha-\alpha^{1-e^{-t_{i}^{\theta}}}},\qquad x=0,1,2\dots\ ;\ \alpha\neq1,\qquad(13)$$

$$ah_{DAPTW-G}(x;\alpha,\delta,\theta)=ln\left[\frac{S_{DAPTW-G}(x)}{S_{DAPTW-G}(x+1)}\right]=ln\left[\frac{\alpha-\alpha^{1-e^{-t_{i}^{\theta}}}}{\alpha-\alpha^{1-e^{-t_{i*}^{\theta}}}}\right],x=0,1,2\dots\ ;\ \alpha\neq1,\qquad(14)$$

and

$$rh_{DAPTW-G}(x;\alpha,\delta,\theta)=\frac{P_{DAPTW-G}(x)}{F_{DAPTW-G}(x)}=\frac{\alpha^{1-e^{-t_{i*}^{\theta}}}-\alpha^{1-e^{-t_{i}^{\theta}}}}{\alpha^{1-e^{-t_{i*}^{\theta}}}-1},\qquad x=0,1,2\dots\ ;\ \alpha\neq1.\qquad(15)$$

2.1 Some statistical properties

1. Quantile function

The qth quantile function of $DAPTW-G$, say x_q, is the solution of $F_{DAPTW-G}(x_q;\alpha,\delta,\theta)-q=0, x_q>0$, then

$$x_q = \left\lceil G^{-1}\left[\frac{\left(-ln\left(1-\frac{ln(1+q(\alpha-1))}{ln(\alpha)}\right)\right)^{\frac{1}{\theta}}}{1-\left(-ln\left(1-\frac{ln(1+q(\alpha-1))}{ln(\alpha)}\right)\right)^{\frac{1}{\theta}}}\right]\right\rceil, \quad (16)$$

where $\lceil x \rceil$denotes the smallest integer greater than or equal to x, $0 < u < 1$ and G^{-1} denotes the baseline of quantile function.

Special quantiles may be obtained using (16). For example, u= 0.5, the median of the $DAPTW - G$ distribution is

$$\text{Median} = \left\lceil G^{-1}\left[\frac{\left(-ln\left(1-\frac{ln(1+0.5(\alpha-1))}{ln(\alpha)}\right)\right)^{\frac{1}{\theta}}}{1-\left(-ln\left(1-\frac{ln(1+0.5(\alpha-1))}{ln(\alpha)}\right)\right)^{\frac{1}{\theta}}}\right]\right\rceil, \quad (17)$$

the Bowley skewness and Moors kurtosis based on quantiles can be obtained as

Bowley skewness= $\frac{x_{\frac{3}{4}}+x_{\frac{1}{4}}-2x_{\frac{1}{2}}}{x_{\frac{3}{4}}-x_{\frac{1}{4}}}$ and Moors kurtosis= $\frac{x_{\frac{3}{8}}-x_{\frac{1}{8}}+x_{\frac{7}{8}}-x_{\frac{5}{8}}}{x_{\frac{3}{4}}-x_{\frac{1}{4}}}$.

2. Moments, skewness, kurtosis and index of dispersion

The rth moments of the rv $x \sim \text{DAPTW} - \text{G}$ family may be obtained as follows:

$$\mu_r' = E(X^r) = \sum_{x=1}^{\infty}(x^r - (x-1)^r)\, s_{DAPTW-G}(x)$$

$$= \sum_{x=1}^{\infty}(x^r - (x-1)^r)(\alpha-1)^{-1}(\alpha - \alpha^{1-e^{-t_i{}^{\theta}}}), \qquad x = 1,2 \dots ; \alpha \neq 1, r = 1, 2, \dots . \qquad (18)$$

Using (18) the mean (μ_1') and variance $V(x)$can be respectively formulated as

$$\acute{\mu}_1 = \mu = \sum_{x=1}^{\infty}\left[(\alpha-1)^{-1}(\alpha - \alpha^{1-e^{-t_i{}^{\theta}}})\right] \quad (19)$$

and

$$V(x) = \sum_{x=1}^{\infty}\left[(2x-1)(\alpha-1)^{-1}(\alpha - \alpha^{1-e^{-t_i{}^{\theta}}})\right] - (\acute{\mu}_1)^2. \quad (20)$$

Also, one can use the first fourth rth moments to get skewness (sk) and kurtosis (ku) as follows:

$sk = \frac{\acute{\mu}_3 + 3\acute{\mu}_2\, \acute{\mu}_1 + 2\, \acute{\mu}_1{}^3}{(V(x))^{\frac{3}{2}}}$ and $ku = \frac{\acute{\mu}_4 - 4\acute{\mu}_3\, \acute{\mu}_1 + 6\acute{\mu}_2\, \acute{\mu}_1{}^2 - 3\, \acute{\mu}_1{}^4}{(V(x))^2}$. (21)

The *index of dispersion* (ID) is the variance-to-mean ratio and *coefficient of variation* (CV) is the ratio of the standard deviation to the mean, they are applied in various situations. The ID is widely used in ecology as a standard measure for measuring repulsion (under dispersion) or clustering (over dispersion).

If $ID < 1$ $(ID > 1)$ the distribution is under-dispersed (over-dispersed), whereas the distribution is equi-dispersed when $ID = 1$. The CV is a relative variability measure, expressing the dispersion of data values around the mean. It should be computed only for data measured on a ratio scale, that is, scales that have a meaningful zero and hence allow relative comparison of two measurements (i.e., division of one measurement by the other).

3. Order statistics

Let $F_i(x; \alpha, \delta, \theta)$; the cdf of the i^{th} order statistics for random sample $X_1, X_2, \ldots, X_n$, from the $DAPTW - G$ family of distributions is given by

$$F_{i:n}(x; \alpha, \delta, \theta) = \sum_{r=i}^{n} \binom{n}{r} [F_{DAPTW-G}(x; \alpha, \delta, \theta)]^r\, [1 - F_{DAPTW-G}(x; \alpha, \delta, \theta)]^{n-r}. \quad (22)$$

Using the binomial expansion for $[1 - F_{DAPTW-G}(x; \alpha, \delta, \theta)]^{n-r}$ and substituting (10) in (22).

Hence

$$F_{i:n}(x; \alpha, \delta, \theta) = \sum_{r=i}^{n} \binom{n}{r} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j \left[\frac{\alpha^{1-e^{-t_{i*}{}^{\theta}}} - 1}{\alpha - 1}\right]^{r+j}, \quad \alpha \neq 1. \quad (23)$$

Special cases

Case I: If i=1 in (23) one can obtain the cdf of the first order statistics, as given below

$$F_1(x; \alpha, \delta, \theta) = 1 - [1 - F_{DAPTW-G}(x; \alpha, \lambda, \xi)]^n = 1 - \left[1 - \frac{\alpha^{1-e^{-t_{i*}{}^{\theta}}} - 1}{\alpha - 1}\right]^n, \alpha \neq 1. \quad (24)$$

Case II: If i=n in (23) one can obtain the cdf of the largest order statistics, as given below

$$F_n(x;\alpha,\delta,\theta)=[F_{DAPTW-G}(x;\alpha,\delta,\theta)]^n=\left[\frac{\alpha^{1-e^{-t_{i*}}{}^{\theta}}-1}{\alpha-1}\right]^n,\qquad \alpha\neq 1.\quad (25)$$

Also, the pmf of i^{th} order statistics of $DAPTW-G$ family of distributions, is defined by

$$P_{i:n}(x;\alpha,\delta,\theta)=\frac{n!}{(r-1)!(n-r)!}\int_{F(x-;\xi)}^{F(x;\xi)}v^{r-1}(1-v)^{n-r}dv.\quad (26)$$

Using the binomial expansion for $(1-v)^{n-r}$

$$P_{i:n}(x;\alpha,\delta,\theta)=$$

$$\frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}\frac{(-1)^j}{s+j}\left\{[F_{DAPTW-G}(x;\alpha,\delta,\theta)]^{s+j}-[F_{DAPTW-G}(x-;\alpha,\delta,\theta)]^{s+j}\right\}.$$

$$=\frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}\frac{(-1)^j}{s+j}\left\{\left[\frac{\alpha^{1-e^{-t_{i*}}{}^{\theta}}-1}{\alpha-1}\right]^{s+j}-\left[\frac{\alpha^{1-e^{-t_i}{}^{\theta}}-1}{\alpha-1}\right]^{s+j}\right\}\quad (27)$$

The pmf of the smallest order statistics is obtained by substituting i=1 in (27) as follows:

$$P_1(x;\alpha,\delta,\theta)=\left[1-\frac{\alpha^{1-e^{-t_i}{}^{\theta}}-1}{\alpha-1}\right]^n-\left[1-\frac{\alpha^{1-e^{-t_{i*}}{}^{\theta}}-1}{\alpha-1}\right]^n,\ \alpha\neq 1.\quad (28)$$

The pmf of the largest order statistics is obtained by substituting i=n in (27) as follows:

$$P_n(x;\alpha,\lambda,\xi)=\left[\frac{\alpha^{1-e^{-t_{i*}}{}^{\theta}}-1}{\alpha-1}\right]^n-\left[\frac{\alpha^{1-e^{-t_i}{}^{\theta}}-1}{\alpha-1}\right]^n,\ \alpha\neq 1.\quad (29)$$

Also, (23) can be used to obtain the pmf of $DAPTW-G$, (see Arnold *et al.* (2008)).

