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MODELING OF SMART PIEZOELECTRIC COMPOSITE BEAMS USING A SIMPLE HIGHER ORDER SHEAR DEFORMATION THEORY

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ABSTRACT

The ability to change and control the shape of the structure has been a challenging problem. In the current work the shape control of isotropic as well as an orthotropic fiber-reinforced composite beam with embedded piezoelectric actuators is investigated. A finite element formulation is developed for modeling laminated composite beams with a distributed piezoelectric actuators subjected to both mechanical and electrical loads. A simple higher order shear deformation theory with virtual displacement method is used to formulate the equations of motion. The model is valid for both segmented and continuous piezoelectric elements which can be either surface bonded or embedded in the laminated beams. A two-node with four mechanical degrees of freedom is used in the finite element formulation. The electric potential is treated as a generalized electric coordinates like the generalized displacement coordinates at the mid-plane of the actuator. A MATLAB code is developed to compute the static deformations and the natural frequency of the structure system. The obtained results from the proposed model are compared to the available analytical and the finite element results of other researchers.

KEY WORDS

Finite element - piezoelectric materials –higher order beam theory – composite materials mechanics - smart structure system.

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NOMENCLATURE

Symbol	Definition
A	Beam cross section area.
A_{ij}	Elements of extensional stiffness matrix.
B_{ij}	Elements of coupling stiffness matrix.
b	Width of beam element.
C_{ijkl}	Elastic constants.
$c_1,\ c_2, c_3\ and\ \ c_4$	Constant coefficients.
D_{ij}	Elements of bending stiffness matrix.
D_i	Electric field potential energy.
E_k	Electric field ($E_k = -\nabla\varphi$).
E	Young's modulus for isotropic materials.
E_1	Young's modulus in the fiber direction.
E_2	Young's modulus in the transversal direction to the fiber.
e_{ijk}	Piezoelectric constituents' constants.
$\{F\}$	Element load vector.
$f_a\ ,\ f_t$	Axial and transversal forces.
G	Shear modulus.
H	Electric enthalpy.
h	Thickness of beam element.
$[K]$	Element stiffness matrix.
$\lfloor K_{qq} \rfloor$	Mechanical stiffness matrix.
$\lfloor K_{\varphi\varphi} \rfloor$	Electric stiffness matrix.
$\lfloor K_{q\varphi} \rfloor$	Coupled mechanical - electric stiffness matrix.
k	Layer number of the laminated beam.
L	Length of beam element.
$[M]$	Element Mass matrix.
N	Total number of layers in the laminated beam.
Q_{ij}	Components of the lamina stiffness matrix.
q , and $\ddot{q}$	Nodal displacements and its second derivative.
S_1	Surface area in x-y plane
T	Kinetic energy.
U	Total strain energy for the structure system
$\hat{U}$	Internal strain energy of the structure substrate.
U_e	Electric field potential energy.
u, v, w	Displacements of any point in the x-, y-, and z directions.
$u_1\ , u_2, u_3\ and\ u_4$	Axial displacements at the nodes of beam element.
$u^\circ, v^\circ,\ w^\circ$	Reference surface displacements along x-, y-, and z- axes.
W	Work done due to external loads.

$w_1, w_2, and w_3$	Transversal displacements at the nodes of beam element.
γ°_{xy}	Reference surface Transversal shear strain in x-z plane.
γ_{xz}	Shear strain in x-z plane.
ε^{s}_{ij}	Permittivity constants.
$\varepsilon_x,\ \varepsilon_y,\ \varepsilon_z$	Linear strains in the x-,y-, and x-directions.
$\varepsilon^{\circ}{}_{x},\ \varepsilon^{\circ}{}_{y}$	Reference surface extensional strains in the x-, and y-directions.
ζ_i	Transversal displacement shape functions.
$\kappa^{\circ}_x,\ \kappa^{\circ}_y$	Reference surface curvatures in the x-, and y-directions.
ξ_i	Axial displacement shape functions.
ρ	Mass of the structure material.
σ	Surface charge.
σ_x	Normal stress in the x-direction.
σ_{xz}	Shear stress in the x-z plane.
φ	Electric potential.
ϕ_x	Angle of rotation.
$\phi_1\ and\ \phi_2$	Rotation angles at nodes.
$\varsigma_1 and \varsigma_2$	Electrical potential shape functions.
ψ_i	Rotation displacement shape functions.
HSDT	Third-order shear deformation Theory.
SSDT	Second-order shear deformation Theory.
FSDT	First-order shear deformation Theory.
CBT	Classical beam Theory.

INTRODUCTION

The development of smart composite has generated great interest. Several researchers have studied the interaction between the mechanical properties and the electric field. Crawley et. al [1-2] developed piezoelectric elements for a laminated beams and plates. Allik and Hughes [3], presented a tetrahedral finite element for a three dimensional electroelasticity. Based on this model Tzou [4], proposed a method for solving isotropic plate using isoparametric hexahedron solid element.

Benjeddou et. al [5], presented theoretical formulation and finite element implementation of adaptive un-symmetric sandwich beams to deal with either extension or shear actuation mechanism for both static and dynamic analysis of beams. Shear actuators correspond to an elastic core sandwiched between two transversely polarized active surface layers, whereas extension actuators consist of an axially polarized core, sandwiched between two elastic surface layers. Vibration modes were found to be equivalent in both mechanisms, but shear actuators were found to be less deformed than extension ones. Their results defined features shear actuation mechanism over the conventional extension actuation mechanism, particularly for brittle piezo-ceramics.

