

**Military Technical College  
Kobry El-Kobbah,  
Cairo, Egypt.**



**15<sup>th</sup> International Conference  
on Applied Mechanics and  
Mechanical Engineering.**

## **COMPOUND BENDING OF INHOMOGENEOUS THIN PLATES IN NONUNIFORM TEMPERATURE FIELD**

A. N. Tyurehodzhayev\* and G. K. Kalzhanova\*\*

### **ABSTRACT**

The particular interest in the mechanics of deformable solid are the problems associated with the bends of flexible plates and various flexible shells working in non-uniform temperature field. Such problems are commonly encountered in applied problems of the construction, oil-field, mechanical engineering, water and air transport. During mathematical review of such kind of problems you have to deal with systems of linear differential equations with variable coefficients and nonlinear differential equations, and making analytical solution of which represents considerable mathematical difficulties. Analytical solutions of such problems can be made by the method of partial discretization, the method that has been derived by one of the authors of this article based on the theory of generalized functions.

The paper considers the problem of thermoelasticity of inhomogeneous circular flexible plate in the axially symmetric temperature field by taking into account the influence of bending tension and change of elastic properties of plate material along its thickness. The problem of compound bending of inhomogeneous circular plate exposed to the action of lateral load, under temperature changes in thickness of the plate with the influence of bending tension come to the investigation of decoupled system of differential equations, obtaining of analytical solution of which using the existing mathematical apparatus was not possible.

### **KEY WORDS**

Thin circular plates, Compound bending, Lateral load, Radial force, Deflection of the plate middle plane, Method of partial discretization of differential equations.

---

\* Professor, Department of Applied Mechanics and Principles of Machinery Engineering, the Kazakh National Technical University named after K.I. Satpayev, Almaty, Kazakhstan.

\*\* Head of Educational-Methodical Department, Zhetysu State University named after I. Zhansugurov, Taldykorgan, Kazakhstan.

## NOTATION SYSTEM

$F$	Stress function.
$u_z$	Deflection of the plate middle plane.
$h$	Plate thickness.
$E$	Modulus of elasticity.
$\nu$	Poisson's ratio.
$q_z$	External distributed lateral load.
$N_r$	Normal force.
$g_r$	angle of normal rotation.
$Q_r$	Lateral force.
$u_z$	Deflection of the middle plane.

## DEVELOPED ANALYTICAL SOLUTION OF COMPOUND BENDING PROBLEM

Solving these equations in regard to  $\frac{d}{dr} \nabla^2 F$  and  $\frac{d}{dr} \nabla^2 u_z$  and considering formulas (2), we obtain

$$r \frac{d}{dr} \nabla^2 F = \frac{D_{Nv}D - D_N D_v}{D_N D_M - D^2} \cdot \frac{dF}{dr} \cdot \frac{du_z}{dr} + \frac{D_{Nv}D - D_N D_v}{D_N D_M - D^2} \left( \int q_z r dr - C \right),$$

$$r \frac{d}{dr} \nabla^2 u_z = \frac{D_N}{D_N D_M - D^2} \cdot \frac{dF}{dr} \cdot \frac{du_z}{dr} + \frac{D_N}{D_N D_M - D^2} \left( \int q_z r dr - C \right).$$

It is commonly supposed [1], that the system of equations (1) in case when accounting of bending tension influence is not resulting in decoupled equations. In fact, the system of resolving equations (1) can be reduced to a system of differential equations with nonlinear terms, each of which includes only one resolution function:

$$\frac{d^3 u_z}{dr^3} + \frac{1}{r} \frac{d^2 u_z}{dr^2} - \left( \frac{BC_1}{2} + \frac{1 + BC_2}{r^2} \right) \frac{du_z}{dr} - \frac{A}{r} \left( \frac{du_z}{dr} \right)^2 = \frac{B}{r} \left( \int q_z r dr - C \right), \quad (3)$$

$$\frac{d^3 F}{dr^3} + \frac{1}{r} \frac{d^2 F}{dr^2} + \left( \frac{BC_1}{2} - \frac{1 - BC_2}{r^2} \right) \frac{dF}{dr} - \frac{B}{r} \left( \frac{dF}{dr} \right)^2 = \frac{A}{r} \left( \int q_z r dr - C \right), \quad (4)$$

where

$$A = \frac{D_{Nv}D - D_N D_v}{D_N D_M - D^2}, \quad B = \frac{D_N}{D_N D_M - D^2}.$$

