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## **TRACKING CONTROL OF AN UNDERACTUATED AUTONOMOUS SHIP**

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### **ABSTRACT**

In this paper, we will construct a control that forces position and orientation of the underactuated autonomous ship moves according to a reference feasible trajectory. To achieve this objective, we use as a design tool of puts the Backstepping methodology and Lyapunov function. Experimental results are given to show the tracking performance. We will illustrate trajectories with time varying velocity (sinusoidal path). Then, we will test the tracking robustness in presence of drag forces disturbances.

### **KEY WORDS**

Underactuated Ship, Tracking Control, Backstepping Method, Lyapunov Theorem.

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## INTRODUCTION

The past few decades, control of robotic marine vehicles for autonomous navigation has become an intense research area. The interest in the field is widely motivated by the emerging applications such as defense and patrolling of coastal perimeters, naval system applications, harbor operations, marine biology,...etc.

Besides their various missions, marine vessels raise some challenges in control systems theory, because their dynamics often fall in the class of underactuated systems (i.e. Systems where the control vector has lower dimension than the configuration vector). This configuration is by far most common among the ships.

Several control approaches have been presented in the literature to solve the trajectory tracking control problems that require the design of control laws that force the vehicle to track a time parameterized reference (trajectory). Do and Pan [1] addressed the tracking problem of ships that are not actuated in the sway direction and the mass and damping matrices are not assumed to be diagonal and they used the Backstepping technique to design a controller that forces ships to globally track a reference trajectory. Repoulas and Papadopoulos [2], and Santhakumar and Aoskan [3] presented a tracking control algorithm for underactuated autonomous underwater vehicles (AUVs) moving on the horizontal plane (constant depth motion). But in (Repoulas and Papadopoulos [2]), the mass and damping matrices of the AUVs were assumed to be diagonal. However Santhakumar and Aoskan [3] studied the AUVs with non-zero off diagonal terms in the system matrices. In (Lefeber et. al. [4], Pettersen and Egeland [5] Pettersen and Nijmeijer [6], Pettersen and Nijmeijer [7] ), the yaw velocity was required to be nonzero. This restrictive assumption implies that a straight line cannot be tracked. Moreover, Do et. al. [8] succeed to remove the requirement that yaw velocity be nonzero.

Motivated by the variety approaches that have been proposed to solve the trajectory tracking control problem for underactuated autonomous vehicles, we propose in this paper the combined problem of trajectory planning and designing a controller to track a reference trajectory of a ship Repoulas and Papadopoulos [2]. The Backstepping technique of Krstić et. al. [9], that is recursive design methodology for non linear feedback control in order to develop control algorithm, is used. The present paper is composed of 5 sections: In section 2 the equations that describe the ship motion are recalled. Then, in section 3 the tracking control law is developed after planning the reference trajectories. The experimental results are presented in section 4 and in section 5, conclusion is drawn.

## THE SHIP MODEL

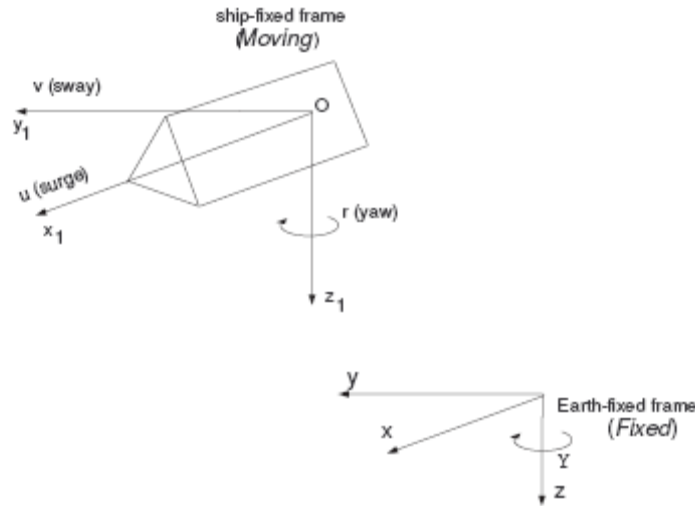
Figure 1 presents the two coordinate frames which we consider in our study, where  $x_1y_1z_1$  the Ship-fixed reference and the origin  $O_1$  is the center of gravity.  $xyz$  is the Earth-fixed reference frame. We consider an underactuated ship, that has two controls which provide surge force and yaw moment. The motion matrix model of the ship is described as Pettersen and Egeland [5]:

$$\begin{cases} M\dot{v} + Cv + Dv = \tau \\ \dot{\eta} = J(\eta)\tau \end{cases} \quad (1)$$

where:

- $\eta = [x, y, z]^T$  is the configuration vector that denotes position and orientation in earth-fixed coordinates,
- $v = [u, v, r]^T$  is the velocity vector, where  $u$  and  $v$  are the linear velocities respectively in surge and sway, and  $r$  is the angular velocity in yaw,
- $J(\eta)$  is the matrix that describes the ship rotations in the  $(x_1Oy_1)$  plan around  $(Oz_1)$  axe and it has full rank where its expression is presented by:

$$J(\eta) = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2)$$



**Fig.1.** Ship-fixed frame and earth-fixed frame.

Then, the equations describing dynamic characteristics of the ship were derived from Newtonian dynamics laws, with

$$M = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & -m_{22}v \\ 0 & 0 & m_{11}u \\ m_{22}v & -m_{11}u & 0 \end{bmatrix}, D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \quad (3)$$

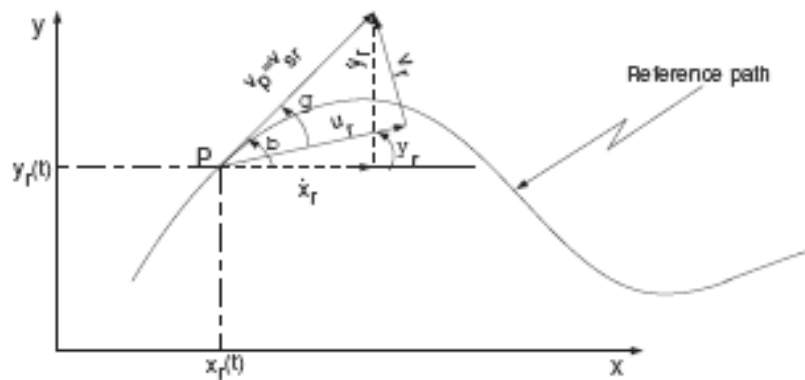
where  $M$  is the inertia matrix,  $D$  is the damping matrix,  $C$  represents centrifugal and Coriolis effects. The matrices  $M$  and  $D$  are constant and positive definite.

- $\tau = [\tau_u, 0, \tau_r]^T$ ,  $\tau_u$  is control force in surge and  $\tau_r$  is control moment in yaw.

