MATHEMATICS

Collapse of Air Bubbles as Oxygen Provider in Wastewater Aeration Tank

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ABSTRACT

This paper aims to investigate the role of surrounding parameters on the collapsing of convective air bubbles in the biological stage of wastewater treatment. The diffusion field is assumed to be quasi-static. The mathematical model describing the current problem consists of diffusion, Fick's, and Laplace equations. The system is solved analytically to obtain the evolution formulae of both the collapsing bubble radius and concentration distribution around the collapsing bubble in terms of surrounding parameters. The validity of this model is examined by applying the resultant evolution equation to fit earlier experimental data of the collapsing of an air bubble in water, and good agreement with this experimental data is achieved. The results show that increasing values of the diffusion coefficient, initial concentration difference, wastewater temperature, and initial void fraction result in a decrease in the collapsing time of the air bubble in wastewater whereas increasing values of the surface tension result in an increase of collapsing time. In addition, the collapsing of the air bubble is sensitive to small changes in the values of the diffusivity coefficient and initial concentration difference which is affected by the rate of oxygen consumption by the microorganisms during the biological stage of wastewater treatment. On the contrary, the collapsing process is slightly affected by changes in the surface tension values.
Introduction

Polluted water comes in various types, and how it is treated depends on the type and features of the wastewater. For instance: Black water and grey water are two categories for domestically polluted water. Toilets or ewage are the sources of black water. Showers, sinks, and floor washing all produce gray water (FAO, 2003).

Any water that has been impacted by human activity is considered to be polluted. Water that has been used for household, mechanical, commercial, or agricultural purposes, surface runoff or storm water, and any sewer input or sewer penetration is considered polluted water. Due to global water shortage challenges, it is essential to consider non-conventional water sources to meet the growing demand for freshwater. Polluted water is seen as a potential alternative to deal with the shortage of water supply that results from a variety of factors, such as population growth (Noori et al., 2014). Because of there is a wide variety of wastewater sources, which can include both organic and inorganic materials, it is necessary to periodically monitor the water's state in order to assess any potential risks to the environment as a whole (FAO, 2003). To protect the environment, water resources, and the general public's health, wastewater recycling must be done satisfactorily.

Studying wastewater treatment attracted and still do many authors, of these studies, Chang et al. studied the treatment performance of integrated vertical-flow constructed wetland plots

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**Nomenclature**

**Roman Symbols:**
- $C$ [mol/m$^3$]: Gas concentration
- $\Delta C_0$ [mol/m$^3$]: Initial supersaturation
- $C_0$ [mol/m$^3$]: initial gas concentration at $t = t_0$
- $C_R$ [mol/m$^3$]: Instant gas concentration at the bubble interface
- $C_{\infty}$ [mol/m$^3$]: Final gas concentration at $t = t_f$
- $D_L$ [m$^2$/s]: Dissolved gas Diffusion coefficient
- $P$ [N/m$^2$]: Pressure
- $P_B$ [N/m$^2$]: Bubble Pressure
- $P_\infty$ [N/m$^2$]: Ambient Pressure at a great distance from the bubble interface
- $r$ [m]: The distance from the origin of the bubble
- $R$ [m]: Bubble wall radius
- $\dot{R}$ [m/s]: Instantaneous bubble wall velocity
- $R/(\text{mol.K})$: General gas constant
- $T$ [K]: Temperature
- $t$ [s]: Elapsed time

**Greek symbols:**
- $\alpha$: Constant defined by Eq. (18)
- $\sigma$ [N/m]: Surface tension
- $\varphi_0$: Void fraction defined by Eq. (17)

**Subscripts:**
- 0: Initial value
- B: Gas bubble
- f: Final value
- $\infty$: At a great distance from the bubble
for domestic wastewater (Chang, et al. 2012). Abou-Elela et al. studied the municipal wastewater treatment in horizontal and vertical flows constructed wetlands (Abou-Elela et al., 2013). Rajasulochana and Preethy presented a review on the efficiency of various techniques in the treatment of waste and sewage water (Rajasulochana and Preethy, 2016). Badejo et al., investigated the municipal wastewater management using Vetiveria zizanioides planted in vertical flow constructed wetland (Badejo et al., 2018).