And functions  $F$  and  $u_z$  are coupled by relation

$$F = \frac{A}{B} u_z + \frac{C_1 r^2}{4} + C_2 \ln r + C_0. \quad (5)$$

Differential equations of third order (3) - (4) with respect to the normal force  $N_r$ , acting in the cylindrical section, and angle of the normal rotation  $\vartheta_r$ , defined at axisymmetric field by ratio

$$N_r = \frac{1}{r} \frac{dF}{dr}, \quad \vartheta_r = -\frac{du_z}{dr},$$

will be rewritten as follows:

$$\frac{d^2 \vartheta_r}{dr^2} + \frac{1}{r} \frac{d\vartheta_r}{dr} - \left( \frac{BC_1}{2} + \frac{1+BC_2}{r^2} \right) \vartheta_r + \frac{A}{r} \vartheta_r^2 = -\frac{B}{r} \left( \int q_z r dr - C \right), \quad (6)$$

$$\frac{d^2 N_r}{dr^2} + \frac{3}{r} \frac{dN_r}{dr} + \left( \frac{BC_1}{2} - \frac{BC_2}{r^2} \right) N_r - B N_r^2 = \frac{A}{r^2} \left( \int q_z r dr - C \right). \quad (7)$$

The radial force and deflection angle are coupled by ratio

$$N_r = \frac{C_1}{2} + \frac{C_2}{r^2} - \frac{A}{B} \cdot \frac{\vartheta_r}{r}. \quad (8)$$

The exact solution of such equations using existing mathematical apparatus is not possible. Applying the method of partial discretization of differential equations, developed by Professor A.N. Tyurehodzhayev, let's define the general solution of these equations.

It should be noted that in this case it is sufficient to solve one of the equations (6) - (7). For example, we can determine the angle of rotation of the normal, and then find the radial force by formula (8).

Applying the method of partial discretization to the differential equation (6), we obtain the following expression for the general solution of this equation as:

$$\begin{aligned} \vartheta_r(r) = & C_3 r + \frac{C_4}{r} - \frac{A}{4} r \sum_{k=1}^n (r_k + r_{k+1}) \left[ \frac{\vartheta_r^2(r_k)}{r_k} H(r - r_k) - \frac{\vartheta_r^2(r_{k+1})}{r_{k+1}} H(r - r_{k+1}) \right] + \frac{BC_1}{8} r \times \\ & \times \sum_{k=1}^n (r_k + r_{k+1}) [\vartheta_r(r_k) H(r - r_k) - \vartheta_r(r_{k+1}) H(r - r_{k+1})] + \frac{BC_2}{4} r \sum_{k=1}^n (r_k + r_{k+1}) \left[ \frac{\vartheta_r^2(r_k)}{r_k^2} H(r - r_k) - \right. \\ & \left. - \frac{\vartheta_r^2(r_{k+1})}{r_{k+1}^2} H(r - r_{k+1}) \right] + \frac{A}{4r} \sum_{k=1}^n (r_k + r_{k+1}) [r_k \vartheta_r^2(r_k) H(r - r_k) - r_{k+1} \vartheta_r^2(r_{k+1}) H(r - r_{k+1})] - \\ & - \frac{BC_1}{8r} \sum_{k=1}^n (r_k + r_{k+1}) [r_k^2 \vartheta_r(r_k) H(r - r_k) - r_{k+1}^2 \vartheta_r(r_{k+1}) H(r - r_{k+1})] - \frac{BC_2}{4r} \sum_{k=1}^n (r_k + r_{k+1}) \times \\ & \times [\vartheta_r(r_k) H(r - r_k) - \vartheta_r(r_{k+1}) H(r - r_{k+1})] - \frac{B}{2} r \int \frac{1}{r} \left( \int q_z r dr - C \right) dr + \frac{B}{2r} \int r \left( \int q_z r dr - C \right) dr, \quad (9) \end{aligned}$$

where  $H(z)$  - the unit function of Heaviside.