Therefore, the developed complete mathematical model of the ship is expressed as:

$$\begin{cases} \dot{x} = uC_\psi - vS_\psi \\ \dot{y} = uS_\psi + vC_\psi \\ \dot{\psi} = r \\ \dot{u} = \frac{1}{m_{11}} [m_{22}vr - d_{11}u + \tau_u] \\ \dot{v} = \frac{-1}{m_{22}} [d_{22}v + m_{11}ur] \\ \dot{r} = \frac{1}{m_{33}} [(m_{11} - m_{22})vu - d_{33}r + \tau_r] \end{cases} \quad (4)$$

The next section is composed of two parts. In the first part, we will focus on planning the reference trajectory Repoulas and Papadopoulos [2] and in the second part we will use the recursive Backstepping controller design that allows us to guide the ship on the course.



**Fig. 2.** Ship moving on a planar path.

## PATH PLANNING

In this section, the planning of reference trajectories are analyzed. Since the ship system is underactuated, it is not expected to force the ship to track an arbitrary path. So the only restriction on this trajectory is that it must be sufficiently "smooth" Repoulas and Papadopoulos [2].

### Path Kinematics

Consider Fig. 2, Note that the point  $P(x_r, y_r)$  is an arbitrary point of the reference trajectory, where  $x_r$  and  $y_r$  are the inertial coordinates. So, In order to make the ship follow the reference trajectory we have to make the gravity center coincide with the point  $P$ .  $v_p$  is the magnitude of the velocity vector  $v_p$  of the reference point  $P$  at time  $t$ . The expression of  $v_p$  is given by:

$$v_p = \|v_p\| = \sqrt{\dot{x}_r^2 + \dot{y}_r^2} \quad (5)$$

The speed  $v_p$  is tangential to the reference path on  $P$  by making an angle  $\beta$  with the inertial axis ( $Ox$ ). Where:

$$\beta = \arctan\left(\frac{\dot{y}_r}{\dot{x}_r}\right) \quad (6)$$

The first and second derivatives of the angle  $\beta$  and the velocity  $v_p$  are given respectively by:

$$\begin{cases} \dot{\beta} = \frac{\dot{x}_r \ddot{y}_r - \dot{y}_r \ddot{x}_r}{\dot{x}_r^2 + \dot{y}_r^2} \\ \ddot{\beta} = f_{\beta}(\dot{x}_r, \ddot{x}_r, \ddot{y}_r, \dot{y}_r, \ddot{y}_r, \ddot{x}_r) \end{cases} \quad (7)$$

$$\begin{cases} \dot{v}_p = \frac{\dot{x}_r \ddot{x}_r + \dot{y}_r \ddot{y}_r}{\sqrt{\dot{x}_r^2 + \dot{y}_r^2}} \\ \ddot{v}_p = f_{v_p}(\dot{x}_r, \ddot{x}_r, \ddot{y}_r, \dot{y}_r, \ddot{y}_r, \ddot{x}_r) \end{cases} \quad (8)$$

### Ship's Dynamics on the Path

After presenting the inertial reference trajectory variables, we will exhibit the algebraic and differential equations that relate the inertial trajectory variables to the local frame velocities, accelerations and orientation. Since the ship tracks the reference trajectory, let assume that  $v_{S_r}$  is the magnitude of the total linear velocity vector of the gravity  $G$  in the ship-fixed frame, which its expression has the following form:

$$v_{S_r} = \|v_{S_r}\| = \sqrt{u_r^2 + v_r^2} \quad (9)$$

where  $u_r$  and  $v_r$  are the local reference velocity when the ship follows the reference trajectory, see Fig. 2.

The vehicle's total linear velocity  $v_{S_r}$  is tangential to the reference trajectory, consequently the angle  $\psi_r$  does not coincide with angle  $\beta$ , but it differs by angle  $\gamma$ , where:

$$\gamma = \arctan\left(\frac{v_r}{u_r}\right) \quad (10)$$

Having the description of the reference path and as the ship tracks the path, we can have the two following geometric conditions:

**1<sup>st</sup> Condition:** Equality between the magnitude of the vehicle's total velocity  $v_{S_r}$  and that of reference path velocity  $v_p$ . As a consequence, we obtain:

$$\sqrt{\dot{x}_r^2 + \dot{y}_r^2} = \sqrt{u_r^2 + v_r^2} \quad (11)$$

2<sup>nd</sup> **Condition**: Extraction of the reference angle  $\psi_r$  from the angles  $\beta$  and  $\gamma$ , observing Fig. 2, we can deduce:

$$\psi_r = \beta - \gamma \quad (12)$$

Replacing  $\beta$  and  $\gamma$ , respectively, by their expressions in (6) and in (10), yields:

$$\psi_r = \arctan\left(\frac{\dot{y}_r}{\dot{x}_r}\right) - \arctan\left(\frac{v_r}{u_r}\right) \quad (13)$$

Without loss of generality we can assume that the ship moves forward the reference path ( $u_r > 0$ ). Then the sign of the angle  $\gamma$  depends on the sign of the reference sway velocity  $v_r$ , which in turn depends on the curvature of the trajectory, i.e. a negative value corresponds to a counterclockwise (CCW) rotation and a positive value to a clockwise (CW) one Repoulas and Papadopoulos [2].

From the equations (5) and (13),  $v_r$  has the following expression:

$$v_r = \pm \sqrt{\dot{x}_r^2 + \dot{y}_r^2 - u_r^2} = \pm \sqrt{v_p^2 - u_r^2} \quad (14)$$

where  $\pm$  indicates the dependence of the sign of  $v_r$  according to the curvature of the reference trajectory. Differentiating equation (14) yields:

$$\dot{v}_r = \pm \frac{v_p \dot{v}_p - u_r \dot{u}_r}{\sqrt{v_p^2 - u_r^2}} \quad (15)$$

**Remark1**: From the equation (14),  $u_r$  it must verify:

$$-v_p \leq u_r \leq v_p \quad (16)$$

where the equality holds in the case of straight line tracking or when a change in the sign of the curvature occurs (then  $v_r = u_r = 0$ ).

In order to determine the reference angular velocity and acceleration, differentiating  $\psi_r$  given in equation (13), respectively, once and twice, yields:

$$\begin{aligned} r_r = \dot{\psi}_r &= \dot{\beta} \mp \frac{u_r \dot{v}_p - \dot{u}_r v_p}{v_p \sqrt{v_p^2 - u_r^2}} \\ \Rightarrow \dot{r}_r &= f_r(\beta, \dot{\beta}, u_r, \dot{u}_r, \ddot{u}_r, v_p, \dot{v}_p, \ddot{v}_p) \end{aligned} \quad (17)$$

Now, we will focus on computing the surge velocity  $u_r$ . So we distinguish two cases:  
**1<sup>st</sup> case (CW-type reference path curvature)**: Using the fact that  $v_r > 0$ , and substituting the appropriate expressions in equation (14),(15) and (17) in the fifth

equation of the system (4), the following differential equation which solution yields  $u_r$  is obtained :

$$\begin{aligned} \dot{u}_r = & \left[ (m_{11} - m_{22})u_r v_p^3 \right]^{-1} \left[ -v_p^3 (m_{22}v_p \dot{v}_p + d_{22}(v_p^2 - u_r^2)) \right. \\ & \left. + m_{11}u_r (u_r \dot{v}_p v_p^2 - v_p \sqrt{v_p^2 - u_r^2} (\dot{x}_r \ddot{y}_r - \ddot{x}_r \dot{y}_r)) \right] \end{aligned} \quad (18)$$