A bubble is a spherical hollow filled with air or gas that is present in a liquid or a solidified liquid such as glass. Spatiotemporal analysis of bubble dynamics in liquids with dissolved gases arises in several disciplines of study, the development, collapse, and oscillation of gas bubbles are significant. Such disciplines include geophysics, such as the study of volcanic explosions caused by magma, chemical engineering such as plastics, sonochemistry, and biological applications. (Srinivasan et al., 2002; Wienk, 2009; Mohammadein and Mohamed, 2014; Mohammadein and Mohamed, 2016; Mohamed and Mohammadein, 2019; Abu-Bakr et al., 2023; Mohamed et al., 2023).

Cavitation occurs when the pressure on a liquid drops, creating pockets or bubbles in the liquid (Chappell and Payne, 2006). Bubble dynamics not only studied in laminar two-phase flow, but also several studies investigated bubble dynamics in turbulent flow systems for example (Abu-Bakr, 2019). Abu-nab et. al. examined the bubble dynamics in a modern attractive type of fluids which is nanofluid (Abu-nab et al., 2023) to address the impact of the addition of nanoparticles to base fluids on the dynamics of vapour bubbles. When the pressure is raised, the bubbles collapse with a powerful "local" force that can harm nearby objects. Of several studies on bubble collapse, Shimiya and Yano experimentally studied diffusion-controlled collapse and growth of an air bubble entrained in water and in wheat flour particles (Shimiya and Yano, 1987). They show that the bubble collapse is controlled by undersaturated dissolved air. Sobhy represented different studies in his MSc. thesis to investigate the dynamics of the vapour bubble collapse in superheated liquids (Sobhy, 2007). The main results of this study showed that: the collapse of the bubble is proportional with superheating, initial bubble velocity, initial bubble radius,
and thermal diffusivity; the collapse of bubble radius increases with the decreasing of initial void fraction; the bubble collapse is very sensitive to the superheating; the collapse of vapour bubble takes a few time in comparing with the time of growth; and the collapse of bubble radius decreases with the increasing of the mixture viscosity.

Wastewater treatment is an important application of bubble dynamics. Wastewater treatment is carried out in three stages: the skimming stage, the sedimentation stage, the final stage that we are dealing with is the treatment stage with oxygen-laden bubbles (Biological stage) at which air bubbles diffuse to aeration tanks throughout fine air bubble diffuser (See Figure 1) to provide microorganisms in the biological stage of cleaning wastewater by a sufficient amount of oxygen to consume the organic components (Zhou et al., 2013). The oxygen-laden gas bubbles are what the microorganisms feed on inside the smart sludge. They have a suitable atmosphere of food, which is the sludge and air. In other words, the oxygen provided by gas bubbles, combined with the food source, sewage, allows the bacteria to produce enzymes which help break down the waste so that it can settle in the secondary clarifiers or be filtered by membranes. The aeration system in a wastewater or sewage treatment plant consumes an average of 50 to 70 percent of the energy of the entire plant (Spellman, 2013), which maximizes the needs to exert more efforts to find optimal bubble sizes and understanding dynamics as oxygen provider in aeration tanks to increase the efficiency and reduce the cost which represents the main goal of this study and our future studies in this field.

This paper is devoted to depicting the role of bubble collapse as oxygen-carrier to microorganisms used in the biological treatment of wastewater. The obtained results are presented to the specialists as helpful knowledge in designing biological treatment tanks for wastewater treatment.