Substituting formula (9) into the formula (8), we obtain:

$$\begin{aligned}
 N_r(r) = & \frac{C_1}{2} + \frac{C_2}{r^2} - \frac{A}{B} \left\{ C_3 + \frac{C_4}{r^2} - \frac{A}{4} \sum_{k=1}^n (r_k + r_{k+1}) \left[ \frac{\vartheta_r^2(r_k)}{r_k} H(r - r_k) - \frac{\vartheta_r^2(r_{k+1})}{r_{k+1}} H(r - r_{k+1}) \right] + \right. \\
 & + \frac{BC_1}{8} \sum_{k=1}^n (r_k + r_{k+1}) [\vartheta_r(r_k) H(r - r_k) - \vartheta_r(r_{k+1}) H(r - r_{k+1})] + \frac{BC_2}{4} \left[ \frac{\vartheta_r(r_k)}{r_k^2} H(r - r_k) - \right. \\
 & - \frac{\vartheta_r(r_{k+1})}{r_{k+1}^2} H(r - r_{k+1}) \left. \right] + \frac{A}{4r^2} \sum_{k=1}^n (r_k + r_{k+1}) [r_k \vartheta_r^2(r_k) H(r - r_k) - r_{k+1} \vartheta_r^2(r_{k+1}) H(r - r_{k+1})] - \\
 & - \frac{BC_1}{8r^2} \sum_{k=1}^n (r_k + r_{k+1}) [r_k^2 \vartheta_r(r_k) H(r - r_k) - r_{k+1}^2 \vartheta_r(r_{k+1}) H(r - r_{k+1})] - \frac{BC_2}{4r^2} \sum_{k=1}^n (r_k + r_{k+1}) [\vartheta_r(r_k) \times \\
 & \times H(r - r_k) - \vartheta_r(r_{k+1}) H(r - r_{k+1})] - \frac{B}{2} \int \frac{1}{r} \left( \int q_z r dr - C \right) dr + \frac{B}{2r^2} \int r \left( \int q_z r dr - C \right) dr \left. \right\} \quad (10)
 \end{aligned}$$

This method allows finding the solution of equations (6) - (7) for almost any laws of change in the elastic modulus and Poisson's ratio. In order to determine the solutions to specific problems, we need five boundary conditions.

Let's consider ring-shaped plate of constant thickness, the outer contour of which is rigidly anchored and inner - may be displaced in the direction of the plate axis, but it does not turn. The contours of the plate are free of radial forces. Then the constants are determined from the following boundary conditions:

$$N_r|_{r=a} = 0, N_r|_{r=b} = 0, \vartheta_r|_{r=a} = 0, \vartheta_r|_{r=b} = 0. \quad (11)$$

According to (11) the constants  $C_1$  and  $C_2$  will be equal to zero. Given the values  $C_1$  and  $C_2$ , the differential equation (6) will be as follows:

$$\frac{d^2 \vartheta_r}{dr^2} + \frac{1}{r} \frac{d\vartheta_r}{dr} - \frac{\vartheta_r}{r^2} + \frac{A}{r} \vartheta_r^2 = -\frac{B}{r} \left( \int q_z r dr - C \right) \quad (12)$$

By discretizing the last element of the left part of equation (12), we will obtain the following its general solution:

$$\begin{aligned}
 \vartheta_r(r) = & C_3 r + \frac{C_4}{r} - \frac{A}{4} r \sum_{k=1}^n (r_k + r_{k+1}) \left[ \frac{\vartheta_r^2(r_k)}{r_k} H(r - r_k) - \frac{\vartheta_r^2(r_{k+1})}{r_{k+1}} H(r - r_{k+1}) \right] + \\
 & + \frac{A}{4r} \sum_{k=1}^n (r_k + r_{k+1}) [r_k \vartheta_r^2(r_k) H(r - r_k) - r_{k+1} \vartheta_r^2(r_{k+1}) H(r - r_{k+1})] - \\
 & - \frac{1}{2} r \int \frac{B}{r} \left( \int q_z r dr - C \right) dr + \frac{1}{2r} \int B \left( \int q_z r dr - C \right) r dr. \quad (13)
 \end{aligned}$$