**2<sup>nd</sup> case (CCW-type reference path curvature):** Repeating the same approach used in the first case considering  $v_r < 0$ , we obtain:

$$\dot{u}_r = \left[ (m_{11} - m_{22})u_r v_p^3 \right]^{-1} \left[ v_p^3 (m_{22}v_p \dot{v}_p + d_{22}(v_p^2 - u_r^2)) - m_{11}u_r (u_r \dot{v}_p v_p^2 + v_p \sqrt{v_p^2 - u_r^2} (\dot{x}_r \ddot{y}_r - \ddot{x}_r \dot{y}_r)) \right] \quad (19)$$

**Remark 2:** When the curvature of the reference path changes with time, we switch between the above differential equations. What remains now is the computation of the open-loop control efforts. From the forth and the sixth equations of the system (4) the controls' expressions are given by:

$$\begin{cases} \tau_{ur} = m_{11}\dot{u}_r + d_{11}u_r - m_{22}v_r r_r \\ \tau_{rr} = m_{33}\dot{r}_r - \frac{(m_{11} - m_{22})}{m_{33}} r_r u_r + d_{33}r_r \end{cases} \quad (20)$$

In the next section, we will focus on conceiving control rules of auto-pilot ship which are derived for nonlinear controllers designed with the aid of the Backstepping method Krstić et. al. [9].

## CONTROL DESIGN

This section is divided in two subsections. In the first one we will write the error dynamic model, and in the second subsection we will use the Backstepping method Krstić et. al. [9] that provides a non linear control law that relies on the tracking control objective.

This Backstepping technique overcome the difficulty of Lyapunov function construction, it allows us to build iteratively the adapted Lyapunov function to the system and allows us to deduct the command(order) which returns the derivative of the Lyapunov function defined negative.

### Error System

In this section, configuration error and velocity error will define as:

$$\begin{cases} x_e = x - x_r \\ y_e = y - y_r \\ \psi_e = \psi - \psi_r \end{cases} \quad \begin{cases} u_e = u - u_r \\ v_e = v - v_r \\ r_e = r - r_r \end{cases} \quad (21)$$

The error kinematics is obtained by differentiating the system that present the configuration error in (21):

$$\begin{cases} \dot{X}_e = RU_e + \lambda \\ \dot{\psi}_e = r_e \end{cases} \quad (22)$$

where  $R$  is a square matrix, that has the following form:

$$R = \begin{bmatrix} c_\psi & -s_\psi \\ s_\psi & c_\psi \end{bmatrix} \quad (23)$$

$U_e = [u_e, v_e]^T$  is the linear velocity error vector,  $X_e = [x_e, y_e]^T$  is the position error vector and  $\lambda = R_{\psi, \psi_r} U_r$  where

$$R_{\psi, \psi_r} = \begin{bmatrix} c_\psi - c_{\psi_r} & -s_\psi + s_{\psi_r} \\ s_\psi - s_{\psi_r} & c_\psi - c_{\psi_r} \end{bmatrix} \quad (24)$$

and  $U_r = [u_r, v_r]^T$  is the reference linear velocity vector.

Differentiating the second system of (21), we obtain the error dynamics:

$$\begin{cases} \dot{u}_e = \tau_1 \\ \dot{v}_e = -\frac{m_{11}}{m_{22}}(u_e r_e + u_e r_r + u_r r_e) - \frac{d_{22}}{m_{22}}v_e - \dot{v}_r - \frac{m_{11}}{m_{22}}u_r r_r - \frac{d_{22}}{m_{22}}v_r \\ \dot{r}_e = \tau_3 \end{cases} \quad (25)$$

where the expressions of  $\tau_1$  and  $\tau_3$  are:

$$\begin{cases} \tau_1 = \frac{m_{22}}{m_{11}}(v_e r_e + v_e r_r + v_r r_e) - \frac{d_{11}}{m_{11}}u_e - \dot{u}_r + \frac{m_{22}}{m_{11}}v_r r_r - \frac{d_{11}}{m_{11}}v_r + \frac{1}{m_{11}}\tau_u \\ \tau_3 = \frac{(m_{11} - m_{22})}{m_{33}}(v_e u_e + v_e u_r + v_r u_e) - \frac{d_{33}}{m_{33}}r_e - \dot{r}_r + \frac{(m_{11} - m_{22})}{m_{33}}v_r u_r - \frac{d_{33}}{m_{33}}r_r + \frac{1}{m_{33}}\tau_r \end{cases} \quad (26)$$

According to systems (22) and (25), the complete error model of the ship is:

$$\begin{cases} \dot{X}_e = RU_e + \lambda \\ \dot{\psi}_e = r_e \\ \dot{u}_e = \tau_1 \\ \dot{v}_e = -\frac{m_{11}}{m_{22}}(u_e r_e + u_e r_r + u_r r_e) - \frac{d_{22}}{m_{22}}v_e - \dot{v}_r - \frac{m_{11}}{m_{22}}u_r r_r - \frac{d_{22}}{m_{22}}v_r \\ \dot{r}_e = \tau_3 \end{cases} \quad (27)$$

Consequently, the control problem is then transformed into a stabilization problem of the error dynamics (27).



## Error System Stabilization

The error system define in the following section is now used to design the backstepping algorithm which guaranteed the reference trajectory tracking objective. The control design is described step by step as follows:

**step 1:** In the first step, we focus on stabilizing the state variables  $(x_e, y_e)$ . Introducing the following Lyapunov function candidate:

$$V_1 = \frac{1}{2} X_e^T X_e \quad (28)$$

then, the time derivative is expressed as:

$$\dot{V}_1 = X_e^T (R U_e + \lambda) \quad (29)$$

We can treat  $U_e$  as an auxiliary control. We will choose a virtual control  $[\alpha_u, \alpha_v]^T$  instead of  $U_e$  as:

$$[\alpha_u, \alpha_v]^T = -R^T [K + K_1] X_e \quad (30)$$

with  $K$  and  $K_1$  are positive definite square matrices. Substituting (29) into (30), we get:

$$\dot{V}_1 = X_e^T \left[ -(K + K_1) X_e + \lambda \right] = -X_e^T K X_e - k_1 \left( x_e - \frac{\lambda_1}{2k_1} \right)^2 - k_1 \left( y_e - \frac{\lambda_2}{2k_1} \right)^2 + \frac{\|\lambda\|^2}{4k_1} \quad (31)$$

where  $\lambda = [\lambda_1, \lambda_2]^T$  (The quantity  $\lambda$  is a bounded function). When we increase the gain  $K_1$  the quantity  $\frac{\|\lambda\|^2}{4k_1}$  diminishes.