The formulation of this study consists of representing the problem of collapsing of an air bubble in aeration tank by a system of differential equations, which is solved analytically for the given boundary, initial and final values to obtain equations of evolution of bubble radius and concentration distribution of oxygen around the collapsing bubble. The validity of the model results is checked by applying
the equation representing the bubble radius evolution to fit the experimental data of Shimiya and Yano (Shimiya and Yano, 1987) that conducted to express the collapse of an air bubble in water. Then a numerical implementation is proceeded by suitable initial and boundary values for the parameters which affect the collapsing process. The paper is entailed by output graphs and a discussion of the obtained results.

**Mathematical Modelling and Analysis**

Consider an air bubble created by an air bubble diffuser in an aeration tank to provide oxygen to the microorganisms that are used in organic materials degradation in the biological stage of wastewater treatment plant. For small enough air bubbles, spherical symmetry of the bubble shape is considered. The bubble begins to collapse by the action of oxygen consumption by the microorganisms. The initial bubble radius is \( R_0 \) while \( R_f \) represents the final radius of the bubble at complete collapse. We devote our efforts to represent a mathematical relation that correlates the evolution of the bubble radius to the elapsed collapse time in combination with the effect of surrounding physical parameters such as surface tension, ambient pressure, initial concentration difference and temperature for the liquid phase (wastewater) and bubble radius boundaries \( R_0 \) and \( R_f \) for the gas phase (Air bubble).

The mathematical model representing the collapse of a convective air bubble in a quasi-static diffusion field consists of the following main equations which represent the diffusion equation (Mohammadein and Mohamed, 2016), Fick’s equation which represents the mass balance at the bubble interface, and pressure balance equation at the bubble interface respectively (Srinivasan et al., 2002; Mohammadein and Mohamed, 2010):

\[
\frac{R^2 \dot{R}}{r^2} \frac{dC}{dr} = D_L \left( \frac{d^2C}{dr^2} + \frac{2}{r} \frac{dC}{dr} \right),
\]

\[
\frac{1}{RT} \frac{dc}{dt} (V_B P_B) = A_B D_L \left( \frac{dc}{dr} \right)_{r=R},
\]

\[
P_B = P_\infty + \frac{2\sigma_B}{R}.
\]

With the following initial, final and boundary conditions

\[
C(r) = \begin{cases} 
C_R & \text{at } r = R \text{ and } t = t_f, \\
C_0 & \text{at } r = R_0 \text{ and } t = t_0, \\
C_\infty & \text{at } r = R_f \text{ and } t = t_f.
\end{cases}
\]

If the air bubble is small enough to overcome deformation in its spherical
shape, then $A_B = 4\pi R^2$ and $V_B = \frac{4}{3}\pi R^3$ and consequently, equation (2) after differentiation and simplifications can be reduced to the following form

$$\frac{\dot{R}}{D_L 2^T} \left( P_m + \frac{4\sigma_L}{3R} \right) = \left( \frac{dC}{dt} \right)_{r=R}. \tag{5}$$

That is, the concentration gradient at the bubble interface is simplified to the following form

$$\left( \frac{dC}{dt} \right)_{r=R} = \frac{3RP_m + 4\sigma_L}{3D_L 2^T R} \dot{R}. \tag{6}$$

The following equation represents the general solution of equation (1)

$$C(r) = k_1 \frac{D_L}{R^2 R} e^{-\left( \frac{R^2}{D_L} \right)^{r_1^2}} + k_2. \tag{7}$$

The constants $k_1$ and $k_2$ can be obtained by applying the boundary conditions given by equation (4), consequently, equation (1) has the following solution form

$$C(r) = \left[ (C_0 - C_R) e^{\left( \frac{R^2}{D_L} \right) - R} \left( C_0 e^{\left( \frac{R^2}{D_L} \right) R} - C_R e^{\left( \frac{R^2}{D_L} \right) R} \right) \right] \sqrt{\left( \frac{R^2}{D_L} \right)} e^{\left( \frac{R^2}{D_L} \right) R} \tag{8}$$

Equation (8) derives an alternative form for concentration gradient at the bubble interface as follows

$$\left( \frac{dC}{dt} \right)_{r=R} = \frac{(C_0 - C_R) \dot{R}}{D_L} \left( e^{\left( \frac{R^2}{D_L} \right) R} - e^{\left( \frac{R^2}{D_L} \right) R} \right). \tag{9}$$