Consequently, for deflection  $u_z(r)$  we will have:

$$\begin{aligned}
 u_z(r) = & -\frac{C_3 r^2}{2} + C_4 \ln r + \frac{A}{8} r^2 \sum_{k=1}^n (r_k + r_{k+1}) \left[ \frac{\vartheta_r^2(r_k)}{r_k} H(r - r_k) - \frac{\vartheta_r^2(r_{k+1})}{r_{k+1}} H(r - r_{k+1}) \right] + \\
 & + \frac{A}{4} \sum_{k=1}^n (r_k + r_{k+1}) [r_k \ln r_k \vartheta_r^2(r_k) H(r - r_k) - r_{k+1} \ln r_{k+1} \vartheta_r^2(r_{k+1}) H(r - r_{k+1})] - \\
 & - \frac{A}{4} \left( \ln r + \frac{1}{2} \right) \sum_{k=1}^n (r_k + r_{k+1}) [r_k \vartheta_r^2(r_k) H(r - r_k) - r_{k+1} \vartheta_r^2(r_{k+1}) H(r - r_{k+1})] + \\
 & + \frac{1}{2} \int r \left[ \int \frac{B}{r} \left( \int q_z r dr - C \right) dr \right] dr - \frac{1}{2} \int \frac{1}{r} \left[ \int B \left( \int q_z r dr - C \right) r dr \right] dr + C_5.
 \end{aligned} \tag{14}$$

The constant  $C_5$  is determined under the condition of rigid anchorage of outer plate contour

$$u_z|_{r=b} = 0 \tag{15}$$

The ratio (8) considering the values of constants  $C_1, C_2$  will be as follows:

$$N_r = -\frac{A}{B} \cdot \frac{\vartheta_r}{r} \tag{16}$$

Substituting formula (13) into (16), we will obtain:

$$\begin{aligned}
 N_r(r) = & -\frac{A}{B} \left\{ C_3 + \frac{C_4}{r^2} - \frac{A}{4} \sum_{k=1}^n (r_k + r_{k+1}) \left[ \frac{\vartheta_r^2(r_k)}{r_k} H(r - r_k) - \frac{\vartheta_r^2(r_{k+1})}{r_{k+1}} H(r - r_{k+1}) \right] + \right. \\
 & + \frac{A}{4r^2} \sum_{k=1}^n (r_k + r_{k+1}) [r_k \vartheta_r^2(r_k) H(r - r_k) - r_{k+1} \vartheta_r^2(r_{k+1}) H(r - r_{k+1})] - \\
 & \left. - \frac{1}{2} \int \frac{B}{r} \left( \int q_z r dr - C \right) dr + \frac{1}{2r^2} \int B \left( \int q_z r dr - C \right) r dr \right\}
 \end{aligned} \tag{17}$$

Let's consider a sample when lateral load is evenly distributed over the entire surface of plate with intensity  $q$ . The constant  $C$  is found from:

$$rQ_r - rN_r \vartheta_r = -\int q_z r dr + C,$$

where  $Q_r$  - lateral force. Taking into account the conditions (11) and assuming that the inner contour of uniformly loaded plate is free of lateral forces, we obtain:

$$C = \frac{qa^2}{2}.$$

Then equation (12) will be rewritten as:

$$\frac{d^2 \vartheta_r}{dr^2} + \frac{1}{r} \frac{d\vartheta_r}{dr} - \frac{\vartheta_r}{r^2} + \frac{A}{r} \vartheta_r^2 = -\frac{Bqr}{r} + \frac{Bqa^2}{2r} \tag{18}$$

The final solution of equation (18) has the form:

$$\begin{aligned} \vartheta_r(r) = & -\frac{Bqr(r^2 - a^2)}{16} - \frac{b(r^2 - a^2)}{r(b^2 - a^2)} \left\{ \frac{A}{4b} \sum_{k=1}^n (r_k + r_{k+1}) [r_k \vartheta_r^2(r_k) - r_{k+1} \vartheta_r^2(r_{k+1})] - \right. \\ & - \frac{A}{4} b \sum_{k=1}^n (r_k + r_{k+1}) \left[ \frac{\vartheta_r^2(r_k)}{r_k} - \frac{\vartheta_r^2(r_{k+1})}{r_{k+1}} \right] - \frac{Bqb(b^2 - a^2)}{16} + \frac{Bqba^2}{4} \ln \frac{b}{a} \left. \right\} + \\ & + \frac{A}{4r} \sum_{k=1}^n (r_k + r_{k+1}) [r_k \vartheta_r^2(r_k) H(r - r_k) - r_{k+1} \vartheta_r^2(r_{k+1}) H(r - r_{k+1})] - \\ & - \frac{A}{4} r \sum_{k=1}^n (r_k + r_{k+1}) \left[ \frac{\vartheta_r^2(r_k)}{r_k} H(r - r_k) - \frac{\vartheta_r^2(r_{k+1})}{r_{k+1}} H(r - r_{k+1}) \right] + \frac{Bqa^2}{4} r \ln \frac{r}{a}. \end{aligned} \quad (19)$$