**step 2:** The components of the vector  $U_e$  are not a true control, we need to introduce new error variables  $z_u$  and  $z_v$  that we would derive to zero

$$\begin{cases} z_u = u_e - \alpha_u \\ z_v = v_e - \alpha_v \end{cases} \quad (32)$$

Then, (31) gives:

$$\dot{V}_1 = X_e^T \left[ -(K + K_1) X_e + R Z_u + \lambda \right] \quad (33)$$

where  $Z_u = [z_u, z_v]^T$ .

The task now is to derive  $z_u$  to zero, so we consider the following Lyapunov function candidate

$$V_2 = V_1 + \frac{1}{2} z_u^2 \quad (34)$$

Using (33) and (34), the derivative of  $V_2$  can be derived as follows:

$$\begin{aligned} \dot{V}_2 = & -X_e^T K X_e - k_1 \left( x_e - \frac{\lambda_1}{2k_1} \right)^2 - k_1 \left( y_e - \frac{\lambda_2}{2k_1} \right)^2 + \frac{\|\lambda\|^2}{4k_1} + z_v (y_e c_\psi - x_e s_\psi) \\ & + z_u [\tau_1 + (k + k_1)u_e + (k + k_1)(\lambda_1 c_\psi + \lambda_2 s_\psi) + x_e c_\psi + y_e s_\psi] \end{aligned} \quad (35)$$

According to (35), the control law  $\tau_1$  is designed as follows:

$$\tau_1 = -c_1 z_u - (k + k_1)u_e - (k + k_1)(\lambda_1 c_\psi + \lambda_2 s_\psi) - x_e c_\psi - y_e s_\psi + f_u \quad (36)$$

where  $c_1$  is a positive constant and  $f_u$  is a design variable for subsequent use. Substituting equation (36) into (35) the following equation can be obtained:

$$\dot{V}_2 = -X_e^T K X_e - k_1 \left( x_e - \frac{\lambda_1}{2k_1} \right)^2 - k_1 \left( y_e - \frac{\lambda_2}{2k_1} \right)^2 + \frac{\|\lambda\|^2}{4k_1} + z_v (y_e c_\psi - x_e s_\psi) - c_1 z_u^2 + z_u f_u \quad (37)$$

So far, the controlled subsystem of the linear kinematics and the error  $z_u$  is transformed to:

$$\begin{cases} \dot{X}_e = -(K + K_1)X_e + RZ_u + \lambda \\ \dot{z}_u = -c_1 z_u - x_e c_\psi - y_e s_\psi + f_u \end{cases} \quad (38)$$

**step 3:** In this step we consider the stabilization of the subsystem that is controlled by the assumed auxiliary control  $r_e$  (the rotational kinematics and the sway dynamic error  $z_v$ ). Before starting, we make some manipulations on  $z_v$ , in order to show the virtual control  $r_e$  and  $z_u$ . The dynamic  $z_v$  is written as:

$$\begin{aligned} \dot{z}_v = & -\frac{m_{11}}{m_{22}} u r_e - \frac{m_{11}}{m_{22}} z_u r_r - \frac{m_{11}}{m_{22}} \alpha_u r_r - \frac{d_{22}}{m_{22}} v_e - \dot{v}_r - \frac{m_{11}}{m_{22}} u_r r_r - \frac{d_{22}}{m_{22}} v_r + (k + k_1)v_e \\ & - (k + k_1)(\lambda_1 s_\psi - \lambda_2 c_\psi) \end{aligned} \quad (39)$$

which gives:

$$\dot{z}_v = -\frac{m_{11}}{m_{22}} u r_e - \frac{m_{11}}{m_{22}} z_u r_r + \delta_v \quad (40)$$

where

$$\delta_v = -\frac{m_{11}}{m_{22}} \alpha_u r_r - \frac{d_{22}}{m_{22}} v_e - \dot{v}_r - \frac{m_{11}}{m_{22}} u_r r_r - \frac{d_{22}}{m_{22}} v_r + (k + k_1)v_e - (k + k_1)(\lambda_1 s_\psi - \lambda_2 c_\psi) \quad (41)$$

Assuming  $u \neq 0$  is a natural condition for tracking, it means, if  $u = 0$  we can not act on our system with  $r_e$  as a control. We note a virtual control law  $\alpha_r$  for  $r_e$ . To design  $\alpha_r$ , add terms concerning  $\psi_e$  and  $z_v$  to  $V_2$  to form the following Lyapunov Function:

$$V_3 = V_2 + \frac{1}{2}(z_u^2 + \psi_e^2) \quad (42)$$

The time derivative of  $V_3$  is:

$$\begin{aligned} \dot{V}_3 = & -X_e^T K X_e - k_1 \left( x_e - \frac{\lambda_1}{2k_1} \right)^2 - k_1 \left( y_e - \frac{\lambda_2}{2k_1} \right)^2 + \frac{\|\lambda\|^2}{4k_1} + z_v (y_e c_\psi - x_e s_\psi + \delta_v) - c_1 z_u^2 \\ & + z_u \left( f_u - \frac{m_{11}}{m_{22}} z_v r_r \right) + \alpha_r \left( \psi_e - \frac{m_{11}}{m_{22}} u z_v \right) \end{aligned} \quad (43)$$

A simpler virtual control law  $\alpha_r$  is chosen:

$$\alpha_r = -c_r \left( \psi_e - \frac{m_{11}}{m_{22}} u z_v \right) \quad (44)$$

where  $c_r > 0$  is a design constant. Taking into account (44), the time derivative of  $V_3$  becomes:

$$\begin{aligned} \dot{V}_3 = & -X_e^T K X_e - k_1 \left( x_e - \frac{\lambda_1}{2k_1} \right)^2 - k_1 \left( y_e - \frac{\lambda_2}{2k_1} \right)^2 + \frac{\|\lambda\|^2}{4k_1} + z_v (y_e c_\psi - x_e s_\psi + \delta_v) - c_1 z_u^2 \\ & + z_u \left( f_u - \frac{m_{11}}{m_{22}} z_v r_r \right) - c_r \left( \psi_e - \frac{m_{11}}{m_{22}} u z_v \right)^2 \end{aligned} \quad (45)$$

**step 4:** As we know that  $r_e$  is not a true control, we will introduce the final error variable  $z_r = r_e - \alpha_r$  instead of  $\alpha_r$  and we use  $\tau_3$  to stabilize the subsystem.

$$\begin{cases} \dot{z}_v = -\frac{m_{11}}{m_{22}} u (z_r + \alpha_r) - \frac{m_{11}}{m_{22}} z_u r_r + \delta_v \\ \dot{\psi}_e = z_r + \alpha_r \\ \dot{z}_r = \tau_3 - \dot{\alpha}_r \end{cases} \quad (46)$$

With

$$\dot{\alpha}_r = -c_r \left[ 1 + \left( \frac{m_{11}}{m_{22}} u \right)^2 \right] (z_r + \alpha_r) + c_r \frac{m_{11}}{m_{22}} u \delta_v - c_r \left( \frac{m_{11}}{m_{22}} \right)^2 u z_u r_r \quad (47)$$