4. Rényi entropy

The importance of reducing uncertainty and increasing system lifetime is widely recognized, with longer lifetimes and lower uncertainties being key indicators of higher system reliability. To this end the Rényi entropy can be used to measure the uncertainty associated with a non-negative rv x with the pmf $P(x)$, and it is denoted by $H_{\rho}(x)$

Rényi entropy has many applications in variety of fields such as measuring uncertainty in dynamical systems, urban and regional planning, business, economics, finance, operations research, queueing theory, spectral analysis, image reconstruction, biology and manufacturing. It is defined by

$$H_{\rho}(\rho) = (1-\rho)^{-1} log\left\{\sum_{x=0}^{\infty}\left(P_{DAPTW-G}(x)\right)^{\rho}\right\}$$

$$= (1-\rho)^{-1} log\left\{\sum_{x=0}^{\infty}\left(\frac{\alpha^{1-e^{-t_{i*}}{}^{\theta}} - \alpha^{1-e^{-t_i^{\theta}}}}{\alpha-1}\right)^{\rho}\right\}, \alpha \neq 1\ \rho > 0, \rho \neq 1\,. \quad (30)$$

The Shannon entropy can be defined by $E\left[-log\left(P_{DAPTW-G}(X)\right)\right]$, and it can be calculated as a special case of the Rényi entropy when $\rho \to 1$.

5. Mean time to failure, mean time between failure, and Availability

Mean Time to Failure (MTTF) is the average time between non-repairable failures. It is a very basic measure that helps predict the lifecycle for components that cannot be repaired, such a light bulb or a backup tape. It is particularly useful as a reliability metric. It can be used to estimate how long a component of critical machinery or equipment may last, evaluate the effectiveness and quality of parts and components, forecast replacement needs and plan preventive maintenance tasks and perform proper inventory management to ensure resources and replacement parts are available. Shorter MTTF means more frequent replacements. And more replacements mean more downtime, less uptime, higher costs, and other impacts on productivity.

The MTTF is given as follows:

$$MTTF = \sum_{x=1}^{\infty} S_{APTW-G}(x) = \sum_{x=1}^{\infty}\frac{\alpha-\alpha^{1-e^{-t_i^{\theta}}}}{\alpha-1}\ ,\ \ x > 0;\ \ \alpha \neq 1. \quad (31)$$

The *Mean Time between Failure* (MTBF) is the mean (or average) time expected between failures of a given device and is normally measured in hours. It is used with items that can be either repaired or replaced and is given bellow

$$MTBF = \frac{-x}{log\ [S_{APTW-G}(x)]} = \frac{-x}{log\ [\frac{\alpha-\alpha^{1-e^{-t_i^{\theta}}}}{\alpha-1}]}\ , \qquad x > 0;\ \ \alpha \neq 1. \qquad (32)$$

The *Availability* (Av) is the probability that a system is operational when called upon to perform its function. It is quantified as a percentage, and is given bellow

$$Av = \frac{MTTF}{MTBF}$$

$$= \frac{\sum_{x=1}^{\infty} S_{APTW-G}(x)\, log\, [S_{APTW-G}(x)]}{-x} = \frac{\sum_{x=1}^{\infty} \frac{\alpha - \alpha^{1-e^{-t_i{}^{\theta}}}}{\alpha-1}}{-x}. \qquad (33)$$

2.2 Maximum likelihood estimation for discrete alpha power transformed Weibull -G family of distributions

In this section, the unknown parameters of the $DAPTW - G$ family of distributions are derived using the ML method based on Type-II censored samples.

Let $(x_1, x_2, \dots, x_n)$ be a random sample from $DAPTW - G$ family of distributions with pmf as $P_{DAPTW-G}(x; \alpha, \delta, \theta)$. The likelihood function of $DAPTW - G$ family of distributions based on Type-II censored sample corresponding (8) and (12) is:

$$L(\alpha, \delta, \theta; \underline{x}) \propto \{\prod_{i=1}^{r} P(x_{(i)})\}[S(x_{(r)})]^{n-r},$$

$$= \left\{\prod_{i=1}^{r} \frac{\alpha^{1-e^{-t_{i*}{}^{\theta}}} - \alpha^{1-e^{-t_i{}^{\theta}}}}{\alpha-1}\right\}\left[\frac{\alpha-\alpha^{1-e^{-t_i{}^{\theta}}}}{\alpha-1}\right]^{n-r}, \qquad (34)$$

the natural logarithm of the likelihood function is given by

$$l \equiv lnL(\alpha, \delta, \theta; \underline{x}) \propto ln \prod_{i=1}^{r}\left[\frac{\alpha^{1-e^{-t_{i*}{}^{\theta}}} - \alpha^{1-e^{-t_i{}^{\theta}}}}{\alpha-1}\right] + (n-r) ln \left[\frac{\alpha-\alpha^{1-e^{-t_i{}^{\theta}}}}{\alpha-1}\right],$$

$$= \sum_{i=1}^{r} ln\left[\alpha^{1-e^{-t_{i*}{}^{\theta}}} - \alpha^{1-e^{-t_i{}^{\theta}}}\right] + (n-r) ln\left[\alpha - \alpha^{1-e^{-t_i{}^{\theta}}}\right] - n\ln(\alpha-1). \quad (35)$$

The ML estimators of the parameters α, δ and θ can be derived by solving the nonlinear likelihood equations obtained by differentiating (35) with respect to α, δ and θ, setting these equations to zero and solving them, immediately yields the ML estimators for the $DAPTW - G$ family parameters. These equations cannot be solved analytically; therefore, an iterative procedure like Newton–Raphson is required to solve them numerically.

3. Some Special Models of Discrete Alpha Power Transformed Weibull -G Family of distributions

In this section the $DAPTW-G$ family is applied to a specific class of distribution functions such as Lindley and Rayleigh distributions.

3.1 Discrete alpha power transformed Weibull – Lindley distribution

The cdf of the Lindley distribution with parameter a is,

$$F(x;\delta)=1-\left(1+\frac{ax}{1+a}\right)e^{-ax}\ ,\ x>0\ ;\ a>0\ , \qquad (36)$$

Applying Lindley distribution,

$$t_i=\frac{G(x;\delta)}{\bar{G}(x;\delta)}=\left(\frac{e^{ax}}{1+\frac{ax}{1+a}}-1\right),t_{i*}=\frac{G(x+1;\delta)}{\bar{G}(x+1;\delta)}=\left(\frac{e^{a(x+1)}}{1+\frac{a(x+1)}{1+a}}-1\right). \qquad (37)$$

Using (10) and (12)-(14), the pmf, cdf, hrf and ahrf of the ($DAPTW$ −*Lindley*) DAPTW − L distribution are, respectively, given by

$$P_{DAPTW-L}(x;\alpha,\delta,\theta)=\begin{cases}\dfrac{\alpha^{1-e^{-\left(\frac{e^{a(x+1)}}{1+\frac{a(x+1)}{1+a}}-1\right)^{\theta}}}-\alpha^{1-e^{-\left(\frac{e^{ax}}{1+\frac{ax}{1+a}}-1\right)^{\theta}}}}{\alpha-1}, & \alpha>0,\ \alpha\neq1,\\[2ex] e^{-\left(\frac{e^{ax}}{1+\frac{ax}{1+a}}-1\right)^{\theta}}-e^{-\left(\frac{e^{a(x+1)}}{1+\frac{a(x+1)}{1+a}}-1\right)^{\theta}}, & \alpha=1,\end{cases}$$

(38)

and

$$F_{DAPTW-L}(x;\alpha,\delta,\theta)=\begin{cases}\dfrac{\alpha^{1-e^{-\left(\frac{e^{a(x+1)}}{1+\frac{a(x+1)}{1+a}}-1\right)^{\theta}}}-1}{\alpha-1}, & \alpha>0,\ \alpha\neq1,\\[2ex] 1-e^{-\left(\frac{e^{a(x+1)}}{1+\frac{a(x+1)}{1+a}}-1\right)^{\theta}}, & \alpha=1\ ,\end{cases} \qquad (39)$$

One can note that when $\alpha=1$, the distribution reduces to the DOW *-Lindley* (DOW-L) which is a member of DOW-G family of distributions introduced by El-Morshedy *et al.* (2021).

$$h_{DAPTW-L}(x;\alpha,\delta,\theta)=\frac{P_{DAPTW-L}(x)}{S_{DAAPTW-L}(x)}=\frac{\alpha^{1-e^{-\left(\frac{e^{a(x+1)}}{1+\frac{a(x+1)}{1+a}}-1\right)^{\theta}}}-\alpha^{1-e^{-\left(\frac{e^{ax}}{1+\frac{ax}{1+a}}-1\right)^{\theta}}}}{\alpha-\alpha^{1-e^{-\left(\frac{e^{ax}}{1+\frac{ax}{1+a}}-1\right)^{\theta}}}}\ ,\ x=$$