The deflection of middle plane of ring-shaped loaded plate will be equal to:

$$\begin{aligned} u_z(r) = & -\frac{Bq(b^4 - r^4)}{64} - \frac{Bqa^2(b^2 - r^2)}{32} - \frac{b}{b^2 - a^2} \left( a^2 \ln \frac{r}{b} + \frac{b^2 - r^2}{2} \right) \left\{ \frac{A}{4b} \times \right. \\ & \times \sum_{k=1}^n (r_k + r_{k+1}) [r_k \vartheta_r^2(r_k) - r_{k+1} \vartheta_r^2(r_{k+1})] - \frac{A}{4} b \sum_{k=1}^n (r_k + r_{k+1}) \left[ \frac{\vartheta_r^2(r_k)}{r_k} - \frac{\vartheta_r^2(r_{k+1})}{r_{k+1}} \right] - \frac{Bqb(b^2 - a^2)}{16} + \\ & + \frac{Bqba^2}{4} \ln \frac{b}{a} \left. \right\} - \frac{A}{4} \left( \ln r + \frac{1}{2} \right) \sum_{k=1}^n (r_k + r_{k+1}) [r_k \vartheta_r^2(r_k) H(r - r_k) - r_{k+1} \vartheta_r^2(r_{k+1}) H(r - r_{k+1})] + \\ & + \frac{A}{4} \sum_{k=1}^n (r_k + r_{k+1}) [r_k \ln r_k \vartheta_r^2(r_k) H(r - r_k) - r_{k+1} \ln r_{k+1} \vartheta_r^2(r_{k+1}) H(r - r_{k+1})] + \frac{A}{8} r^2 \sum_{k=1}^n (r_k + r_{k+1}) \times \\ & \times \left[ \frac{\vartheta_r^2(r_k)}{r_k} H(r - r_k) - \frac{\vartheta_r^2(r_{k+1})}{r_{k+1}} H(r - r_{k+1}) \right] + \frac{A}{4} \left( \ln b + \frac{1}{2} \right) \sum_{k=1}^n (r_k + r_{k+1}) [r_k \vartheta_r^2(r_k) - r_{k+1} \vartheta_r^2(r_{k+1})] - \\ & - \frac{A}{4} \sum_{k=1}^n (r_k + r_{k+1}) [r_k \ln r_k \vartheta_r^2(r_k) - r_{k+1} \ln r_{k+1} \vartheta_r^2(r_{k+1})] - \frac{A}{8} b^2 \sum_{k=1}^n (r_k + r_{k+1}) \left[ \frac{\vartheta_r^2(r_k)}{r_k} - \frac{\vartheta_r^2(r_{k+1})}{r_{k+1}} \right] - \\ & - \frac{Bqa^2}{8} \left( r^2 \ln \frac{r}{a} - b^2 \ln \frac{b}{a} \right). \end{aligned} \quad (20)$$