Then, after the set of transformation, the controlled dynamic system so far is:

$$\begin{cases} \dot{X}_e = -(K + K_1)X_e + RZ_u + \lambda \\ \dot{z}_u = -c_1 z_u - x_e c_\psi - y_e s_\psi + f_u \\ \dot{z}_v = -\frac{m_{11}}{m_{22}} u z_r + c_r \frac{m_{11}}{m_{22}} u \psi_e - c_r \left( \frac{m_{11}}{m_{22}} u \right)^2 z_v - \frac{m_{11}}{m_{22}} z_u r_r + \delta_v \\ \dot{\psi}_e = z_r - c_r \psi_e + c_r \frac{m_{11}}{m_{22}} u z_v \\ \dot{z}_r = \tau_3 - \dot{\alpha}_r \end{cases} \quad (48)$$

Again, to stabilize the error variable  $z_r$ , we consider the following Lyapunov function candidate:

$$V_4 = V_3 + \frac{1}{2} z_r^2 \quad (49)$$

and computing its time derivative, we obtain:

$$\begin{aligned} \dot{V}_4 = & -X_e^T K X_e - k_1 \left( x_e - \frac{\lambda_1}{2k_1} \right)^2 - k_1 \left( y_e - \frac{\lambda_2}{2k_1} \right)^2 + \frac{\|\lambda\|^2}{4k_1} + z_v (y_e c_\psi - x_e s_\psi + \delta_v) - c_1 z_u^2 \\ & + z_u \left( f_u - \frac{m_{11}}{m_{22}} z_v r_r \right) - c_r \left( \psi_e - \frac{m_{11}}{m_{22}} u z_v \right)^2 + z_r \left[ \tau_3 + c_r \left[ 1 + \left( \frac{m_{11}}{m_{22}} u \right)^2 \right] \left( z_r - c_r \psi_e + c_r \frac{m_{11}}{m_{22}} u z_v \right) \right. \\ & \left. - c_r \frac{m_{11}}{m_{22}} u \delta_v + c_r \left( \frac{m_{11}}{m_{22}} \right)^2 u z_u r_r + \psi_e - \frac{m_{11}}{m_{22}} u z_v \right] \end{aligned} \quad (50)$$

Choosing

$$\tau_3 = -c_r \left[ 1 + \left( \frac{m_{11}}{m_{22}} u \right)^2 \right] r_r + c_r \frac{m_{11}}{m_{22}} u \delta_v - c_r \left( \frac{m_{11}}{m_{22}} \right)^2 u z_u r_r - \psi_e + \frac{m_{11}}{m_{22}} u z_v - c z_r \quad (51)$$

and

$$f_u = \frac{m_{11}}{m_{22}} z_v r_r - c_4 z_u z_v^2 \quad (52)$$

with  $c$  and  $c_4$  are positive constants. Substituting equations (51) and (52) in (50) we obtain:

$$\begin{aligned} \dot{V}_4 = & -X_e^T K X_e - k_1 \left( x_e - \frac{\lambda_1}{2k_1} \right)^2 - k_1 \left( y_e - \frac{\lambda_2}{2k_1} \right)^2 + \frac{\|\lambda\|^2}{4k_1} + z_v (y_e c_\psi - x_e s_\psi + \delta_v) \\ & - c_1 z_u^2 - c z_r^2 - c_r \psi_e^2 - c_4 z_u^2 z_v^2 + 2c_r \frac{m_{11}}{m_{22}} u z_v \psi_e - c_r \left( \frac{m_{11}}{m_{22}} u z_v \right)^2 \end{aligned} \quad (53)$$

In the above expression we remark that the last three terms have uncertain signs, so we will examine them by considering all of them are positive. For the analysis we will

use the inequality of Young (Ghommam [10]), with the quantities  $\varepsilon_i$ ,  $i = 1.....6$  are positive constants.

$$z_v (y_e c_\psi - x_e s_\psi) \leq \frac{1}{4\varepsilon_1} |y_e|^2 + \varepsilon_1 |z_v|^2 + \frac{1}{4\varepsilon_1} |x_e|^2 + \varepsilon_1 |z_v|^2 \quad (54)$$

$$2c_r \left( \frac{m_{11}}{m_{22}} u \right) z_v \psi_e \leq \frac{1}{\varepsilon_2} \left( \frac{m_{11}}{m_{22}} u \right)^2 |\psi_e|^2 + \varepsilon_2 c_r^2 |z_v|^2 \quad (55)$$

Now, we will expand the expression of  $z_v \delta_v$ , we obtain the following expression:

$$z_v \delta_v = \frac{m_{11}}{m_{22}} r_r (k + k_1) (x_e c_\psi - y_e s_\psi) z_v + \left( (k + k_1) - \frac{d_{22}}{m_{22}} \right) v_e z_v - \left( \dot{v}_r + \frac{m_{11}}{m_{22}} u_r r_r + \frac{d_{22}}{m_{22}} v_r \right) z_v - (k + k_1) (v_r c_{\psi_e} - u_r s_{\psi_e} - v_r) z_v \quad (56)$$

Then, we will bound the four terms of (56) one by one as follows:

$$\begin{aligned} \frac{m_{11}}{m_{22}} r_r (k + k_1) (x_e c_\psi - y_e s_\psi) z_v &\leq \frac{1}{4\varepsilon_3} (k + k_1)^2 |r_r|^2 |x_e|^2 + \varepsilon_3 \left( \frac{m_{11}}{m_{22}} \right)^2 |z_v|^2 \\ &+ \frac{1}{4\varepsilon_3} (k + k_1)^2 |r_r|^2 |y_e|^2 + \varepsilon_3 \left( \frac{m_{11}}{m_{22}} \right)^2 |z_v|^2 \end{aligned} \quad (57)$$

$$\left( (k + k_1) - \frac{d_{22}}{m_{22}} \right) v_e z_v \leq \frac{1}{4\varepsilon_4} \left( (k + k_1) - \frac{d_{22}}{m_{22}} \right)^2 |v_e|^2 + \varepsilon_4 |z_v|^2 \quad (58)$$

$$-\left( \dot{v}_r + \frac{m_{11}}{m_{22}} u_r r_r + \frac{d_{22}}{m_{22}} v_r \right) z_v \leq \varepsilon_5 \left( \dot{v}_r + \frac{m_{11}}{m_{22}} u_r r_r + \frac{d_{22}}{m_{22}} v_r \right)^2 + \frac{1}{4\varepsilon_5} |z_v|^2 \quad (59)$$

$$-(k + k_1) (v_r c_{\psi_e} - u_r s_{\psi_e} - v_r) z_v \leq \varepsilon_6 (k + k_1)^2 (v_r c_{\psi_e} - u_r s_{\psi_e} - v_r)^2 + \frac{1}{4\varepsilon_6} |z_v|^2 \quad (60)$$