Han et. al [6], proposed a refined finite element model which can effectively describe the variable in-plane displacements and stepped geometry. They concluded that the developed finite element computer program can effectively predict the characteristics of composite beams with a bonded piezo-ceramic actuator.

Clinton et. al [7], developed finite element model with two node Hermatian element and layer-wise nodes to get the static response of a smart beam with of n-layers. They deduced the following: (i) the linearity between tip displacements and the applied voltage of piezoelectric beam might not necessarily hold for other structural configurations with different properties and boundary conditions, (ii) as the substrate stiffness decreases the obtained actuation increases, (iii) the actuator positioning near the fixed end of a cantilever produces greater effect on the curvature, (iv) as the number of actuators increases the deflection and curvature increases.

Aldraihem and Khdeir [8] formulated and developed analytical models and exact solutions for beams with thickness-shear and extension piezoelectric actuators. Based on the first-order beam theory (FOBT) and higher-order beam theory (HOBT). They concluded that: (i) the obtained deflections from FOBT and HOBT are slight different for extension-mode actuator, and pronounced difference for shear-mode actuators. (ii) the obtained deflection for cantilever beam using HOBT is 27% greater than that of the FOBT however, the pattern of the deflection curves is the same. iii) the FOBT is very sensitive to the value of the shear correction factor k, for k=2/3, the tip deflection of a cantilever beam computed by the FOBT matches that computed by the HOBT. Iv) interesting results are obtained for clamped - clamped beam with shear-mode actuator, unlike the extension-mode actuator which cannot produce bending deflection in the clamped - clamped beam.

Khdeir and Osama [9] developed analytical solution for static analysis of smart beams with extension and shear mode actuators. They used the state-space concept in conjunction with the Jordan canonical form by making use of Heaviside discontinuity functions. They studied the effects of actuator length and location on the deflected shapes of the two structures. Their results showed good agreement with the extension mode actuators, but indicated for shear mode actuator.

Krommer [10], presented a simple Bernoulli–Euler type beam theory for smart piezoelectric composite beams. He formulated a purely mechanical beam theory taking into account the coupling to the electric field by means of the direct piezoelectric effect. His calculations showed the accuracy of one-dimensional theory in comparable to the two-dimensional finite element.

Vel and Baillargeon [11], proposed model using piezoelectric shear actuators/sensors where the electric field is applied perpendicular to the direction of polarization to cause shear deformation of the material. They presented exact analysis and active vibration suppression of piezoelectric laminated composite plates. In addition, experimental and finite element for active vibration suppression of a sandwich cantilever beams using piezoelectric shear actuators. Their finite element simulations showed good comparison with the experimental results.

Trindade and Maio [12], reported studies for using thickness-shear piezoelectric patches connected to resistive shunt circuits for the passive vibration control of sandwich beams. They developed theoretical and finite element formulations to evaluate the vibration damping performance. The obtained results for cantilever beam with shear piezoelectric patch connected to resistive shunt circuit gave a reduction up to 15 dB in resonant vibration amplitude of the third mode.

Yeilaghi et. al [13], developed finite element model based on the Euler-Bernoulli theory to study the shape and vibration control of the function graded materials beam with piezoelectric materials using new feedback control algorithm. Their results showed the effects of volume fraction on the shape and vibration of the beam. Furthermore, it is observed that the natural frequencies and peak responses can be controlled by the displacement control gain, and active damping can be provided by adjusting the velocity control gain.

Q. Wang et al [14], provide a basic mechanics model for the flexural analysis of a sandwich beam coupled with a piezoelectric layer. Their model is based on the Euler beam model for a long and thin beam structure, together with the electric potential satisfying the surface free charge condition for free vibration analysis. They concluded that the dynamic characteristics of the entire structure are related to the position of the piezoelectric layer. They investigated how the mode shape distribution of the electric potential in the piezoelectric layer in the longitudinal direction is related to the transverse displacement, and they concluded that the distribution of electric potential obtained serves as a guide for selecting the trial function for the mode shapes of the electric potential required in numerical methods for coupled piezoelectric structures.

In the present work, a finite element model is developed based on a simple higher order shear deformation theory made by Reddy [15]. The model represents the parabolic distribution of transverse shear stresses and the non-linearity of in-plane displacements across the thickness. The model is able to compute static and dynamic responses of laminated composite structures with distributed piezoelectric actuators.

THEORETICAL FORMULATION

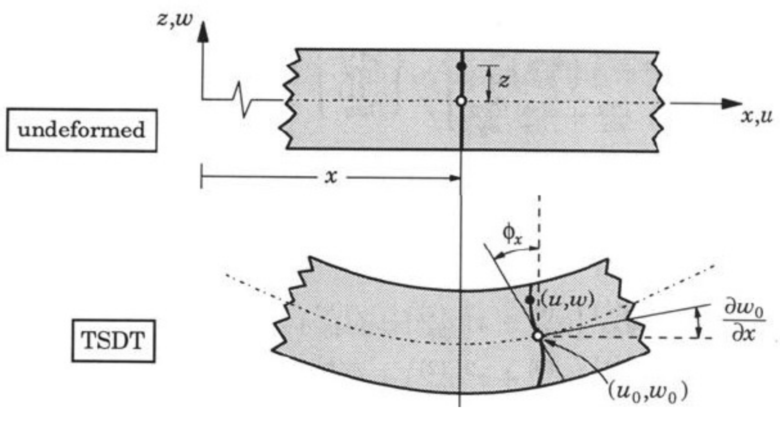

Figure (1): Un-deformed and Deformed cross section [16].