From equations (6) and (9), we obtain

$$\frac{(C_0 - C_R) \dot{R}}{e^{\left( \frac{R^2}{D_L} \right) R}} = \frac{3RP_m + 4\sigma_L}{3R^T R} \dot{R}. \tag{10}$$

From equation (10), we obtain

$$3RT (C_0 - C_R) R = (3RP_m + 4\sigma_L) \left( R \dot{R} \left( 1 - \frac{R}{R_0} \right) - 1 \right). \tag{14}$$

At $t = t_f$, $R = R_f$, $C_R = C_\infty$, and $C_\infty - C_0 = \Delta C_0$, the bubble collapse profile to have the following parabolic form (Mohammadein et al., 2017).

$$R(t) = \sqrt{R_0^2 - 2\alpha D_L t}, \tag{12}$$

where $\alpha > 0$ for bubble collapse, then the bubble growth rate is given by

$$\dot{R} = -\frac{\alpha D_L}{R}. \tag{13}$$

Substituting equation (13) into equation (11), we obtain

$$3RT (C_0 - C_R) R = (3RP_m + 4\sigma_L) \left( R \dot{R} \left( 1 - \frac{R}{R_0} \right) - 1 \right). \tag{14}$$

Using final conditions in equations (15) into equation (14), we obtain
Substituting $R_f/R_0 = \varphi_0^{1/3}$, where $0 < \varphi_0 < 1$ into equation (18), we finally obtain

$$1 - \frac{3RT\Delta C_0 R_f}{3R_f P_\infty + 4\sigma_L} = e^{-\alpha \left(1 - \varphi_0^{3/8}\right)}.$$  

(17)

Applying necessary simplifications, we can express the constant $\alpha$ as follows

$$\alpha = \frac{\ln \left(1 - \frac{3RT\Delta C_0 R_f}{3R_f P_\infty + 4\sigma_L}\right)}{\varphi_0^{3/8} - 1}.$$  

(18)

In view of equation (12), which represents the parabolic profile of the bubble collapse, the parameter $\alpha$ controls the reduction rate in bubble radius throughout the collapsing process, the higher the values of the parameter $\alpha$, the higher the reduction rates of the bubble radius, hence faster collapsing rates. Equation (18) shows that the value of $\alpha$ depends on the values of the physical parameters and boundary values which it depends on such as surface tension, temperature, ambient pressure and initial pressure difference for wastewater (liquid phase); and initial and final bubble radius boundaries (gas phase). Inserting the expression of $\alpha$ into equation (12) we obtain the following formula that calculates the instantaneous bubble radius at any instant of collapse time in terms of surrounding parameters’ values

$$R(t) = R_0^2 - \sqrt{\frac{2D_L \ln \left(1 - \frac{3RT\Delta C_0 R_f}{3R_f P_\infty + 4\sigma_L}\right)}{\varphi_0^{3/8} - 1} t}.$$  

(19)

By using equation (13), one gets the following reduced form of the concentration distribution around the collapsing air bubble in wastewater treatment, where the constant $\alpha$ is given by equation (18)

$$C(r) = \left[(C_0 - C_R) e^{\frac{\alpha R}{r}} - \left(C_0 e^{\alpha r} - C_R e^{\frac{\alpha R}{r}} \right)\right] \left(e^{\frac{\alpha R}{r}} - e^{\alpha r}\right).$$  

(20)

Equation (19) gives the instantaneous bubble radius as a function of time with the surrounding physical and initial parameters, where $D_L$ represents the oxygen diffusivity in water, $\Delta C_0$ is the initial concentration difference, $P_\infty$ is the surrounding pressure away from the bubble, $\sigma_L$ is the surface tension $\varphi_0$ is the initial void fraction.
The problem of collapsing of an air bubble in wastewater treatment is modeled by the system of equations (1-3), this system is solved analytically and the boundary and initial values equation (4) are applied to obtain the evolution of the bubble radius and concentration distribution around the collapsing air bubble equations (19) and (20).