Taking into account the results from (54) to (60), the derivative of  $V_4$  in (53), will be increased by:

$$\begin{aligned} \dot{V}_4 &\leq - \left[ k - \frac{1}{4\varepsilon_1} - \frac{1}{4\varepsilon_3} (k + k_1)^2 r_{\max}^2 \right] x_e^2 - \left[ k - \frac{1}{4\varepsilon_1} - \frac{1}{4\varepsilon_3} (k + k_1)^2 r_{\max}^2 \right] y_e^2 - k_1 \left( x_e - \frac{\lambda_1}{2k_1} \right)^2 \\ &- k_1 \left( y_e - \frac{\lambda_2}{2k_1} \right)^2 + \frac{\|\lambda_{\max}\|^2}{4k_1} - c_1 z_u^2 - c_r \left( \frac{m_{11}}{m_{22}} u_{\max} z_v \right)^2 - \left( c_r - \frac{1}{\varepsilon_2} \left( \frac{m_{11}}{m_{22}} u_{\max} \right)^2 \right) \psi_e^2 - c_r z_r^2 \\ &- \left( c_4 z_u^2 - 2\varepsilon_1 - \varepsilon_2 c_r^2 - 2\varepsilon_3 \left( \frac{m_{11}}{m_{22}} \right)^2 - \varepsilon_4 - \frac{1}{4\varepsilon_5} - \frac{1}{4\varepsilon_6} \right) z_v^2 + \frac{1}{4\varepsilon_4} \left[ (k + k_1) - \frac{d_{22}}{m_{22}} \right]^2 v_{e \max}^2 + \varepsilon_5 |\zeta_1|^2 \\ &+ \varepsilon_6 (k + k_1)^2 |\zeta_2|^2 \end{aligned} \quad (61)$$

where  $r_{\max}, \lambda_{\max}, v_{\max}, u_{r\max}$  and  $v_{e\max}$  are the maximum values for the time varying quantities. In the inequality (61) the term  $c_4 z_u^2 - 2\varepsilon_1 - \varepsilon_2 c_r^2 - 2\varepsilon_3 \left( \frac{m_{11}}{m_{22}} \right)^2 - \varepsilon_4 - \frac{1}{4\varepsilon_5} - \frac{1}{4\varepsilon_6}$  must be positive, for that we set  $2\varepsilon_1 + \varepsilon_2 c_r^2 + 2\varepsilon_3 \left( \frac{m_{11}}{m_{22}} \right)^2 + \varepsilon_4 + \frac{1}{4\varepsilon_5} + \frac{1}{4\varepsilon_6}$ . So, it must that  $c_4 z_u^2 > \zeta \Rightarrow z_u > \sqrt{\frac{\zeta}{c_4}}$  which is verified for a large and positive  $c_4$  and small  $\zeta$  consequently we have:

$$\begin{aligned} \dot{V}_4 \leq & -\varpi_1 x_e^2 - \varpi_1 y_e^2 - k_1 \left( x_e - \frac{\lambda_1}{2k_1} \right)^2 - k_1 \left( y_e - \frac{\lambda_2}{2k_1} \right)^2 - c_1 z_u^2 - \varpi_3 \psi_e^2 - c_r z_r^2 - [c_4 z_u^2 - \zeta] z_v^2 \\ & - \varpi_2 z_v^2 + \mu \end{aligned} \quad (62)$$

where

$$\begin{aligned} \varpi_1 &= \left[ k - \frac{1}{4\varepsilon_1} - \frac{1}{4\varepsilon_3} (k + k_1)^2 r_{\max}^2 \right] \\ \varpi_2 &= c_r \left( \frac{m_{11}}{m_{22}} u_{\max} \right)^2 \\ \varpi_3 &= c_r - \frac{1}{\varepsilon_2} \left( \frac{m_{11}}{m_{22}} u_{\max} \right)^2 \\ \mu &= \frac{1}{4\varepsilon_4} \left[ (k + k_1) - \frac{d_{22}}{m_{22}} \right]^2 v_{e\max}^2 + \varepsilon_5 |\zeta_1|^2 + \varepsilon_6 (k + k_1)^2 |\zeta_2|^2 + \frac{\|\lambda_{\max}\|^2}{4k_1} \end{aligned}$$

which are all positive. So, from the inequality (62), we can deduce that:

$$\begin{aligned} \dot{V}_4 &\leq -\varpi_1 x_e^2 - \varpi_1 y_e^2 - c_1 z_u^2 - \varpi_2 z_v^2 - \varpi_3 \psi_e^2 - c_r z_r^2 + \mu \\ &\leq \nu V_4 + \mu \end{aligned} \quad (63)$$

where  $\nu = \min\{\varpi_1, c_1, \varpi_2, \varpi_3, c_4\}$ .

By employing the comparison lemma (Khalil [11]), we can yield:

$$V_4(t) \leq V_4(0)e^{-2\nu t} + \frac{\mu}{2\nu}, t \geq 0 \quad (64)$$

Which implies that the states of error  $(x_e, y_e, \psi_e, z_u, z_v, z_r)$  remain in a bounded set about the origin.

## ANALATYCAL RESULTS

In this section, we carry out some computer simulations that illustrate the performance that can be achieved with the motion control algorithms mentioned before. The simulations are implemented on Mat lab. We assume the underactuated ship vehicle model with the following parameters of kinematics and the dynamics that are the same as those in Dong and Guo [12]:

$$m_{11} = 200Kg, m_{22} = 250Kg, m_{33} = 80Kg, d_{11} = 70Kg/s, d_{22} = 100Kg/s \text{ and } d_{33} = 50Kg/s$$

Here, we will track a sinusoidal path (time varying trajectory). Responses were obtained for two cases: in the first case we will show simulation results of the tracking without disturbances and in the second case we will present simulation results in the disturbed condition, in this case we assume that the disturbances are the sea drag forces which are taken equal to  $0.05u$  and  $0.05v$  acting in the surge and sway respectively. We choose a desired trajectory which its parametric system describing by:

$$\begin{cases} x_r = 0.03t \\ y_r = 10\sin(0.03t) \end{cases} \quad (65)$$

For simulation uses, we pick the following initial condition for the real system:

$$[x(0), y(0), \psi(0), u(0), v(0), r(0)] = [0.12, 2.5, 0, 0, 0, 0]$$

and the initial desired conditions are:

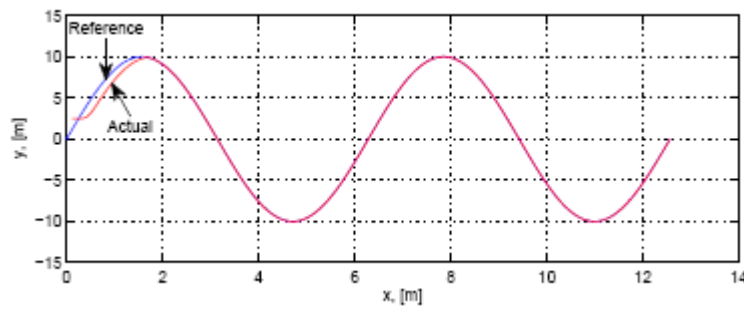
$$[x_r(0), y_r(0), \psi_r(0), u_r(0), v_r(0), r_r(0)] = [0, 0, 1.3175, 0.3, 0.03, 0.2]$$

The control parameters used in the Backstepping control are set to  $k_1 = 0.01, k = 0.03, c_1 = 10, c_r = 3, c = 10$  and  $c_4 = 1$  to satisfy convergence conditions.