The displacements field equations of the beam are presented as [17]:

$$u(x,z)=u_{\circ}(x)+z\left[c_{\circ}\frac{dw}{dx}+c_1\phi(x)\right]+c_2z^2\psi(x)+c_3\left(z/h\right)^3\left[\phi(x)+\frac{dw}{dx}\right] \quad (1)\text{ a}$$

$$v(x,z)=0 \quad (1)\text{ b}$$

$$w(x,z)=w_{\circ}(x) \quad (1)\text{ c}$$

where $u, v,$ and w are the displacements field equations along the $x, y,$ and z coordinates, respectively, u_0 and w_o denote the displacements of a point $(x,y,0)$ at the mid plane, and $\phi(x)$ and $\psi(x)$ are the rotation angles of the cross-section as shown in Figure 1. Selecting the constant values of Eqn. (1) a as: $c_{\circ}=0,\quad c_1=1\quad c_2=0\quad c_3=-\left(4/3\right)h$. The displacements field equations for the simple higher order shear deformation theory, made by Reddy, at any point through the thickness can be expressed by [15]:

$$\begin{aligned} u(x,y,z) &= u_{\circ}(x)+z\,\phi_x-\frac{4}{3h^2}z^3\left[\phi_x+\frac{\partial w}{\partial x}\right] \\ v(x,y,z) &= 0 \\ w(x,y,z) &= w_{\circ}(x) \end{aligned} \quad (2)$$

The Reddy's displacements field accounts for a parabolic distribution of the shear strain and the non linearity of in-plane displacements across the thickness and thus does not involve the shear correction factor.

For a one-dimensional beam, the width in the y-direction is stress free, and from the plane stress assumption the remaining strains components are ε_{xx} and γ_{xz} which are represented by:

$$\varepsilon_{xx}(x,y,z)\equiv\frac{\partial u(x,z)}{\partial x}=\frac{\partial u}{\partial x}+z\frac{\partial\phi_x}{\partial x}-z^3\frac{4}{3h^2}\left[\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right]=\varepsilon_{\circ(x)}+z\kappa_{\circ(x)}+z^3\kappa_{2(x)} \quad (3)\text{ a}$$

$$\gamma_{xz}(x,y,z)\equiv\frac{\partial w(x,y,z)}{\partial x}+\frac{\partial u(x,y,z)}{\partial z}=\phi_x-z^2\frac{4}{h^2}\left[\phi_x+\frac{\partial w}{\partial x}\right]+\frac{\partial w}{\partial x}=\gamma_{\circ(xz)}+z^2\kappa_{2(xz)} \quad (3)\text{ b}$$

where;

$$\varepsilon_{\circ(x)}=\frac{\partial u}{\partial x},\ \gamma_{\circ(xz)}=\phi_x+\frac{\partial w}{\partial x},\ \kappa_{\circ(x)}=\frac{\partial\phi_x}{\partial x} \quad (4)\text{ a}$$

$$k_{2(x)}=-g_1\left[\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right],\quad k_{2(xz)}=-g_2\left[\phi_x+\frac{\partial w}{\partial x}\right] \quad (4)\text{ b}$$

and $g_1=\frac{4}{3h^2}$, $g_2=\frac{4}{h^2}$, $\varepsilon_{\circ(x)}$ is the reference surface extensional strain in the x-direction, $\gamma_{\circ(xz)}$ is the in-plane shear strain , $\kappa_{\circ(x)}$ and $\kappa_{2(x)}$ are the reference surface curvatures in the x-direction, $\kappa_{2(xz)}$ is the reference surface curvature in the z-direction. Thus the strains components ε_{xx}, and γ_{xz} can be expressed as:

$$\varepsilon_{xx}(x,y,z)=\varepsilon_{\circ(x)}+z\kappa_{\circ(x)}+z^3\kappa_{2(x)}$$

$$\gamma_{xz}(x,y,z)=\gamma_{\circ(xz)}+z^2\kappa_{2(xz)} \quad (5)$$

PIEZOELECTRIC CONSTIUTIVE RELATIONS

The piezoelectric constitutive equations are given by [18-19]:

$$\sigma_{ij}=c_{ijkl}\varepsilon_{kl}-e_{kij}E_k \quad (6)$$

$$D_i=e_{ikl}\varepsilon_{kl}+\varepsilon^s_{ik}E_k \quad (7)$$

where, $i,j=1,..,6$ and $k=1,..,3$.

In the present model the piezoelectric element is considered as isotropic material prior to the poling. But it behaves as transversely isotropic after the poling process. That is the 1-2 plane is isotropic, while the 3rd direction has properties different from the other two. For a k^{th} layer. The plane stress approximation is made by setting $\sigma_{33}=0$, and the strain ε_{33} is eliminated from Eqns. (6) and (7), the constitutive relations can be written in a matrix form for a material having orthorhombic mm^2 symmetry including piezoelectric effect as follows [20]:

$$\begin{Bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{23}\\\sigma_{13}\\\sigma_{12}\end{Bmatrix}_k=\begin{bmatrix}Q_{11}&Q_{12}&0&0&0\\Q_{12}&Q_{22}&0&0&0\\0&0&Q_{44}&0&0\\0&0&0&Q_{55}&0\\0&0&0&0&Q_{66}\end{bmatrix}_k\begin{Bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{23}\\\varepsilon_{13}\\\varepsilon_{12}\end{Bmatrix}_k-\begin{bmatrix}0&0&e'_{31}\\0&0&e'_{32}\\0&e'_{24}&0\\e'_{15}&0&0\\0&0&0\end{bmatrix}_k\begin{Bmatrix}E_1\\E_2\\E_3\end{Bmatrix}_k \quad (8)$$