Numerical implementation to the results for suitable values for the physical and initial values for air diffusion to microorganisms in biological treatment of wastewater. The used values are indicated in Table (1):

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_0 ) [m]</td>
<td>( 5.0 \times 10^{-3} )</td>
<td>( \rho_0 ) [N/m²]</td>
<td>101325</td>
</tr>
<tr>
<td>( R_f ) [m]</td>
<td>( 5.0 \times 10^{-4} )</td>
<td>( \sigma ) [N/m]**</td>
<td>0.0549</td>
</tr>
<tr>
<td>( D_L ) [m²/s]</td>
<td>( 2.42 \times 10^{-9} )</td>
<td>( T ) [K]</td>
<td>310</td>
</tr>
<tr>
<td>( \mathcal{R} ) [(J/(mol.K)]*</td>
<td>8.314472</td>
<td>( \varphi_0 )</td>
<td>0.001</td>
</tr>
<tr>
<td>( \Delta C_0 ) [mol/m³]</td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* (Lide, 2005) ** (Sirdhar and Rami, 1984)

By using the Mathematica program, the following series of graphs are obtained which depict the effect of surrounding parameters on the bubble collapse in wastewater treatment.

Fig. (2): The collapse behaviour of air bubble in aeration tank at different values of the diffusivity coefficient \( D_L \).

Fig. (3): The collapse behaviour of air bubble in aeration tank at different values of the initial supersaturation \( \Delta C_0 \).
Fig. (4): The collapse behaviour of air bubble in aeration tank at different values of waste water temperature $T$.

Fig. (5): The collapse behaviour of air bubble in aeration tank at different values of waste water surface tension $\sigma_L$.

Fig. (6): The collapse behaviour of air bubble in aeration tank at different values of the initial void fraction $\varphi_0$.

Fig. (7): The collapse behaviour of air bubble in aeration tank at different values of the ambient pressure $P_\infty$.

Fig. (8): Comparison between the current theoretical model with the experimental data of (Shimiya and Yano, 1987)
Figure (2) shows the collapse behavior of the air bubble in wastewater at three different values of the diffusion coefficient $D_L$, it is found that the bubble collapse is inversely proportional with the diffusion coefficient. Figure (3) shows the collapse behavior of the air bubble in wastewater at three different values of the initial concentration difference "driving concentration force" $\Delta C_f$ it is found that the bubble collapse is inversely proportional with the initial concentration difference. Figure (4) shows the collapse behavior of the air bubble in wastewater at three different values of the wastewater temperature $T$, it is found that the bubble collapse is inversely proportional with the wastewater temperature. Figure (5) shows the collapse behavior of the air bubble in wastewater at three different values bubble-biofluid surface tension $\sigma_L$ it is found that the bubble collapse is directly proportional with the wastewater-air surface tension. Figure 6 shows the collapse behavior of the air bubble in wastewater at three different values of the initial void fraction $\varphi_0$ it is found that the bubble collapse is inversely proportional with the initial void fraction. We noticed that the increment in the diffusion coefficient, initial concentration difference, wastewater temperature and initial void fraction results in a decrement in the collapsing time of the air bubble in wastewater whereas the increment in the surface tension which consequently increases the initial pressure difference results in an increase of collapsing time. Figure (7) shows the collapse behavior of the air bubble in wastewater at three different values wastewater ambient pressure $P_\infty$, it is found that the bubble collapse is directly proportional with the wastewater ambient pressure.

Figures (2-7) illustrates also that the collapsing of the air bubble is sensitive to small changes in the values of the diffusivity coefficient and initial concentration difference that affected by the rate of oxygen consumption by the microorganisms during the biological stage of the wastewater treatment. On the contrary, the collapsing process is slightly affected by changes in the surface tension values.