$0,1,2\ldots\ ;\ \alpha\neq1,$ (40)

and

$$ah_{DAPTW-L}(x;\alpha,\delta,\theta)=\ ln\left[\frac{S_{DAPTW-L}(x)}{S_{DAPTW-L}(x+1)}\right]=ln\left[\frac{\alpha-\alpha^{1-e^{-\left(\frac{e^{ax}}{1+\frac{ax}{1+a}}-1\right)^{\theta}}}}{\alpha-\alpha^{1-e^{-\left(\frac{e^{a(x+1)}}{1+\frac{a(x+1)}{1+a}}-1\right)^{\theta}}}}\right],x=$$

$0,1,2\ldots\ ;\ \alpha\neq1.$ (41)

3.2 Discrete alpha power transformed Weibull – Rayleigh distribution

The cdf of Rayleigh distribution with parameter β is,

$$F(x;\delta)=1-e^{-\frac{\beta}{2}x^2}\ ,\ x>0\ ;\ \beta>0\ , \qquad (42)$$

In this case, $t_i=\frac{G(x;\delta)}{\bar{G}(x;\delta)}=\left(e^{\frac{\beta}{2}x^2}-1\right),\quad t_{i*}=\frac{G(x+1;\delta)}{\bar{G}(x+1;\delta)}=\left(e^{\frac{\beta}{2}(x+1)^2}-1\right).$ (43)

Using (10) and (12)-(14), the pmf, cdf, hrf and ahrf of the ($DAPTW-Rayleigh$) DAPTW − R distribution are, respectively, given by

$$P_{DAPTW-R}(x;\alpha,\delta,\theta)=\begin{cases}\dfrac{\alpha^{1-e^{-\left(e^{\frac{\beta}{2}(x+1)^2}-1\right)^{\theta}}}-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}}}{\alpha-1}, & \alpha>0,\ \alpha\neq1,\\ e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}-e^{-\left(e^{\frac{\beta}{2}(x+1)^2}-1\right)^{\theta}}, & \alpha=1,\end{cases}$$

(44)

and

$$F_{DAPTW-R}(x;\alpha,\delta,\theta)=\begin{cases}\dfrac{\alpha^{1-e^{-\left(e^{\frac{\beta}{2}(x+1)^2}-1\right)^{\theta}}}-1}{\alpha-1}, & \alpha>0,\ \alpha\neq 1,\\ 1-e^{-\left(e^{\frac{\beta}{2}(x+1)^2}-1\right)^{\theta}}, & \alpha=1,\end{cases} \qquad (45)$$

Similarly, when $\alpha = 1$, the distribution reduces to the DOW - *Rayleigh* (DOW-R) which is a member of DOW-G family of distributions introduced by El-Morshedy *et al.* (2021).

$$h_{DAPTW-R}(x;\alpha,\delta,\theta)=\frac{P_{DAPTW-R}(x)}{S_{DAAPTW-R}(x)}=\frac{\alpha^{1-e^{-\left(e^{\frac{\beta}{2}(x+1)^2}-1\right)^{\theta}}}-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}}}{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}}}\ ,\ x=$$

$0,1,2\ldots\ ;\ \alpha\neq 1,$ (46)

$$ah_{DAPTW-R}(x;\alpha,\delta,\theta)=\ ln\left[\frac{S_{DAPTW-R}(x)}{S_{DAPTW-R}(x+1)}\right]=ln\left[\frac{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}}}{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}(x+1)^2}-1\right)^{\theta}}}}\right],x=$$

$0,1,2\ldots\ ;\ \alpha\neq 1.$ (47)

4. Discrete Alpha Power Transformed Weibull – Exponential Distribution

The cdf of the exponential distribution with parameter β is,

$$F(x;\delta)=1-e^{-\beta x}\ ,\ x>0\ ;\ \beta>0\ , \qquad (48)$$

if, $t_i=\frac{G(x;\delta)}{\bar{G}(x;\delta)}=\left(e^{\beta x}-1\right),\ t_{i*}=\frac{G(x+1;\delta)}{\bar{G}(x+1;\delta)}=\left(e^{\beta(x+1)}-1\right).$ (49)

Using (10) and (12)-(14), the pmf, cdf, hrf, ahrf and rhrf of the ($DAPTW-$ *Exponential*) DAPTW − E distribution are, respectively, given by

$$P_{DAPTW-E}(x;\alpha,\delta,\theta)=\begin{cases}\dfrac{\alpha^{1-e^{-\left(e^{\beta(x+1)}-1\right)^{\theta}}}-\alpha^{1-e^{-\left(e^{\beta x}-1\right)^{\theta}}}}{\alpha-1}, & \alpha>0,\ \alpha\neq 1,\\ e^{-\left(e^{\beta x}-1\right)^{\theta}}-e^{-\left(e^{\beta(x+1)}-1\right)^{\theta}}, & \alpha=1,\end{cases}$$

(50)

and

$$F_{DAPTW-E}(x;\alpha,\delta,\theta)=\begin{cases}\dfrac{\alpha^{1-e^{-\left(e^{\beta(x+1)}-1\right)^{\theta}}}-1}{\alpha-1}, & \alpha>0,\ \alpha\neq 1,\\ 1-e^{-\left(e^{\beta(x+1)}-1\right)^{\theta}}, & \alpha=1\,,\end{cases} \qquad (51)$$

One can note that when $\alpha = 1$, the family reduces to the *DOW - Exponential* (DOW-E) which is a member of *DOW* -G family of distributions introduced by El-Morshedy *et al.* (2021).

The hrf, ahrf *and* rhrf can be formulated as

$$h_{DAPTW-E}(x;\alpha,\delta,\theta)=\frac{P_{DAPTW-E}(x)}{S_{DAAPTW-E}(x)}=\frac{\alpha^{1-e^{-\left(e^{\beta(x+1)}-1\right)^{\theta}}}-\alpha^{1-e^{-\left(e^{\beta x}-1\right)^{\theta}}}}{\alpha-\alpha^{1-e^{-\left(e^{\beta x}-1\right)^{\theta}}}}\ ,x=0,1,2\ldots\ ;\ \alpha\neq 1, \qquad (52)$$

$$ah_{DAPTW-E}(x;\alpha,\delta,\theta)=ln\left[\frac{S_{DAPTW-E}(x)}{S_{DAPTW-E}(x+1)}\right]=ln\left[\frac{\alpha-\alpha^{1-e^{-\left(e^{\beta x}-1\right)^{\theta}}}}{\alpha-\alpha^{1-e^{-\left(e^{\beta(x+1)}-1\right)^{\theta}}}}\right],x=0,1,2\ldots\ ;\ \alpha\neq 1, \qquad (53)$$

and

$$rh_{DAPTW-E}(x;\alpha,\delta,\theta)=\frac{P_{DAPTW-E}(x)}{F_{DAPTW-E}(x)}=\frac{\alpha^{1-e^{-t_{i*}}{}^{\theta}}-\alpha^{1-e^{-t_i}{}^{\theta}}}{\alpha^{1-e^{-\left(e^{\beta(x+1)}-1\right)^{\theta}}}-1},\qquad x=0,1,2\ldots\ ;\ \alpha\neq 1. \qquad (54)$$

Figures 1- 3 display some plots of pmf, hrf and ahrf of the $DAPTW-E$ distribution for various values of the parameters.

Figure 1 indicates that the pmf of $DAPTW-E$ can be either unimodal or bimodal and can be decreasing, increasing, decreasing followed by unimodal, left and right skewed with heavy tail, among other useful pmf. Figures 2 and 3 show some plots of the hrf and ahrf for various values of the parameters which are decreasing, increasing and bathtub shapes.