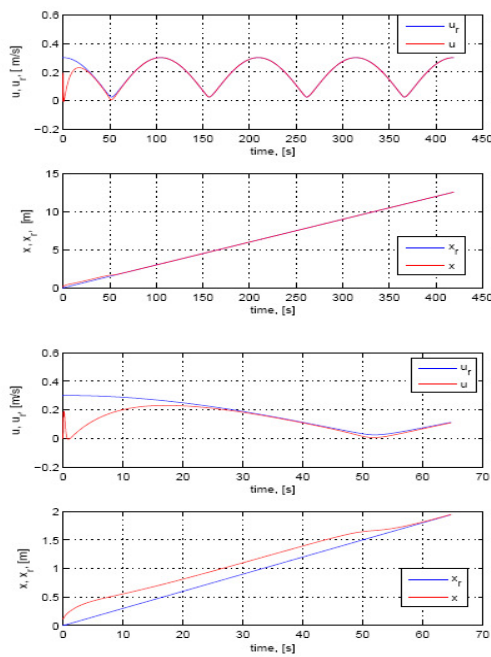
### Sinusoidal Tracking with No Disturbances

Figure 3 shows the desired trajectory and the actual trajectory of the ship in  $X - Y$  plane. We see that the ship converged to the desired trajectory. For each figure of Figs. (4-7), we have figures on the left side and on the right side. The figures on the left regard simulation time of  $450s$  needed for two full period for the sinus function, while the figures on the right regard the first  $70s$  in order to observe the transient responses.

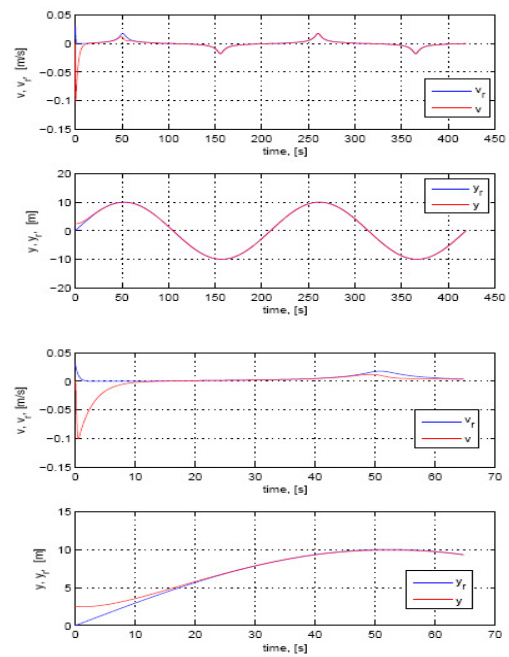
The simulations results, depicted in Figs. (4-5), are for response of positions  $(x, y)$  and linear velocities  $(u, v)$ , we observe that after a transition period of  $65s$  all presented variables converge to the desired variables. Figure 6 shows the convergence of  $\psi$  and yaw velocity  $r$  to  $\psi_r$  and  $r_r$ , respectively after around  $2s$ . In Figure 7, we see the behavior of the control surge force  $\tau_u$  and the control yaw torque  $\tau_r$  needed for tracking and they also converge to the open-loop control efforts  $\tau_{ur}$  and  $\tau_{rr}$  respectively after around  $2s$ .



**Fig. 3.** Ship reference and actual path.



**Fig. 5.** (left side) Time responses of  $y$  and  $v$  together with  $y_r$  and  $v_r$

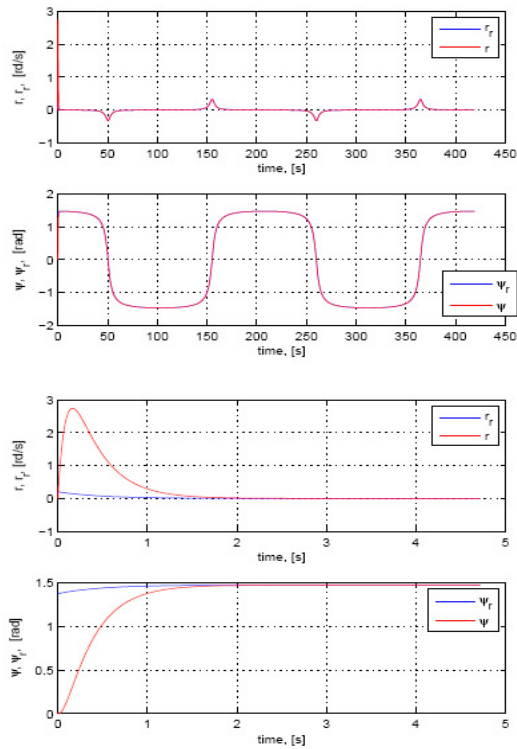


**Fig. 4.** ( right side) Time responses of  $x$  and  $u$  together with  $x_r$  and  $u_r$

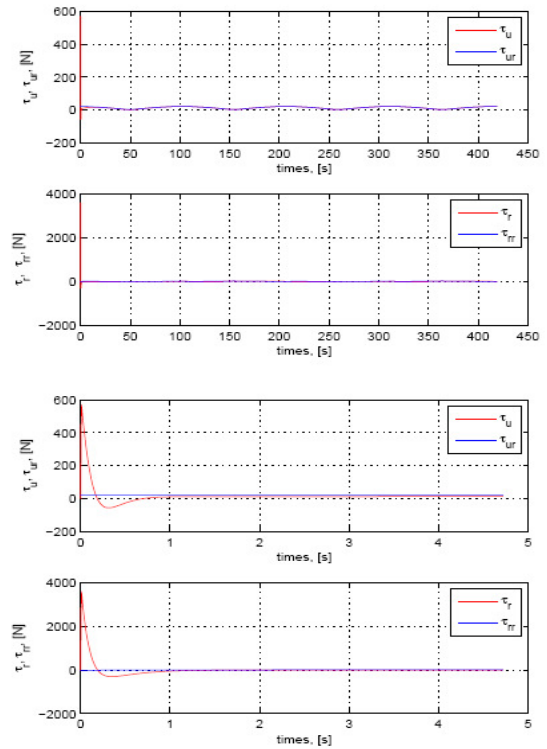
### Sinusoidal Tracking Path with Disturbances

In the disturbed condition, we present some numerical results concerning the robustness of the designed control law. Figs. (8-12) are simulation results with disturbance in which we assume that there are added drag forces of the order  $(0.05u, 0.05v)$ . As in the preceding section, we present in each figure the response after two full periods on the left side and on the right the first 65 s. We remark that the proposed control law is robust for the chosen disturbances.