$$\begin{Bmatrix}D_1\\D_2\\D_3\end{Bmatrix}_k=\begin{bmatrix}0&0&0&e'_{15}&0\\0&0&e'_{24}&0&0\\e'_{31}&e'_{32}&0&0&0\end{bmatrix}_k\begin{Bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{23}\\\varepsilon_{13}\\\varepsilon_{12}\end{Bmatrix}_k+\begin{bmatrix}\varepsilon'^s_{11}&0&0\\0&\varepsilon'^s_{22}&0\\0&0&\varepsilon'^s_{33}\end{bmatrix}_k\begin{Bmatrix}E_1\\E_2\\E_3\end{Bmatrix}_k \quad (9)$$

where; $\{D\}$ is the electric displacement vector (C/m^2), $\{\varepsilon\}$ strain vector, $[\varepsilon^s]$ is the dielectric matrix at constant mechanical strain (F/m), $\{E\}$ is the electric field vector (V/m), $\{\sigma\}$ is the stress vector (N/m^2) and $[Q]$ is the elasticity matrix for a constant electric field (N/m^2).

and, $$e'_{31}=e_{31}-\frac{c_{13}}{c_{33}}e_{33};\quad e'_{32}=e_{32}-\frac{c_{23}}{c_{33}}e_{33};\quad e'_{24}=e_{24};\quad e'_{15}=e_{15}; \quad (10)\ \text{a}$$

$$\varepsilon_{11}'^{s} = \varepsilon_{11}^{s}; \qquad \varepsilon_{22}'^{s} = \varepsilon_{22}; \qquad \varepsilon_{33}'^{s} = \varepsilon_{33} + \frac{e_{33}^{2}}{c_{33}} ;$$

where, c_{ij} are the elastic coefficients given in Appendix A [21].

The reduced stiffness coefficients Q_{ij} related to the engineering constants for two cases as follows [16]:

Case I: Isotropic Beam:

$$Q_{11}^{k} = Q_{22}^{k} = \frac{E}{1-\nu^{2}} \qquad Q_{12}^{k} = \frac{\nu E}{1-\nu^{2}} \qquad Q_{44}^{k} = Q_{55}^{k} = Q_{66}^{k} = G \tag{10 b}$$

Where; E , and ν are the isotropic material properties.

Case II: Anisotropic Beam:

$$\begin{gathered} Q_{11}^{k} = \frac{E_{1}^{k}}{1-\upsilon_{12}^{k}\upsilon_{21}^{k}} \qquad Q_{12}^{k} = \frac{\upsilon_{12}^{k}E_{2}^{k}}{1-\upsilon_{12}^{k}\upsilon_{21}^{k}} \qquad Q_{22}^{k} = \frac{E_{2}^{k}}{1-\upsilon_{12}^{k}\upsilon_{21}^{k}} \\ Q_{44}^{k} = G_{23} \qquad Q_{55}^{k} = G_{13}^{k} \qquad Q_{66}^{k} = G_{12}^{k} \end{gathered} \tag{10 c}$$

The electric field components are related to the electrostatic potential φ by the equation [22]:

$$E_{k} = -\varphi,_{k} \tag{11}$$

The transformation of Eqns. (8) and (9) can be written as [16]:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix}_{k} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & 0 & 0 & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & 0 & 0 & \overline{Q}_{26} \\ 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\ 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & 0 & 0 & \overline{Q}_{66} \end{bmatrix}_{k} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}_{k} - \begin{bmatrix} 0 & 0 & \overline{e}_{31} \\ 0 & 0 & \overline{e}_{32} \\ \overline{e}_{14} & \overline{e}_{24} & 0 \\ \overline{e}_{15} & \overline{e}_{25} & 0 \\ 0 & 0 & \overline{e}_{36} \end{bmatrix}_{k} \begin{Bmatrix} E_{x} \\ E_{y} \\ E_{z} \end{Bmatrix}_{k} \tag{12}$$

$$\begin{Bmatrix} D_{x} \\ D_{y} \\ D_{z} \end{Bmatrix}_{k} = \begin{bmatrix} 0 & 0 & \overline{e}_{14} & \overline{e}_{15} & 0 \\ 0 & 0 & \overline{e}_{24} & \overline{e}_{25} & 0 \\ \overline{e}_{31} & \overline{e}_{32} & 0 & 0 & \overline{e}_{36} \end{bmatrix}_{k} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}_{k} + \begin{bmatrix} \overline{\varepsilon}_{xx}^{s} & \overline{\varepsilon}_{xy}^{s} & 0 \\ \overline{\varepsilon}_{xy}^{s} & \overline{\varepsilon}_{yy}^{s} & 0 \\ 0 & 0 & \overline{\varepsilon}_{zz}^{s} \end{bmatrix}_{k} \begin{Bmatrix} E_{x} \\ E_{y} \\ E_{z} \end{Bmatrix}_{k} \tag{13}$$

where $\overline{Q}_{ij}$ and $\overline{e}_{ij}$ are the transformed reduced stiffness coefficients, and piezoelectric modules, respectively given in Appendix A.