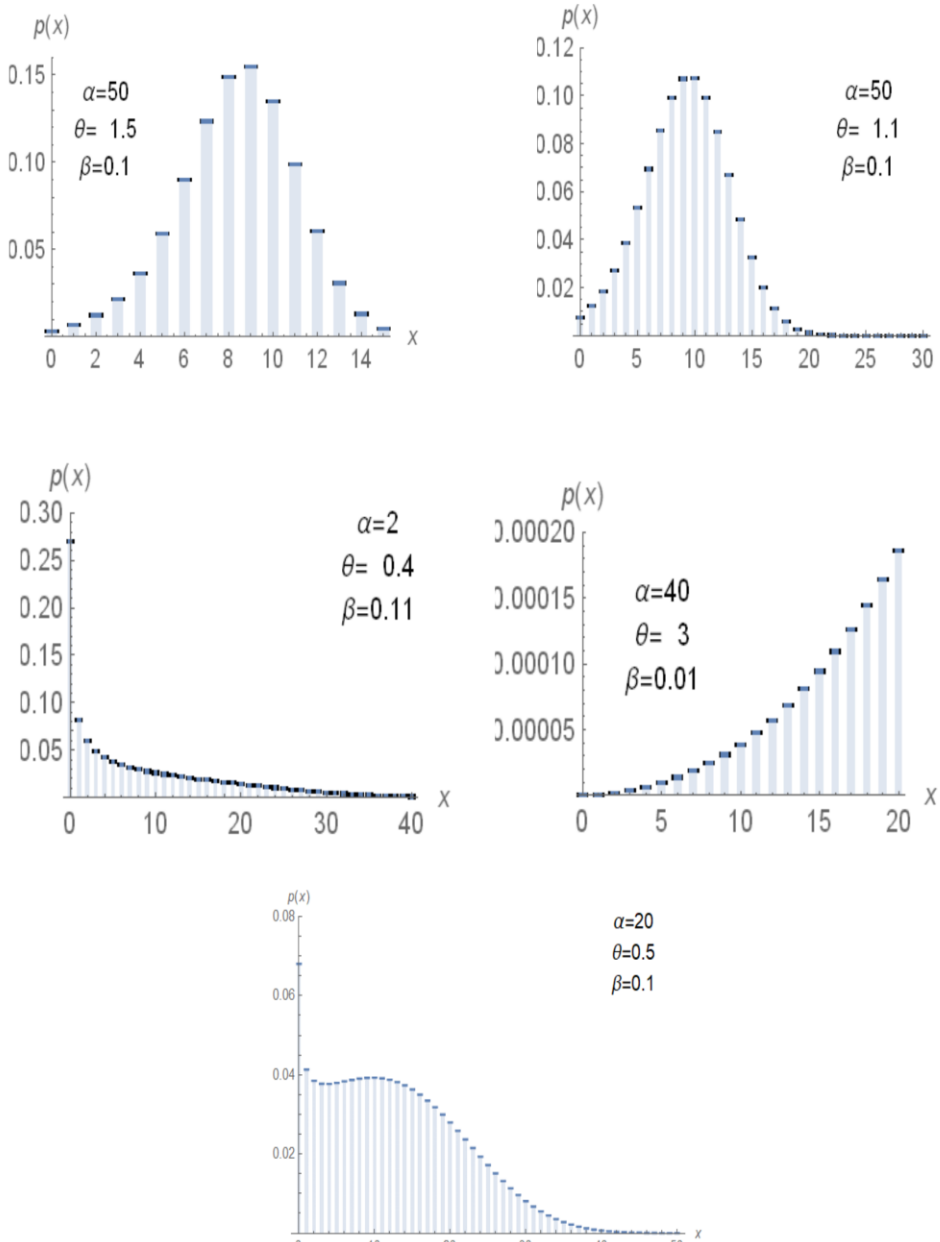


Figure 1. Plots of the pmf of $DAPTW-E$ for different values of the parameters

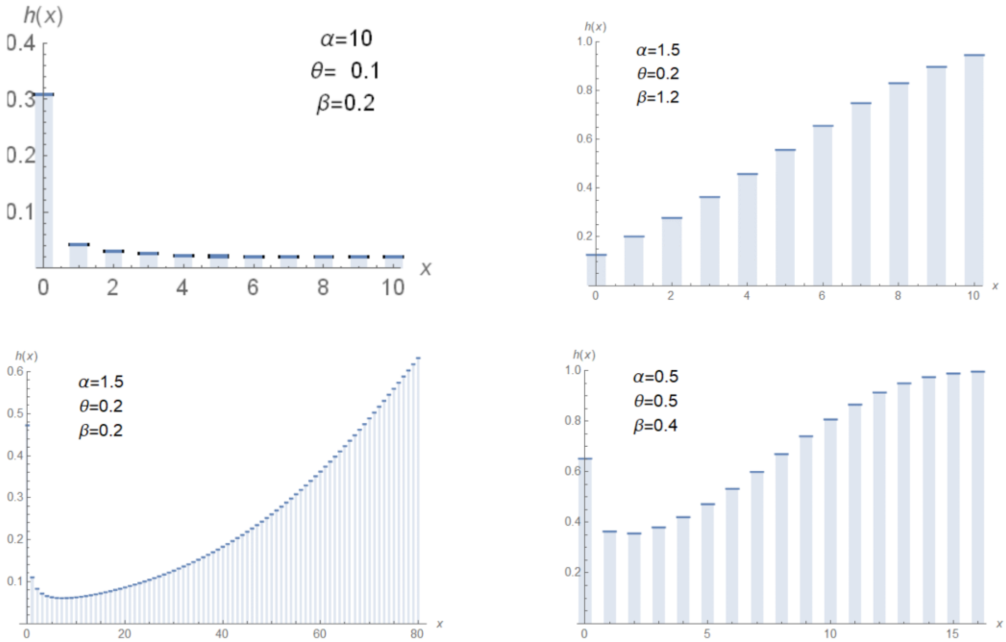
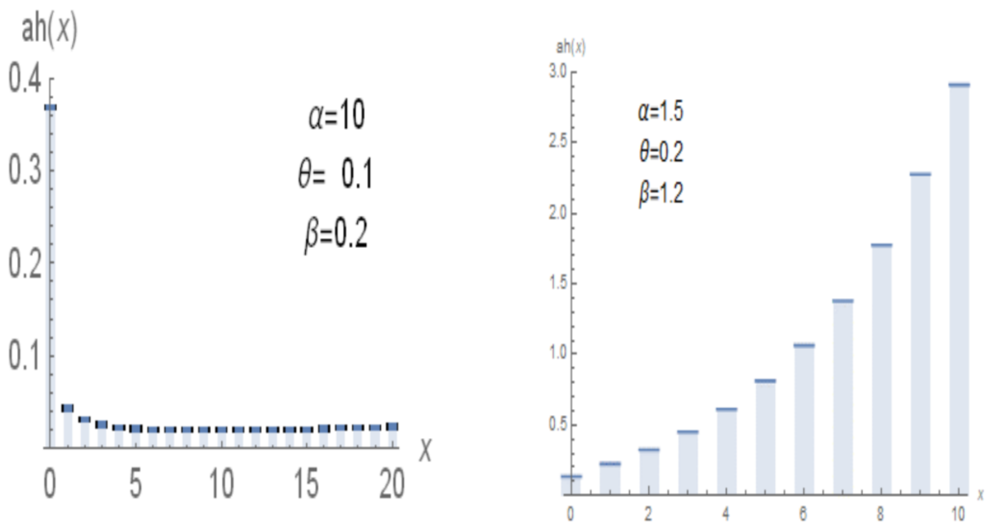


Figure 2. Plots of the hrf of $DAPTW-E$ for different values of the parameters



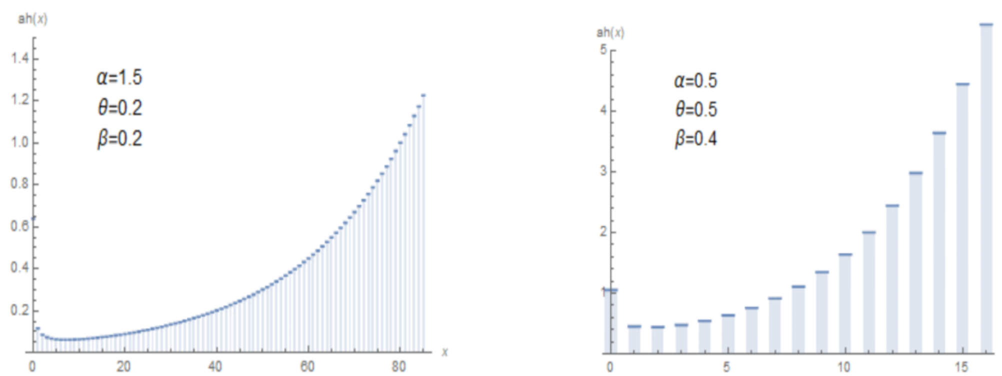


Figure 3. Plots of the ahrf of $DAPTW - E$ for different values of the parameters

4.1 Some statistical properties of discrete alpha power transformed Weibull – exponential distribution

In this subsection, some basic properties of the DAPTW-E distribution such as quantile function, moments, order statistics, Rényi entropy, mean time to failure, mean time between failure and Availability are derived.

4.1.1 Quantile function

The qth quantile x_q of DAPTW-E distribution, for $\alpha \neq 1$, can be obtained by using (16) as

$$x_q = \left[\left[\frac{1}{\beta} * log\left\{\left(-ln\left(1 - \frac{ln\left(1+q(\alpha-1)\right)}{ln(\alpha)}\right)\right)^{\frac{1}{\theta}} + 1\right\}\right]\right], \qquad 0 < q < 1. \qquad (55)$$

Hence, the median can be obtained if q=0.5 as given below

$$x_{0.5} = \left[\left[\frac{1}{\beta} * log\left\{\left(-ln\left(1 - \frac{ln\left(1+0.5(\alpha-1)\right)}{ln(\alpha)}\right)\right)^{\frac{1}{\theta}} + 1\right\}\right]\right]. \qquad (56)$$

4.1.2 Moments, skewness, kurtosis and index of dispersion

The rth moments of DAPTW-E distribution cannot be expressed in closed form, so a software program should be used to calculate these statistics to recognize the properties of DAPTW-E distribution. So, Mathematica 11 program is used to exhibit some of them for different values of the parameters lists in Table 1.