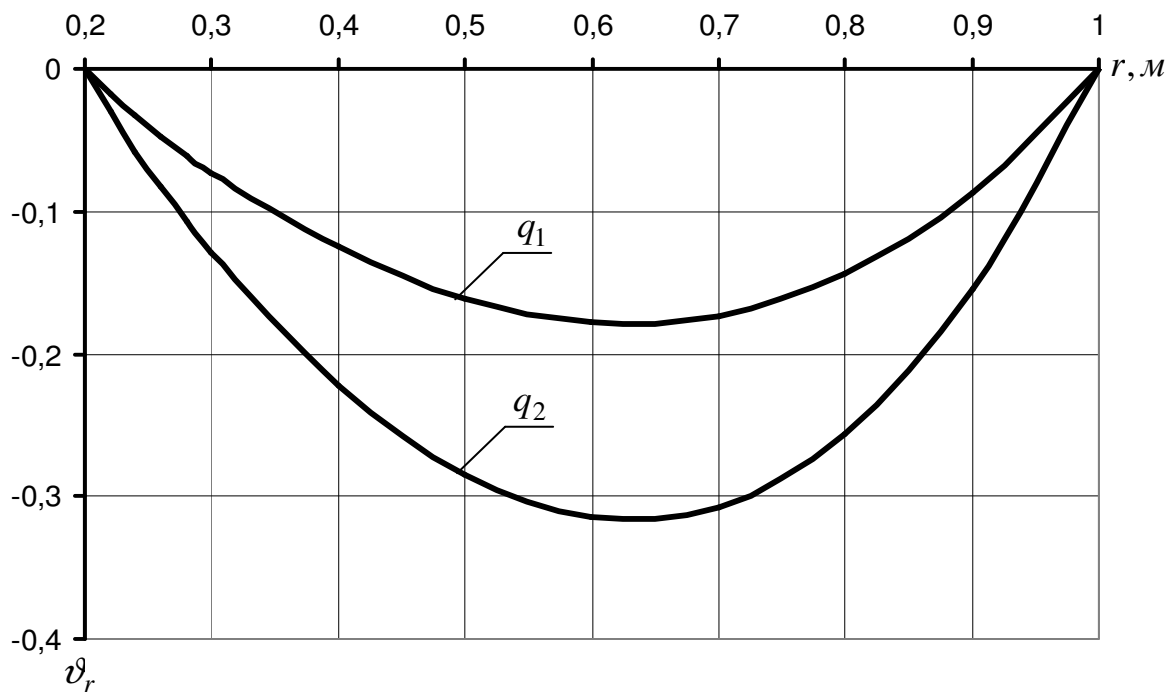
The radial force is determined by formula:

$$\begin{aligned} N_r = & \frac{Aq(r^2 - a^2)}{16} + \frac{A}{B} \frac{b(r^2 - a^2)}{r^2(b^2 - a^2)} \left\{ \frac{A}{4b} \sum_{k=1}^n (r_k + r_{k+1}) [r_k \vartheta_r^2(r_k) - r_{k+1} \vartheta_r^2(r_{k+1})] - \frac{A}{4} b \sum_{k=1}^n (r_k + r_{k+1}) \times \right. \\ & - \frac{A}{4} b \sum_{k=1}^n (r_k + r_{k+1}) \left[ \frac{\vartheta_r^2(r_k)}{r_k} - \frac{\vartheta_r^2(r_{k+1})}{r_{k+1}} \right] - \frac{Bqb(b^2 - a^2)}{16} + \frac{Bqba^2}{4} \ln \frac{b}{a} \left. \right\} - \frac{A}{4Br^2} \sum_{k=1}^n (r_k + r_{k+1}) \times \\ & [r_k \vartheta_r^2(r_k) H(r - r_k) - r_{k+1} \vartheta_r^2(r_{k+1}) H(r - r_{k+1})] + \frac{A}{4B} \sum_{k=1}^n (r_k + r_{k+1}) \left[ \frac{\vartheta_r^2(r_k)}{r_k} H(r - r_k) - \right. \\ & \left. - \frac{\vartheta_r^2(r_{k+1})}{r_{k+1}} H(r - r_{k+1}) \right] - \frac{Aqa^2}{4} \ln \frac{r}{a} \end{aligned} \quad (21)$$