In the proposed model the following assumptions are used: (1) the width in y direction is stress free and a plane stress assumption is used. Therefore, it is possible to set $\sigma_{yy} = \sigma_{yz} = \sigma_{xy} = \gamma_{yz} = \gamma_{xy} = 0$, and $\varepsilon_{yy} \neq 0$ in Eqn. (12) [8], (2) the polarization axis z is aligned with the thickness direction of the beam, thus only D_z in Eqn. (13) is taken into consideration. Thus the constitutive relations Eqn. (12) and Eqn. (13) are reduced to:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \\ D_z \end{Bmatrix}_k = \begin{bmatrix} \tilde{Q}_{11} & 0 & 0 \\ 0 & \tilde{Q}_{55} & 0 \\ \tilde{e}_{31} & 0 & 0 \end{bmatrix}_k \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \\ 0 \end{Bmatrix}_k - \begin{bmatrix} 0 & 0 & \tilde{e}_{31} \\ \tilde{e}_{15} & 0 & 0 \\ 0 & 0 & -\tilde{\varepsilon}^s_{zz} \end{bmatrix}_k \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}_k \tag{14}$$

where; the coefficients in Eqn. (14) are given by:

Case I: Isotropic Beam:

$$\tilde{Q}_{11} = E \qquad \tilde{Q}_{55} = G \qquad \text{, and} \qquad \overline{Q}_{ij} = Q_{ij} \tag{15) a}$$

Case II: Anisotropic Beam:

$$\tilde{Q}_{11} = \overline{Q}_{11} - \frac{\overline{Q}_{12}\overline{Q}_{12}}{\overline{Q}_{22}} \qquad \tilde{Q}_{55} = \overline{Q}_{55} - \frac{\overline{e}_{25}}{\overline{e}_{24}}\overline{Q}_{45} \tag{15) b}$$

And the piezoelectric coefficients are given by:

$$\tilde{\varepsilon}^s_{zz} = \overline{\varepsilon}^s_{zz} + \frac{\overline{e}_{32}\overline{e}_{32}}{\overline{Q}_{22}} \qquad \tilde{e}_{31} = \overline{e}_{31} - \overline{e}_{32}\frac{\overline{Q}_{12}}{\overline{Q}_{22}} \qquad \tilde{e}_{15} = \overline{e}_{15} - \frac{\overline{e}_{25}\overline{e}_{14}}{\overline{e}_{24}} \tag{15) c}$$

By introducing the electric field E_z applied across the thickness of the piezoelectric layers and the other components of the electric fields are zero. And the shear piezoelectric effect e_{15} and the axial electric permittivity ε^s_{11} are neglected. Therefore the constitutive relation, Eqs. (14) is simplified as follows:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \\ D_z \end{Bmatrix}_k = \begin{bmatrix} \tilde{Q}_{11} & 0 & 0 \\ 0 & \tilde{Q}_{55} & 0 \\ \tilde{e}_{31} & 0 & 0 \end{bmatrix}_k \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \\ 0 \end{Bmatrix}_k - \begin{bmatrix} \tilde{e}_{31} \\ 0 \\ -\tilde{\varepsilon}^s_{zz} \end{bmatrix} E_z \tag{16}$$

ENERGY FORMULATION

The kinetic energy of the structure system is given by [23]:

$$T = \frac{1}{2}\int_v \rho\left[\dot{u}^2 + \dot{w}^2\right]dv \tag{17}$$

where, ρ is the mass density of the beam material.

The work done due to external mechanical and electrical loads is represented by [23]:

$$W = \int_0^L f_a u dx + \int_0^L f_t w dx + P_i w_i - \int_{S_1} \sigma\varphi dS_1 \tag{18}$$

where; f_a and f_t are the transversal and axial forces along the length L, respectively. P_i is the concentrated force at point i and w_i is the corresponding generalized displacement, σ (C/m^2) is the surface charge density and φ is the electric potential (volt) applied to the piezoelectric surface areas S_1.

Thus the internal strain energy for the structure system U is the sum of internal strain energy $\hat{U}$, and the electric field potential energy U_e such as [20]:

$$U = \frac{1}{2}\int_v \left(\hat{U} + U_e\right) dv \tag{19}$$

Thus;

$$U = \frac{1}{2}\int_v \left[(\sigma_{xx}\varepsilon_{xx} + \sigma_{xz}\gamma_{xz}) - (D_z E_z)\right] dv \tag{20}$$

Case I: Isotropic Beam:

$$U = \frac{1}{2}\int_v \left[(E\varepsilon_{xx} - \tilde{e}_{31}E_z)\varepsilon_{xx} + (G\gamma_{xz})\gamma_{xz} - (\tilde{e}_{31}\varepsilon_{xx} + \tilde{\varepsilon}^s_{zz}E_z)E_z\right] dv \tag{21 a}$$

$$U = \frac{1}{2}\int_v \left[E\varepsilon_{xx}^2 + G\gamma_{xz}^2 - 2\tilde{e}_{31}E_z\varepsilon_{xx} - \tilde{\varepsilon}^s_{zz}E_z^2\right] dv \tag{21 b}$$

By substituting Eqns. (3)a and (3)b into Eqn. (21) b, one can obtain:

$$U = \int_v \left[\begin{array}{l} \dfrac{1}{2}\left[\begin{array}{l} E\left[\begin{array}{l} \left(\dfrac{\partial u_0}{\partial x}\right)^2 + 2z\left(\dfrac{\partial u_0}{\partial x}\dfrac{\partial \phi_x}{\partial x}\right) + z^2\left(\dfrac{\partial \phi_x}{\partial x}\right)^2 - \dfrac{8}{3h^2}z^3\left(\dfrac{\partial \phi_x}{\partial x} + \dfrac{\partial^2 w}{\partial x^2}\right)\left(\dfrac{\partial u}{\partial x}\right) \\ -\dfrac{8}{3h^2}z^4\left(\dfrac{\partial \phi_x}{\partial x}\right)\left(\dfrac{\partial \phi_x}{\partial x} + \dfrac{\partial^2 w}{\partial x^2}\right) + \dfrac{16}{9h^4}z^6\left(\dfrac{\partial \phi_x}{\partial x} + \dfrac{\partial^2 w}{\partial x^2}\right)^2 \end{array}\right] \\ + G\left[\left(\phi_x + \dfrac{\partial w}{\partial x}\right)^2 - \dfrac{8}{h^2}z^2\left(\phi_x + \dfrac{\partial w_0}{\partial x}\right)^2 + \dfrac{16}{h^4}z^4\left(\phi_x + \dfrac{\partial w_0}{\partial x}\right)^2\right] \end{array}\right] \\ -\left(-\dfrac{\partial \varphi}{\partial z}\right)\tilde{e}_{31}\left[\left(\dfrac{\partial u_0}{\partial x}\right) + z\left(\dfrac{\partial \phi_x}{\partial x}\right) - \dfrac{4}{3h^2}z^3\left(\dfrac{\partial \phi_x}{\partial x} + \dfrac{\partial^2 w}{\partial x^2}\right)\right] - \dfrac{1}{2}\tilde{\varepsilon}^s_{zz}\left[\left(\dfrac{\partial \varphi}{\partial z}\right)^2\right] \end{array}\right] dv \tag{22}$$