Table 1
Some descriptive statistics for DAPTW-E distribution for some values of the parameters

Parameter				Descriptive statistics				
α	**β**	**θ**	**Mean**	**Median**	**Variance**	**ID**	**Sk**	**Kur**
0.5	**0.15**	**0.8**	2.9769	2	9.8062	3.2941	1.2266	4.1530
5			5.0856	5	12.9398	2.5444	0.5152	2.6739
10			5.7002	5	12.9692	2.2752	0.3724	2.5896
1.3	**0.05**	**2**	11.9651	12	23.5741	1.9702	-0.0065	2.5355
	0.1		5.7326	6	5.9560	1.0390	-0.0062	2.5443
	0.2		2.6162	3	1.5513	0.5930	-0.0039	2.5671
0.5	**0.1**	**0.5**	5.1354	2	48.3097	8.9645	1.7376	5.8576
		3	5.3890	5	3.1703	0.58829	-0.1354	2.7081
		4	5.5925	6	2.0195	0.3611	-0.3142	2.9396

From Table 1 it is clear that the DAPTW-E distribution is suitable for modeling different types of data sets where it is suitable for modeling over and under dispersion data sets where $ID > (<)\ 1$. It is also can be used for modeling positive and negative skewed and can be used to model either platykurtic ($Ku < 3$) or leptokurtic ($Ku > 3$) data.

4.1.3 Order statistic

From (23) and (27) the cdf and pmf of the i^{th} order statistics for a random sample $X_1, X_2, \dots, X_n$, from DAPTW-E distribution is given by

$$F_{i:n}(x;\alpha,\beta,\theta) = \sum_{r=i}^{n}\binom{n}{r}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\left[\frac{\alpha^{1-e^{-\left(e^{\beta(x+1)}-1\right)^{\theta}}}-1}{\alpha-1}\right]^{r+j},$$

$\alpha \neq 1$, (57)

and

$$P_{i:n}(x;\alpha,\beta,\theta)=\frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}\frac{(-1)^j}{s+j}\left\{\left[\frac{\alpha^{1-e^{-\left(e^{\beta(x+1)}-1\right)^{\theta}}}-1}{\alpha-1}\right]^{s+j}-\left[\frac{\alpha^{1-e^{-\left(e^{\beta x}-1\right)^{\theta}}}-1}{\alpha-1}\right]^{s+j}\right\}. \quad (58)$$

Special cases

The pmf of the smallest order statistics is obtained as follows:

$$P_1(x;\alpha,\beta,\theta)=\left[1-\frac{\alpha^{1-e^{-\left(e^{\beta x}-1\right)^{\theta}}}-1}{\alpha-1}\right]^{n}-\left[1-\frac{\alpha^{1-e^{-\left(e^{\beta(x+1)}-1\right)^{\theta}}}-1}{\alpha-1}\right]^{n},\ \alpha\neq 1. \quad (59)$$

The pmf of the largest order statistics is

$$P_n(x;\alpha,\beta,\theta)=\left[\frac{\alpha^{1-e^{-\left(e^{\beta(x+1)}-1\right)^{\theta}}}-1}{\alpha-1}\right]^{n}-\left[\frac{\alpha^{1-e^{-\left(e^{\beta x}-1\right)^{\theta}}}-1}{\alpha-1}\right]^{n},\quad \alpha\neq 1. \quad (60)$$

4.1.4 Rényi entropy

The Rényi entropy can be given as

$$H_\rho(\rho)=(1-\rho)^{-1}log\left\{\sum_{x=0}^{\infty}\left(\frac{\alpha^{1-e^{-\left(e^{\beta(x+1)}-1\right)^{\theta}}}-\alpha^{1-e^{-\left(e^{\beta x}-1\right)^{\theta}}}}{\alpha-1}\right)^{\rho}\right\},\alpha\neq 1\ \rho>0,\rho\neq 1\,. \quad (61)$$

The Shannon entropy can be calculated as a special case of the Rényi entropy when $\rho\rightarrow 1$.

4.1.5 Mean time to failure, mean time between failure, and Availability

The MTTF, MTBFand Av are given as follows:

$$MTTF=\sum_{x=1}^{\infty}\frac{\alpha-\alpha^{1-e^{-\left(e^{\beta x}-1\right)^{\theta}}}}{\alpha-1}\,,\qquad x>0;\ \alpha\neq 1, \quad (62)$$

$$MTBF = \frac{-x}{log\,[S_{APTW-G}(x)]} = \frac{-x}{log\,[\frac{\alpha-\alpha^{1-e^{-\left(e^{\beta x}-1\right)^{\theta}}}}{\alpha-1}]}\,, \qquad x>0;\ \ \alpha\neq 1,$$
(63)

and

$$Av \ = \frac{\sum_{x=1}^{\infty} S_{APTW-G}(x)\,log\,[S_{APTW-G}(x)]}{-x} = \frac{\sum_{x=1}^{\infty}\frac{\alpha-\alpha^{1-e^{-\left(e^{\beta x}-1\right)^{\theta}}}}{\alpha-1}}{-x}, \qquad x>0;\ \ \alpha\neq 1.$$
(64)

4.2 Maximum likelihood estimation for discrete alpha power transformed Weibull -exponential distribution

The natural logarithm of the likelihood function of DAPTW-E can be written in the form:

$$\ell \equiv lnL\left(\alpha,\beta,\theta;\underline{x}\right) \propto \sum_{i=1}^{r} ln\left[\alpha^{1-e^{-\left(e^{\beta(x+1)}-1\right)^{\theta}}} - \alpha^{1-e^{-\left(e^{\beta x}-1\right)^{\theta}}}\right]$$
$$+(n-r)ln\left[\alpha-\alpha^{1-e^{-\left(e^{\beta x}-1\right)^{\theta}}}\right] - n\ln(\alpha-1). \qquad (65)$$

The ML estimators can be derived by setting the partial first derivatives of (65) with respect to α,θ and β, respectively, to zeros. The system of the non-linear equations can be solved numerically using the Newton-Raphson method, to obtain the ML estimators $\hat{\alpha},\hat{\theta}$ and $\hat{\beta}$.

5. Numerical Results

This section aims to evaluate the performance of the ML estimates based on simulated and real data through some measurements of accuracy.

5.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented ML estimates based on a simulation study which is describes in the following:

- Select different combinations of true values for the parameters.
- Generate 1000 samples (*number of replication* (NR)) of sample sizes 30, 50 and 100 and 200 from DAPTW-E based on complete sample levels of $\frac{r}{n}\times 100$ percentage of uncensored observations Type-II censoring.

- For each model parameter and for each sample size, the ML estimates are computed.
- Repeat the previous steps 1000 times for each sample size and for selected sets of the parameters.

The ML averages, *relative absolute biases* (RABs), *Relative errors* (REs), *Estimated risk* (ERs) and variances of the ML estimates of the parameters, sf, hrf and ahrf are computed as follows:

1) Averages = $\frac{\sum_{i=1}^{NR} \text{estimates}}{NR}$
2) RABs (estimate) = $\frac{|\text{bias (estimate)}|}{\text{true value}}$,
3) REs =$\frac{\text{ER(estimate)}}{\text{true value}}$,
4) Variances (estimate) = $\text{ER(estimate)} - \text{bias}^2 \text{ (estimate)}$.

The simulation study is performed using Mathematica 11.

The results are presented in Tables 2-5 for different combinations of the parameters based on complete sample and level of $\frac{r}{n} \times 100$ percentage of uncensored observations Type II censoring 70%.

Table 2.

ML averages, relative absolute biases, relative errors, variances of ML estimates, 95%confidence

intervals of the parameters from DAPTW-E distribution for based on complete sample size

$(\alpha = 5, \quad \beta = 0.2, \quad \theta = 3, \quad x_0 = 1 \text{ and } \quad NR = 1000)$