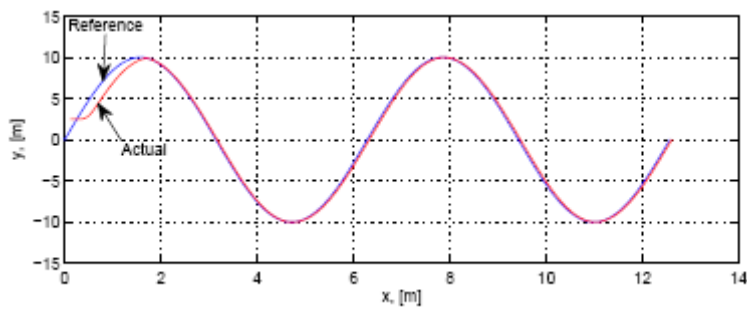




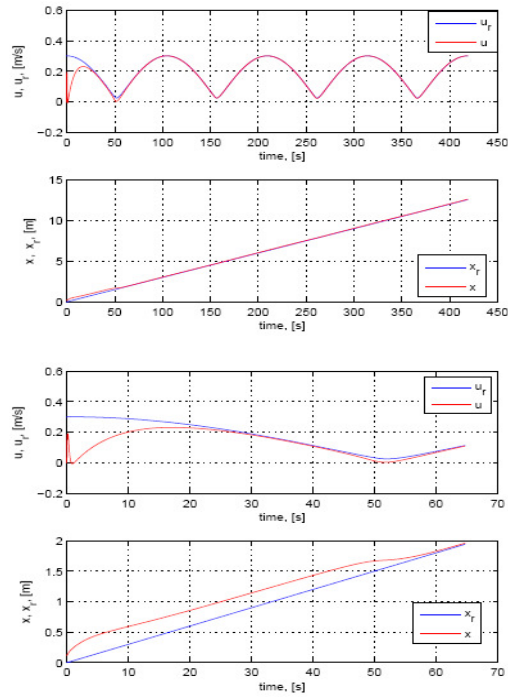
**Fig. 7.** (left side) Time responses of  $\tau_u$  and  $\tau_r$  together with  $\tau_{u_r}$  and  $\tau_{r_r}$



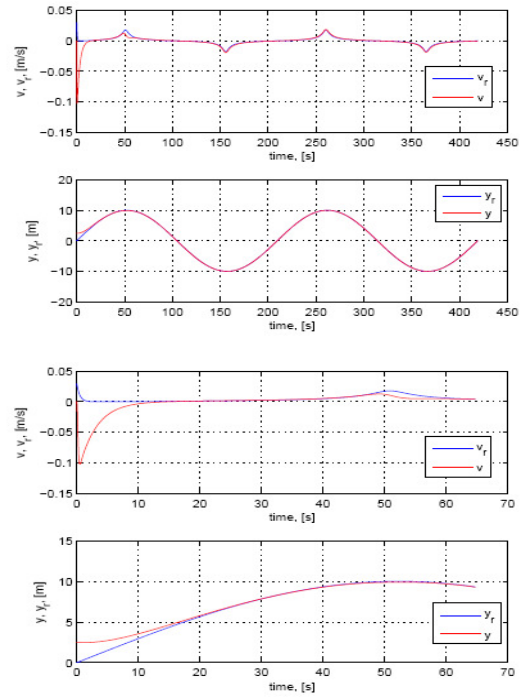
**Fig. 6.** (right side) Time responses of  $\psi$  and  $r$  together with  $\psi_r$  and  $r_r$



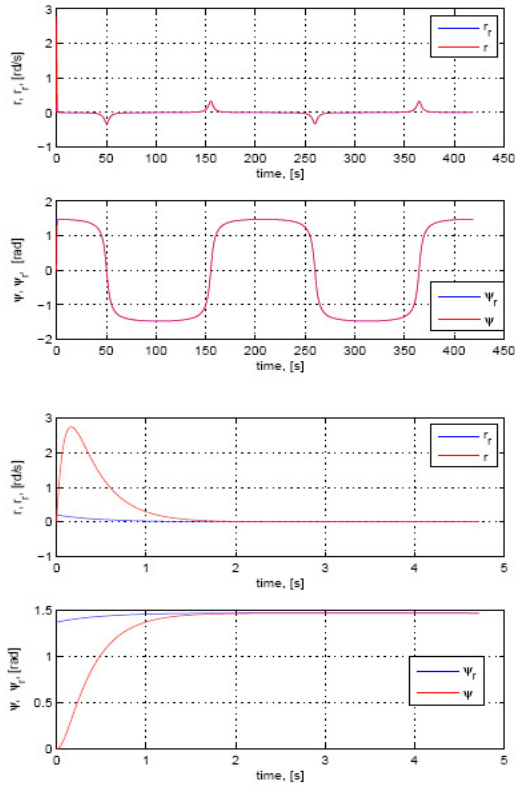
**Fig. 8.** The trajectory of the ship in the  $X - Y$  plane and the desired trajectory.



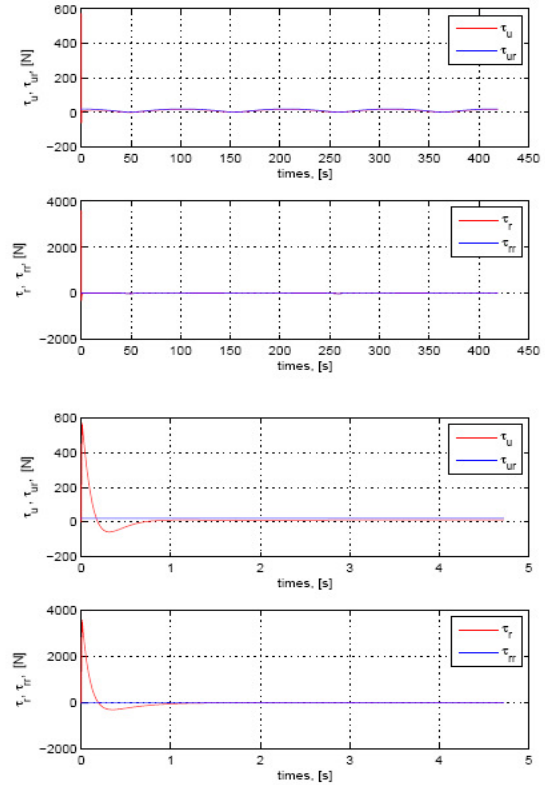
**Fig. 10.** (left side) Time responses of  $y$  and  $v$  together with  $y_r$  and  $v_r$



**Fig. 9.** (right side) Time responses of  $x$  and  $u$  together with  $x_r$  and  $u_r$



**Fig. 12.** (left side) Responses of  $\tau_u$  and  $\tau_r$  together with  $\tau_{ur}$  and  $\tau_{rr}$



**Fig. 11.** (right side) Responses of  $\psi$  and  $r$  together with  $\psi_r$  and  $r_r$

## CONCLUSION

In this paper, we presented an autonomous navigation control technique for an under-actuated ship. A Backstepping methodology combined with Lyapunov function for determining explicit stabilizing feedbacks, for the completed error system that assure the motion of the ship to track realisable trajectories, have been given. Simulation results show a very good tracking performance and effective robustness against considered disturbances.

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