Furthermore, Figure (8) depicts a comparison between the current theoretical model with the experimental data of (Shimiya and Yano, 1987) at following parameters’ values which extracted from the Shimiya and Yano study:
\( R_0 = 4.98 \times 10^{-5} \text{ m} \),
\( R_f = 7.81 \times 10^{-6} \text{ m} \),
\( D = 2.10 \times 10^{-9} \text{ m}^2/\text{s} \),
\( \sigma = 7.32 \times 10^{-2} \text{ N/m} \),
\( T = 290.15 \text{ K} \),
and \( P_\infty = 101300 \text{ Pa} \). Our theoretical model shows good agreement of the experimental data when initial concentration difference is \( 46 \text{ mol/m}^3 \).

This shows the validity of the current theoretical model to predict the evolution of the bubble radius during bubble collapse.

**Conclusion**

The results of the current study reveal the following concluded remarks:

- This theoretical model is in good agreement with the experimental data of (Shimiya and Yano, 1987), this shows the validity of the current model to predict the evolution of the bubble radius during bubble collapse.

- The collapse of air bubbles in aeration tanks is a result of the consumption of oxygen by microorganisms that degrade the organic materials dissolved in wastewater.

- The increment in the diffusion coefficient, initial concentration difference, wastewater temperature, and initial void fraction results in decrement in the collapsing time of the air bubble in wastewater, whereas the increment in the surface tension which consequently increases the initial pressure difference results in increasing of collapsing time.

- The collapsing of the air bubble is sensitive to small changes in the values of the diffusivity coefficient and initial concentration difference that affected by the rate of oxygen consumption by the microorganisms during the biological stage of the wastewater treatment. On the contrary, the collapsing process is slightly affected by changes in the surface tension values.

- This model as one of the earliest mathematical models that deals with the role of bubble dynamics in wastewater treatment, draws different directions to newer studies in this field by considering the effects of more physical properties of wastewater fluid and mechanisms of treatment process.

**References:**


Collapse of Air Bubbles as Oxygen Provider in Wastewater Aeration Tank


تقلص الفقاعات الهوائية كمصدر للأكسجين في خزان تهوية مياه الصرف الصحي
خالد جودة محمد، أحمد جمال الدين عبد الحميد، دينا عاطف البيضوني، شيماء عزت وحيدالدين

قسم الرياضيات – كلية العلوم – جامعة بنها
قسم مسوح المواد الطبيعية في النظم البيئية – معهد الدراسات والبحوث البيئية – جامعة مدينة السادات

يهدف البحث الحالي إلى دراسة دور العوامل المحيطة في تقلص فقاعات الهواء في المرحلة البيولوجية لمعالجة مياه الصرف الصحي على فرضية أن يكون مجال الانتشار شبه ثابت. يتكون النموذج الرياضي الذي يصف المشكلة الحالية من معادلات الانتشار وكيل وليك Lapalce و Fick. تم حل النظام تحليلياً للحصول على صيغ التطور لكل من نصف قطر الفقاعة المتقلصة وتوزيع التركيز حول الفقاعة المتقلصة تحت تأثير البارامترات المحيطة. أظهرت النتائج أن زيادة قيم عامل الانتشار وفرق التركيز الأولي ودرجة حرارة المياه العادية وجزء الفراغ الأولي يؤدي إلى انخفاض زمن تقلص فقاعة الهواء في مياه الصرف الصحي في حين أن زيادة قيم التوتر السطحي تؤدي إلى زيادة زمن التقلص. بالإضافة إلى ذلك، فإن تقلص فقاعة الهواء حساس للتغيرات الصغيرة في قيم عامل الانتشار وفرق التركيز الأولي الذي يؤثر بعدد استهلاك الأكسجين من قبل الكائنات الحية الدقيقة خلال المرحلة البيولوجية لمعالجة مياه الصرف الصحي. وعلى العكس من ذلك فإن عملية التقلص تتأثر قليلاً بالتغير في قيم التوتر السطحي.