n	φ	Average	RAB	RE	Variance	UL	LL	Length
30	α	8.1654	0.6331	0.6347	0.0501	8.6042	7.7266	0.8776
	β	0.1920	0.0398	0.0424	8.5857×10^{-6}	0.1978	0.1863	0.0115
	θ	2.2399	0.2534	0.2551	0.0079	2.4146	2.0652	0.3494
	$S(x_0)$	0.9907	0.0050	0.0053	3.1203×10^{-6}	0.9941	0.9872	0.0069
	$h(x_0)$	0.0501	0.1054	0.1523	2.4758×10^{-5}	0.0598	0.0403	0.0195
	$ah\ (x_0)$	0.0514	0.1085	0.1572	2.7758×10^{-5}	0.0617	0.0410	0.0207
60	α	8.1489	0.6298	0.6305	0.0214	8.4355	7.8623	0.5731
	β	0.1918	0.0410	0.0421	3.7172×10^{-6}	0.1956	0.1880	0.0076
	θ	2.2503	0.2499	0.2506	0.0031	2.3594	2.1413	0.2181
	$S(x_0)$	0.9909	0.0047	0.0048	9.0869×10^{-7}	0.9928	0.9890	0.0037
	$h(x_0)$	0.0495	0.0928	0.1145	9.2395×10^{-6}	0.0554	0.0435	0.0119
	$ah\ (x_0)$	0.0507	0.0953	0.1177	1.0256×10^{-5}	0.0570	0.0445	0.0126
100	α	8.1673	0.6335	0.6338	0.0106	8.3690	7.9657	0.4032
	β	0.1917	0.0413	0.0419	1.7740×10^{-6}	0.1943	0.1891	0.0052
	θ	2.2495	0.2502	0.2505	0.0015	2.3326	2.1731	0.1529
	$S(x_0)$	0.9910	0.0047	0.0047	4.3458×10^{-7}	0.9922	0.9897	0.0026
	$h(x_0)$	0.0493	0.0899	0.1012	4.4226×10^{-6}	0.0535	0.0452	0.0082
	$ah\ (x_0)$	0.0506	0.0923	0.1039	4.9035×10^{-6}	0.0550	0.0463	0.0087
200	α	8.1651	0.6330	0.6332	0.0056	8.3115	8.0187	0.2928
	β	0.1917	0.0415	0.0418	9.7218×10^{-7}	0.1936	0.1898	0.0039
	θ	2.2510	0.2497	0.2498	0.0008	2.3068	2.1953	0.1115
	$S(x_0)$	0.9910	0.0047	0.0047	2.2433×10^{-7}	0.9919	0.9901	0.0019
	$h(x_0)$	0.0493	0.0879	0.0942	2.3549×10^{-6}	0.0523	0.0463	0.0060
	$ah\ (x_0)$	0.0505	0.0902	0.0967	2.6071×10^{-6}	0.0537	0.0474	0.0063
500	α	8.1642	0.6328	0.6329	0.0020	8.2518	8.0766	0.1752
	β	0.1917	0.0416	0.0417	3.4324×10^{-7}	0.1928	0.1905	0.0023
	θ	2.2513	0.2496	0.2496	0.0003	2.2840	2.2186	0.0653
	$S(x_0)$	0.9910	0.0046	0.0047	7.5711×10^{-8}	0.9915	0.9905	0.0011
	$h(x_0)$	0.0492	0.0875	0.0897	8.0124×10^{-7}	0.0510	0.0475	0.0035
	$ah\ (x_0)$	0.0505	0.0898	0.0921	8.9312×10^{-7}	0.0523	0.0486	0.0037

Table 3.

ML averages, relative absolute biases, relative errors, variances of ML estimates, 95%confidence intervals of the parameters from DAPTW-E distribution for based on complete sample size

$(\alpha = 2, \quad \beta = 0.4, \quad \theta = 1.6, \; x_0 = 1 \text{ and } \quad NR = 1000)$

n	φ	Average	RAB	RE	Variance	UL	LL	Length
30	α	3.4293	0.7146	0.7153	0.0036	3.5467	3.3118	0.2349
	β	0.3915	0.0211	0.0418	0.0002	0.4199	0.3632	0.0566
	θ	1.0977	0.3139	0.3165	0.0042	1.2248	0.9707	0.2541
	$S(x_0)$	0.7703	0.0252	0.0356	0.0004	0.8091	0.7314	0.0778
	$h(x_0)$	0.4338	0.2730	0.2735	0.0001	0.4543	0.4133	0.0410
	$ah\,(x_0)$	0.5690	0.3734	0.3739	0.0003	0.6052	0.5327	0.0725
60	α	3.4169	0.7084	0.7086	0.0011	3.4821	3.3517	0.1304
	β	0.3896	0.0260	0.0346	0.0001	0.4075	0.3717	0.0357
	θ	1.1084	0.3073	0.3083	0.0017	1.1883	1.0285	0.1598
	$S(x_0)$	0.7732	0.0216	0.0268	0.0002	0.7979	0.7484	0.0494
	$h(x_0)$	0.4342	0.2723	0.2726	5.8359×10^{-5}	0.4492	0.4192	0.0299
	$ah\,(x_0)$	0.5696	0.3727	0.3730	0.0002	0.5961	0.5431	0.0531
100	α	3.4140	0.7070	0.7070	0.0005	3.4593	3.3686	0.0906
	β	0.3894	0.0265	0.0316	4.7470×10^{-5}	0.4029	0.3759	0.0270
	θ	1.1102	0.3061	0.3067	0.0009	1.1685	1.0519	0.1166
	$S(x_0)$	0.7736	0.0210	0.0242	0.0001	0.7919	0.7552	0.0368
	$h(x_0)$	0.4346	0.2716	0.2718	4.0836×10^{-5}	0.4471	0.4221	0.0250
	$ah\,(x_0)$	0.5703	0.3719	0.3721	0.0001	0.5925	0.5481	0.0444
200	α	3.4113	0.7056	0.7057	0.0002	3.4406	3.3820	0.0586
	β	0.3891	0.0273	0.0299	2.3677×10^{-5}	0.3986	0.3795	0.0191
	θ	1.1124	0.3047	0.3050	0.0004	1.1536	1.0712	0.0824
	$S(x_0)$	0.7741	0.0204	0.0220	4.4353×10^{-5}	0.7872	0.7610	0.0261
	$h(x_0)$	0.4348	0.2713	0.2714	2.1157×10^{-5}	0.4438	0.4258	0.0180
	$ah\,(x_0)$	0.5706	0.3716	0.3717	0.0001	0.5865	0.5546	0.0319
500	α	3.4097	0.7049	0.7049	0.0001	3.4271	3.3924	0.0347
	β	0.3889	0.0276	0.0286	8.5471×10^{-6}	0.3947	0.3832	0.0115
	θ	1.1134	0.3041	0.3042	0.0002	1.1384	1.0885	0.0498
	$S(x_0)$	0.7743	0.0201	0.0207	1.6027×10^{-5}	0.7822	0.7665	0.0157
	$h(x_0)$	0.4349	0.2711	0.2711	8.3004×10^{-6}	0.4406	0.4293	0.0113
	$ah\,(x_0)$	0.5708	0.3713	0.3714	2.6015×10^{-5}	0.5808	0.5608	0.0200

Table 4.
ML averages, relative absolute biases, relative errors, variances of ML estimates, 95%confidence intervals of the parameters from DAPTW-E distribution for based on Type-II censoring

$$(\alpha = 1.3,\quad \beta = 0.3,\qquad \theta = 2,\ x_0 = 1 \text{ and } \quad NR = 1000)$$

n	r	φ	Average	RAB	RE	Variance	UL	LL	Length
30	21	α	2.3667	0.8205	0.8268	0.0174	2.6254	2.1080	0.5174
		β	0.3200	0.0667	0.0817	0.0002	0.3477	0.2923	0.0554
		θ	0.9563	0.5218	0.5293	0.0314	1.3038	0.6089	0.6949
		$S(x_0)$	0.7617	0.1515	0.1595	0.0020	0.8494	0.6740	0.1754
		$h(x_0)$	0.3227	0.1870	0.1999	0.0008	0.3777	0.2677	0.1100
		$ah\ (x_0)$	0.3904	0.2280	0.2398	0.0014	0.4641	0.3166	0.1475
	30	α	2.2667	0.7436	0.7468	0.0082	2.4437	2.0897	0.3540
		β	0.3007	0.0024	0.0435	0.0001	0.3262	0.2752	0.0510
		θ	1.3039	0.3480	0.3549	0.0195	1.5773	1.0305	0.5468
		$S(x_0)$	0.8388	0.0656	0.0743	0.0010	0.9004	0.7773	0.1230
		$h(x_0)$	0.3291	0.1709	0.1739	0.0002	0.3543	0.3038	0.0505
		$ah\ (x_0)$	0.3993	0.2104	0.2137	0.0004	0.4367	0.3618	0.0750
60	42	α	2.3561	0.8124	0.8151	0.0073	2.5231	2.1080	0.3339
		λ	0.3191	0.0636	0.0714	0.0001	0.3382	0.2923	0.0382
		θ	0.9660	0.5170	0.5202	0.0135	1.1935	0.6089	0.4550
		$S(x_0)$	0.7650	0.1478	0.1517	0.0009	0.8250	0.6740	0.1200
		$h(x_0)$	0.3269	0.1763	0.1858	0.0005	0.3725	0.2677	0.0911
		$ah\ (x_0)$	0.3963	0.2162	0.2233	0.0008	0.4518	0.3166	0.1109
	60	α	2.3563	0.7356	0.7366	0.0026	2.3565	2.1560	0.2005
		β	0.2993	0.0023	0.0293	7.6732×10^{-5}	0.3165	0.2821	0.03434
		θ	1.3218	0.3391	0.3419	0.0077	1.4944	1.1493	0.3451
		$S(x_0)$	0.8433	0.0606	0.0647	0.0004	0.8827	0.8038	0.0789
		$h(x_0)$	0.3297	0.1692	0.1714	0.0001	0.3509	0.3085	0.0424
		$ah\ (x_0)$	0.4002	0.2086	0.2109	0.0003	0.4317	0.3692	0.0620
100	70	α	2.3514	0.8088	0.8099	0.0029	2.4567	2.2461	0.2106
		λ	0.3188	0.0627	0.0667	4.6929×10^{-5}	0.3322	0.3054	0.0269
		θ	0.9721	0.5140	0.5145	0.0061	1.1251	0.8190	0.3060
		$S(x_0)$	0.7667	0.1460	0.1477	0.0004	0.8066	0.7267	0.0799
		$h(x_0)$	0.3295	0.1698	0.1708	4.1580×10^{-5}	0.3438	0.3151	0.0287
		$ah\ (x_0)$	0.3998	0.2094	0.2104	0.0001	0.4198	0.3797	0.0401
	100	α	2.2525	0.7327	0.7328	0.0003	2.2875	2.2175	0.0699
		β	0.2993	0.0022	0.0138	1.6777×10^{-5}	0.3074	0.2913	0.0161
		θ	1.3201	0.3348	0.3404	0.0013	1.3896	1.2507	0.1389
		$S(x_0)$	0.8433	0.0605	0.0612	0.0001	0.8589	0.8277	0.0312
		$h(x_0)$	0.3311	0.1658	0.1665	3.4402×10^{-5}	0.3426	0.3196	0.0230
		$ah\ (x_0)$	0.4021	0.2047	0.2055	0.0001	0.4193	0.3850	0.0343