Case II: Anisotropic Beam:

$$U = \frac{1}{2}\int_v \left[\left(\tilde{Q}_{11}\varepsilon_{xx} - \tilde{e}_{31}E_z\right)\varepsilon_{xx} + \left(\tilde{Q}_{55}\gamma_{xz}\right)\gamma_{xz} - \left(\tilde{e}_{31}\varepsilon_{xx} + \tilde{\varepsilon}^s_{zz}E_z\right)E_z\right] dv \tag{23 a}$$

$$U = \frac{1}{2}\int_{v} \left[\left(\tilde{Q}_{11}\varepsilon_{xx}^{2} + \tilde{Q}_{55}\gamma_{xz}^{2} - 2\tilde{e}_{31}E_{z}\varepsilon_{xx} - \tilde{\varepsilon}_{zz}^{s}E_{z}^{2} \right) \right] dv \qquad \text{(23) b}$$

By substituting Eqns. (3) a and (3) b into Eqn. (23) b, one can obtain:

$$U = \int_{v} \left[\begin{array}{l} \frac{1}{2}\left[\begin{array}{l} \tilde{Q}_{11}\left[\begin{array}{l} \left(\frac{\partial u_0}{\partial x}\right)^2 + 2z\left(\frac{\partial u_0}{\partial x}\frac{\partial \phi_x}{\partial x}\right) + z^2\left(\frac{\partial \phi_x}{\partial x}\right)^2 - \frac{8}{3h^2}z^3\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial u}{\partial x}\right) \\ -\frac{8}{3h^2}z^4\left(\frac{\partial \phi_x}{\partial x}\right)\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + \frac{16}{9h^4}z^6\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)^2 \end{array} \right] \\ + \tilde{Q}_{55}\left[\left(\phi_x + \frac{\partial w}{\partial x}\right)^2 - \frac{8}{h^2}z^2\left(\phi_x + \frac{\partial w_0}{\partial x}\right)^2 + \frac{16}{h^4}z^4\left(\phi_x + \frac{\partial w_0}{\partial x}\right)^2 \right] \end{array} \right] \\ -\left(-\frac{\partial \varphi}{\partial z}\right)\tilde{e}_{31}\left[\left(\frac{\partial u_0}{\partial x}\right) + z\left(\frac{\partial \phi_x}{\partial x}\right) - \frac{4}{3h^2}z^3\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \right] - \frac{1}{2}\tilde{\varepsilon}_{zz}^{s}\left[\left(\frac{\partial \varphi}{\partial z}\right)^2\right] \end{array} \right] dv \qquad (24)$$

FINITE ELEMENT FORMULATION

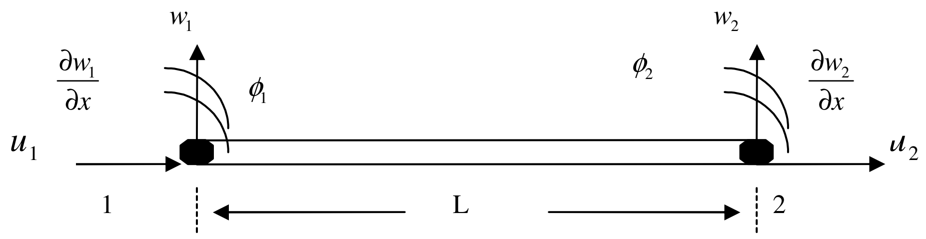

Figure (2): Nodal degrees of freedoms of the element.

The axial displacement at the mid-plane u_0 is expressed as the following:

$$\frac{\partial^2 u_0}{\partial x^2} = 0 \qquad (25)$$

By solving the previous equation and imposing the boundary conditions, the axial displacement can be represented as:

$$u_0(x) = u_1\xi_1 + u_2\xi_2 = \sum_{j=1}^{2} u_j \xi_j \qquad (26)$$

where the Linear interpolation shape functions ψ_j have the form [24]:

$$\xi_1 = 1 - \frac{x}{L} \text{ , and } \xi_2 = \frac{x}{L} \tag{27}$$

The transversal displacement w_0 at the mid-plane is represented as:

$$\frac{\partial^4 w_0}{\partial x^4} = 0 \tag{28}$$

By solving the above equation and applying the boundary conditions to determine the unknown constants, the transversal displacement w_0 at the mid-plane can be expressed in terms of the nodal displacement as:

$$w_0(x) = w_1\zeta_1 + w_2\zeta_2 + w_3\zeta_3 + w_4\zeta_4 = \sum_{j=1}^{4} w_j \zeta_j \tag{29}$$