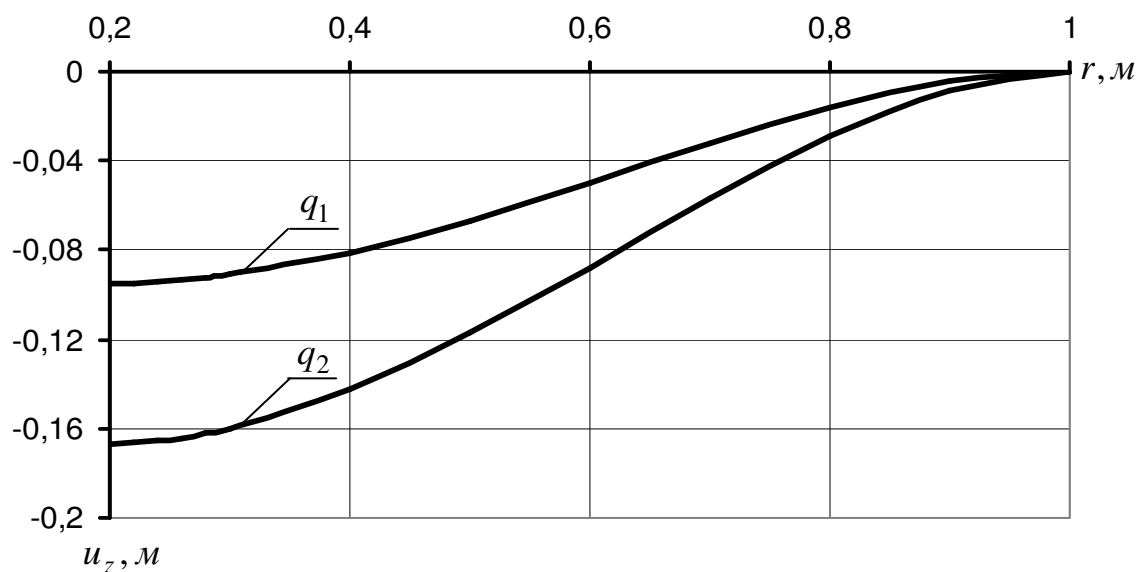
## RESULTS AND DISCUSSIONS

Figure 1 and Figure 2 show the graphs of change in the angle of deflection and normal rotation for two values of the load intensity. Numerical calculations are made for the case of linear variation of elastic modulus and Poisson's ratio of plate thickness for specific values of the plate thickness, load intensity, radius of the inner and outer contour of the plate.

If the effect of bending tension is not taken into account, the system of equations (1) becomes linear and obtaining of its analytic solution is not so difficult. Accounting for this effect also results to consideration of the coupled system of resolving equations with nonlinear terms (1), obtaining an analytical solution of which applying the existing mathematical apparatus was not possible. We managed to split the system (1) into two adequate to it, uncoupled nonlinear equations (3) - (4), and by applying the method of partial discretization to obtain a solution which satisfies the system of equations (1) and boundary conditions (11).



**Figure. 1.** Plate rotation angle curves exposed to non-uniform per thickness heating and uniform lateral load action at different values of  $q$ .



**Figure. 2.** Plate deflection curves exposed to non-uniform per thickness heating and uniform lateral load action at different values of  $q$ .

## SUMMARY

We managed to split coupled system of equations into two decoupled equations and to find an exact solution applying the method of partial discretization of differential equations. The research results are given in the form of formulas and graphs.

## REFERENCES

- [1] S. A. Korneev, "Thermodynamically consistent equations of state of the nonlinear theory of thermoelasticity", *Izvestiya RAN*, 2003, No. 2 – p.p. 71-82.
- [2] V. L. Bazhanov, I. I. Goldenblat, N. A. Nikolaenko and A. M. Sinyukov, "Structures thermal effects calculation", M.: Mechanical Engineering, 1969. - 599.
- [3] A. N. Tyurehodzhayev and B. ZH. Kyrykbaev – "Solution to the problem of bending of a flexible circular plate by applying a method of partial discretization of differential equations", *Izvestiya of the Academy of Sciences of RK*. - 2004, No. 3. – pp. 66-71.
- [4] A. N. Tyurehodzhayev, G. K. Kalzhanova, G. U. Mamatova, "Compound axisymmetric bending of an inhomogeneous circular plate with temperature change along the radius", *Actual problems of mechanics and mechanical engineering: collected papers of the International Scientific Conference*, Almaty, 2005. – pp. 348-353.
- [5] A. N. Tyurehodzhayev, G. K. Kalzhanova, "Analytical solution of problem on bending of inhomogeneous flexible circular plate in the non-uniform temperature field", *Herald of the Academy of Sciences of RK*. - 2005, No. 1. – pp. 24-34.
- [6] A. D. Kovalenko - *Fundamentals of thermoelasticity*. Kiev: "Naukova Dumka", (1970)