Table 5.
ML averages, relative absolute biases, relative errors, variances of ML estimates, 95%confidence intervals of the parameters from DAPTW-E distribution for based on Type-II censoring
$(\alpha = 1.6,\ \ \beta = 0.3,\ \ \theta = 2,\ x_0 = 1 \text{ and } \ NR = 1000)$

n	r	φ	Average	RAB	RE	Variance	UL	LL	Length
30	21	α	2.8893	0.8058	0.8133	0.0311	3.2350	2.5437	0.6914
		β	0.3175	0.0605	0.0798	0.0002	0.3488	0.2875	0.0612
		θ	1.0037	0.5058	0.5148	0.0368	1.3644	0.6125	0.7518
		$S(x_0)$	0.7922	0.1318	0.1423	0.0024	0.8832	0.6922	0.1910
		$h(x_0)$	0.3123	0.1808	0.2012	0.0011	0.3721	0.2422	0.1299
		$ah\ (x_0)$	0.3747	0.2169	0.2359	0.0019	0.4534	0.2826	0.1708
	30	α	2.7637	0.7273	0.7273	0.0098	2.9576	2.5698	0.3879
		β	0.2986	0.0046	0.0434	0.0001	0.3240	0.2732	0.0507
		θ	1.3303	0.3349	0.3415	0.0179	1.5927	1.0678	0.5248
		$S(x_0)$	0.8582	0.0541	0.0631	0.0009	0.9159	0.8005	0.1154
		$h(x_0)$	0.3092	0.1754	0.1800	0.0002	0.3387	0.2796	0.0591
		$ah\ (x_0)$	0.3701	0.2124	0.2172	0.0005	0.4120	0.3282	0.0837
60	42	α	2.8624	0.7890	0.7910	0.0081	3.0388	2.6859	0.3529
		β	0.3165	0.0552	0.0627	0.0001	0.3341	0.2990	0.0350
		θ	1.0142	0.4929	0.4956	0.0109	1.2186	0.8098	0.4088
		$S(x_0)$	0.7945	0.1243	0.1276	0.0007	0.8457	0.7433	0.1023
		$h(x_0)$	0.3150	0.1599	0.1692	0.0004	0.3559	0.2741	0.0819
		$ah\ (x_0)$	0.3787	0.1942	0.2005	0.0005	0.4249	0.3324	0.0925
	60	α	2.7794	0.7371	0.7388	0.0064	2.9367	2.6220	0.3147
		β	0.3017	0.0057	0.0380	0.0001	0.3238	0.2796	0.0441
		θ	1.3107	0.3447	0.3484	0.0105	1.5115	1.1099	0.4016
		$S(x_0)$	0.8540	0.0587	0.0633	0.0005	0.8957	0.8122	0.0835
		$h(x_0)$	0.3152	0.1594	0.1639	0.0002	0.3431	0.2873	0.0558
		$ah\ (x_0)$	0.3788	0.1939	0.1988	0.0004	0.4194	0.3382	0.0811
100	70	α	2.8674	0.7789	0.7809	0.0080	3.0217	2.6707	0.3510
		β	0.3172	0.0485	0.0583	9.3661×10^{-5}	0.3335	0.2956	0.0379
		θ	1.0052	0.4842	0.4872	0.0120	1.2461	0.8173	0.4288
		$S(x_0)$	0.7925	0.1196	0.1232	0.0007	0.8516	0.7460	0.1056
		$h(x_0)$	0.3156	0.1617	0.1625	3.5707×10^{-5}	0.3260	0.3026	0.0234
		$ah\ (x_0)$	0.3792	0.1969	0.1978	0.0001	0.3943	0.3605	0.0338
	100	α	2.7624	0.7265	0.7269	0.0015	2.8386	2.6861	0.1525
		β	0.3008	0.0027	0.0207	3.7970×10^{-5}	0.8129	0.2887	0.0242
		θ	1.3346	0.3327	0.3335	0.0020	1.4231	1.2461	0.1769
		$S(x_0)$	0.8584	0.0539	0.0550	0.0001	0.8776	0.8391	0.0385
		$h(x_0)$	0.3164	0.1560	0.1579	8.2545×10^{-5}	0.3342	0.2986	0.0356
		$ah\ (x_0)$	0.3805	0.1902	0.1923	0.0001	0.4065	0.3546	0.0519

From Tables 2-5, the following observations can be noted,

- The ML averages of the estimates perform better when the sample size n increases.
- The REs, RABs, and variances of the ML estimates decrease in most cases when the sample size n increases. Also, the lengths of the confidence intervals get shorter when the sample size increases.
- The REs, RABs, and variances of the ML estimates decrease when the level of censoring decreases. The lengths of the confidence intervals become narrower when the sample size increases.

5.2 Applications

This subsection aims to demonstrate empirical importance of the proposed DAPTW-E distribution through analyzing two real lifetime data sets.

The fitted model is compared using some criteria, namely, *Akaike Information Criterion* (AIC), *Akaike Information Criterion with correction* (AICC) and *Bayesian Information Criterion* (BIC) with some distributions such as *discrete Weibull* (DW) introduced by Nakagawa and Osaki (1975), *discrete Marshall-Olkin Weibull* (DMOW) proposed by Opone *et al.* (2021), *discrete Marshall-Olkin generalized exponential* (DMOGE) presented by Almetwally *et al.* (2020), *discrete Zubair Weibull* (DZW) derived by AL-Kashlan *et al* (2023)*, discrete alpha power Weibull* (DAPW) obtained by EL- Helbawy *et al.* (2022) and *discrete alpha power Exponential* (DAPE) which is a sub model from DAPW introduced by EL-Helbawy *et al.* (2022).

The best distribution corresponds to the lowest values of AIC, AICC and BIC, also the highest p-value,

where AIC = $-2\ log\ L + 2k$, BIC = $-2\ log\ L + k\ log\ n$ and AICC = AIC $+\frac{2k(k+1)}{n-k-1}$, where k is the number of the parameters and n is the sample size and L is the maximized value of the likelihood function for the estimated model. Tables 6 and 7 display the values of p-value, AIC, BIC and AICC for the two data sets.

Application 1:

The first application represents the failure times of 50 devices (in weeks) put on a certain life test taken from Bodhisuwan and Sangpoom (2016). The data are: 0.1, 0.2, 1, 1, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86 and 86.

Application 2:

The second data set is lifetime data which gives the failure times for 15 electronic components in an acceleration lifetime test provided by Lawless (2003). The data are: 1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23, 30.6, 37.3, 46.3, 53.9, 59.8 and 66.2.

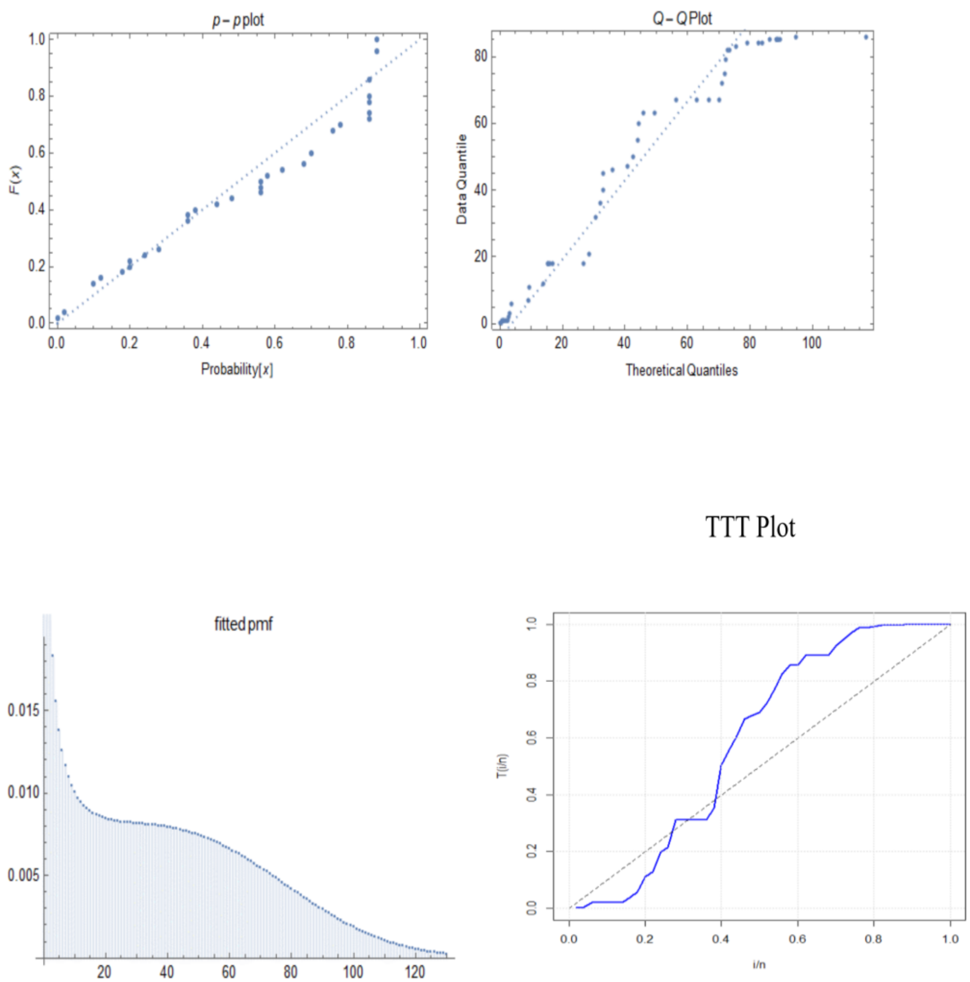


Figure 4: The PP-plot, QQ-plot, fitted pdf and TTT-plot of the DAPW-E distribution for the first data set