The Hermit cubic shape functions ξ_j have the form [24]:

$$\begin{aligned} \zeta_1 &= 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 & \zeta_2 &= x - 2\left(\frac{x^2}{L}\right) + \frac{x^3}{L^2} \\ \zeta_3 &= 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 & \zeta_4 &= -\frac{x^2}{L} + \frac{x^3}{L^2} \end{aligned} \tag{30}$$

The rotation angle of the normal to the mid-plane about the y axis ϕ_x is represented by:

$$\frac{\partial^2 \phi_x}{\partial x^2} = 0 \tag{31}$$

Similarly; by solving Eqn. (31) and applying the boundary conditions the rotation angle is given by:

$$\phi_x(x) = \phi_{x1}\psi_1 + \phi_{x2}\psi_2 = \sum_{j=1}^{2} \phi_{xj}\psi_j \tag{32}$$

where the Linear interpolation shape functions ψ_j have the form [24]:

$$\psi_1 = 1 - \frac{x}{L} \text{ , and } \psi_2 = \frac{x}{L} \tag{33}$$

In the proposed model the electric potential is considered as a function of the thickness and the length of the beam [23]. In case of the electric potential is function of the length, it can be represented by [25]:

$$\varphi(x) = \varphi_1\hat{\varsigma}_1 + \varphi_2\hat{\varsigma}_2 = \sum_{j=1}^{2} \varphi_j \hat{\varsigma}_j \tag{34}$$

where,

$$\hat{\varsigma}_1 = 1 - \frac{x}{L} , \qquad \hat{\varsigma}_2 = \frac{x}{L} \tag{35}$$

And in case the electric potential is function of the thickness of the beam, it can be given as [25]:

$$\varphi(z) = \varphi_1 \breve{\varsigma}_1 + \varphi_2 \breve{\varsigma}_2 = \sum_{j=1}^{2} \varphi_j \breve{\varsigma}_j \tag{36}$$

where,

$$\breve{\varsigma}_1 = \frac{1}{2} + \frac{z}{h} \ , \qquad \breve{\varsigma}_2 = \frac{1}{2} - \frac{z}{h} \tag{37}$$

Thus by the product of equations (35) and (37) and impose the homogenous boundary condition on the bottom surface to eliminate the rigid body modes. Thus the electric potential can be written as [23]:

$$\varphi(x,0,z) = \varphi_1 \varsigma_1 + \varphi_2 \varsigma_2 = \sum_{j=1}^{2} \varphi_j \varsigma_j \tag{38}$$

And the shape functions are finally takes the form [23]:

$$\varsigma_1 = \left(\frac{1}{2} + \frac{z}{h}\right)\left(1 - \frac{x}{L}\right) \ ; \ \varsigma_2 = \left(\frac{1}{2} + \frac{z}{h}\right)\left(\frac{x}{L}\right) \tag{39}$$

VARIATIONAL FORMULATION

By applying the principle of the virtual displacements to a representative physical element of the beam, thus:

$$\delta U = \delta W + \delta T \tag{40}$$

By substituting Eqn. (2) into the kinetic energy Eqn. (17) yields:

$$T = \int_v \rho \left(\dot{u}_0 + z\dot{\phi}_x - \frac{4}{3h^2} z^3 \left(\dot{\phi}_x + \frac{d\dot{w}_0}{dx} \right) + \dot{w}_0 \right) dv \tag{41}$$

The first variation of the kinetic energy Eqn. (41) is expressed as:

$$\delta T = \int_v \rho_0 \begin{bmatrix} \delta\dot{u}_0^T \left(\dot{u}_0 + z\dot{\phi}_x - \frac{4}{3h^2} z^3 \dot{\phi}_x - \frac{4}{3h^2} z^3 \frac{d\dot{w}_0}{dx} \right) + \\ \delta\dot{\phi}_x^T \left(\left(z - \frac{4}{3h^2} z^3 \right) \dot{u}_0 + z^2 \dot{\phi}_x - \frac{8}{3h^2} z^4 \dot{\phi}_x + \frac{16}{9h^4} z^6 \dot{\phi}_x + \left(-\frac{4}{3h^2} z^4 + \frac{16}{9h^4} z^6 \right) \frac{d\dot{w}_0}{dx} \right) + \\ \delta\left(\frac{d\dot{w}_0}{dx}\right)^T \left(\left(-\frac{4}{3h^2} z^3 \right) \dot{u}_0 + \left(-\frac{4}{3h^2} z^4 + \frac{16}{9h^4} z^6 \right) \dot{\phi}_x + \frac{16}{9h^4} z^6 \frac{d\dot{w}_0}{dx} \right) + \delta\dot{w}_0^T (\dot{w}_0) \end{bmatrix} dv \tag{42}$$

Thus the elements of the mass matrix can be expressed as:

$$M_{11} = \int_0^L \delta\dot{u}_0^T I_0 \dot{u}_0 dx \qquad M_{12} = 0 \qquad M_{13} = 0$$
$$M_{21} = 0 \tag{43}$$

$$M_{22} = \int_0^L \delta\dot{\phi}_x^T \left(I_2 - \frac{8}{3h^2} I_4 + \frac{16}{9h^4} I_6 \right) \dot{\phi}_x dx$$

$$M_{23} = \int_0^L \delta\dot{\phi}_x^T \left(-\frac{4}{3h^2} I_4 + \frac{16}{9h^4} I_6 \right) \left(\frac{d\dot{w}_0}{dx} \right) dx$$

$$M_{31} = 0$$

$$M_{32} = \int_0^L \delta \left(\frac{d\dot{w}_0}{dx} \right)^T \left(-\frac{4}{3h^2} I_4 + \frac{16}{9h^4} I_6 \right) \dot{\phi}_x dx$$

$$M_{33} = \int_0^L \delta \left(\frac{d\dot{w}_0}{dx} \right)^T \left(\frac{16}{9h^4} I_6 \right) \left(\frac{d\dot{w}_0}{dx} \right) dx + \int_0^L \delta\dot{w}_0^T I_0 \dot{w}_0 dx$$

where;

$$(I_0, I_2, I_4, I_6) = \int_{-h/2}^{h/2} \rho_0 \left(1, z^2, z^4, z^6\right) dA$$

By substituting the shape functions equations (26), (29), and (32) into Eqn. (43) yield the mass matrix of the beam element.