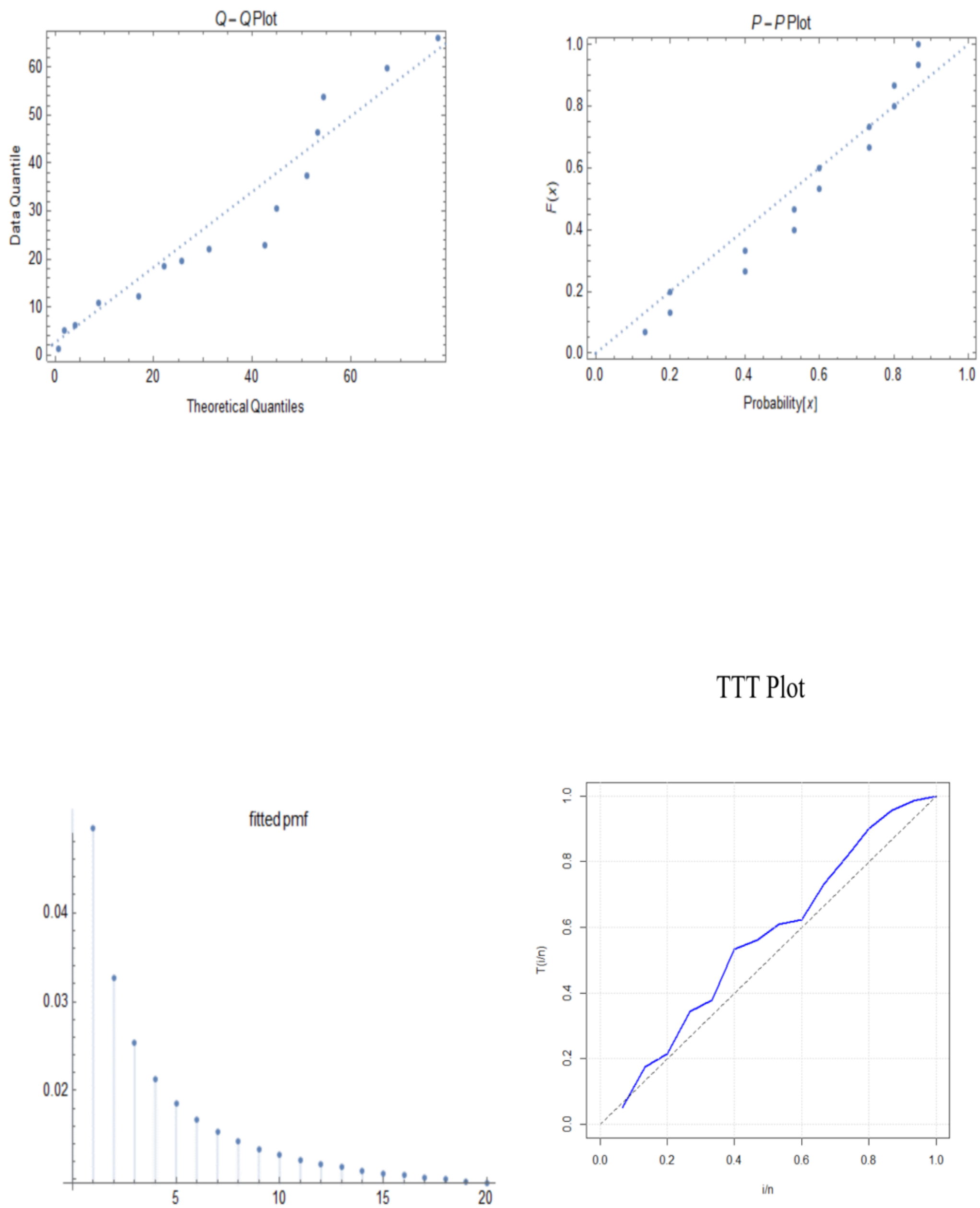


Figure 5: The PP-plot, QQ-plot, fitted pdf and TTT-plot of the DAPW-E distribution for the second data set

Table 6.
Parameter estimates with their corresponding standard errors and goodness of fit for various models fitted for the first data

Model	parameter	Estimate	SE	*K-S*	P-value	-2$\mathscr{L}$	AIC	BIC	AICC
DNAPTW	α β θ	12.0977 0.1100 0.1342	0.9717 1.0635 1.0633	0.14	0.7016	482.747	488.747	494.483	489.268
DAPW	α p β	6.2094 1.0993 0.2396	1.0164 1.0558 1.0625	0.24	0.1086	745.099	751.099	756.836	751.621
DAPE	α β	8.7497 0.0495	0.9970 1.0639	0.26	0.0640	956.23	956.23	960.054	956.485
DMOW	α θ γ	1.6671 0.9914 1.2678	1.0514 1.0567 1.0545	0.18	0.3829	484.392	490.392	496.128	492.341
DMOGE	α θ λ	10.2403 0.9089 5.1447	0.9856 1.0573 1.0246	0.26	0.0651	782.908	788.908	794.644	789.43
DZW	α θ γ	0.7294 0.7039 0.8861	1.0587 1.0589 1.0575	0.2	0.2623	488.483	494.483	500.219	495.005
DW	α β	0.3983 0.1790	1.0612 1.0629	0.22	0.1720	614.456	618.456	622.28	618.711

Table 7.
Parameter estimates with their corresponding standard errors and goodness of fit for various models fitted for the second data

Model	parameter	Estimate	SE	*K-S*	P-value	-2$\mathscr{L}$	AIC	BIC	AICC
DAPTW-E	α β θ	2.5493 0.1056 0.1902	1.4958 1.5402 1.5387	0.2	0.9383	142.904	148.904	151.029	151.086
DAPW	α p β	5.4988 1.0379 0.2702	1.4426 1.5233 1.5372	0.4167	0.0755	210.275	216.275	218.399	218.456
DAPE	α β	1.9226 0.0657	1.5072 1.5410	0.4667	0.0653	235.571	239.571	240.988	240.571
DMOW	α θ γ	2.7667 0.9089 1.0735	1.4918 1.5256 1.5226	0.2667	0.6781	151.151	157.151	159.275	159.333
DMOGE	α θ λ	4.7255 0.8447 0.5010	1.4564 1.5268 1.5330	0.4	0.1844	178.25	184.25	186.374	186.432
DZW	α θ γ	8.3798 0.6776 0.6023	1.3916 1.5298 1.5312	0.25	0.7515	144.177	150.177	152.302	152.359
DW	α β	0.4137 0.2841	1.5346 1.5370	0.3333	0.3855	171.05	175.05	176.466	176.05

Figures 4 and 5 present the PP and QQ plots, fitted pdf and TTT plot for the two real data sets, which indicates that the DAPW-E distribution provides better fit to the data sets. The TTT plot for the first real data set which is displayed in Figure 4 provides evidence that the first data set possesses bathtub hrf, but the TTT plot of the second real data set in Figure 5 indicates that the hrf is decreasing function.

Regarding Tables 6 and 7, it is clear that the DATW-E, DAPW, DAPE, DMOW, DMOGE, DZW and DW distributions perform quite well for analyzing the two data sets. However, the DATW-E distribution is the best distribution among all the tested distributions; it has smallest values of $-2lnL$, AIC, BIC, CAIc, lowest *K–S* values and highest p-values.

6. Conclusion

In this paper, a family of discrete distributions is proposed. Generalizations of discrete Lindley, discrete Rayleigh and discrete exponential, are obtained using this family. Also, many other discrete distributions can be obtained as sub models. As a particular case, discrete alpha power transformed Weibull-exponential distribution is introduced. Some of its properties are studied. The ML estimators for the model parameters are derived. The discrete alpha power transformed Weibull- exponential distribution appears to be more suitable for modeling real data sets and is a better alternative to some distributions. We wish the proposed model is applied to a wider range of applications in medicine, engineering and other fields of research fields.

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