The first variation of the external work equation (18) takes the form:

$$\delta W = \int_0^L f_a \delta u dx + \int_0^L f_t \delta w dx + P_i \delta w_i - \int_{S_1} \sigma \delta \varphi dS_1 \qquad (44)$$

By substituting the displacements Eqn (2) in Eqn.(44) yields;

$$\delta W = \int_0^L f_a \delta \left(u_0 + z\phi_x - \frac{4}{3h^2} z^3 \phi_x - \frac{4}{3h^2} z^3 \frac{dw_0}{dx} \right) dx + \int_0^L f_t \delta w_o + P_i \delta w_{oi} - \int_{S_1} \sigma \delta \varphi dS_1 \qquad (45)$$

By substituting the shape functions equations (26), (29), (32) and (38) into Eqn. (45) and perform the integration over the length of the beam yields the element load vector.

Case I: Isotropic Beam:

The first variation of the strain energy equation (22) takes the form:

$$\delta U=\left[\begin{array}{l}\left[E\left[\begin{array}{l}\left[\left(\frac{\partial u_0}{\partial x}\delta\frac{\partial u_0}{\partial x}\right)+z\left(\frac{\partial u_0}{\partial x}\delta\frac{\partial \phi_x}{\partial x}+\frac{\partial \phi_x}{\partial x}\delta\frac{\partial u_0}{\partial x}\right)+z^2\left(\frac{\partial \phi_x}{\partial x}\delta\frac{\partial \phi_x}{\partial x}\right)\right]\\ -\frac{4}{3h^2}z^3\left(\frac{\partial u}{\partial x}\delta\frac{\partial \phi_x}{\partial x}+\frac{\partial \phi_x}{\partial x}\delta\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\delta\frac{\partial^2 w}{\partial x}+\frac{\partial^2 w}{\partial x}\delta\frac{\partial u}{\partial x}\right)\\ -\frac{4}{3h^2}z^4\left(2\frac{\partial \phi_x}{\partial x}\delta\frac{\partial \phi_x}{\partial x}+\frac{\partial \phi_x}{\partial x}\delta\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial x}\delta\frac{\partial \phi_x}{\partial x}\right)\\ +\frac{16}{9h^4}z^6\left(\frac{\partial \phi_x}{\partial x}\delta\frac{\partial \phi_x}{\partial x}+\frac{\partial \phi_x}{\partial x}\delta\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\delta\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\delta\frac{\partial^2 w}{\partial x^2}\right)\end{array}\right]\right.\\ \left.+G\left[\left(1-\frac{8}{h^2}z^2+\frac{16}{h^4}\right)\left((\phi_x\delta\phi_x)+\left(\phi_x\delta\frac{\partial w_0}{\partial x}+\frac{\partial w_0}{\partial x}\delta\phi_x\right)+\left(\frac{\partial w_0}{\partial x}\delta\frac{\partial w_0}{\partial x}\right)\right)\right]\right]\\ +\tilde{e}_{31}\left[\begin{array}{l}\left(\frac{\partial u_0}{\partial x}\delta\frac{\partial \varphi}{\partial z}+\frac{\partial \varphi}{\partial z}\delta\frac{\partial u_0}{\partial x}\right)+z\left(\frac{\partial \phi_x}{\partial x}\delta\frac{\partial \varphi}{\partial z}+\frac{\partial \varphi}{\partial z}\delta\frac{\partial \phi_x}{\partial x}\right)\\ -\frac{4}{3h^2}z^3\left(\frac{\partial \phi_x}{\partial x}\delta\frac{\partial \varphi}{\partial z}+\frac{\partial \varphi}{\partial z}\delta\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\delta\frac{\partial \varphi}{\partial z}+\frac{\partial \varphi}{\partial z}\delta\frac{\partial^2 w}{\partial x^2}\right)\end{array}\right]-\tilde{\varepsilon}^s_{zz}\left[\left(\frac{\partial \varphi}{\partial z}\delta\frac{\partial \varphi}{\partial z}\right)\right]\end{array}\right]dv \quad (46)$$

Thus the element of the stiffness matrix can be given as:

$$k_{11}=E\int_v \delta\left(\frac{\partial u}{\partial x}\right)^T\left(\frac{\partial u}{\partial x}\right)dv$$

$$k_{12}=-E\int_v\left[z^3\frac{4}{3h^2}\delta\left(\frac{\partial u}{\partial x}\right)^T\left(\frac{\partial^2 w}{\partial x^2}\right)\right]dv$$

$$k_{13}=E\int_v\left[z\delta\left(\frac{\partial u}{\partial x}\right)^T\left(\frac{\partial \phi_x}{\partial x}\right)-z^3\frac{4}{3h^2}\delta\left(\frac{\partial u}{\partial x}\right)^T\left(\frac{\partial \phi_x}{\partial x}\right)\right]dv$$

$$k_{21}=-E\int_v\left[z^3\frac{4}{3h^2}\delta\left(\frac{\partial^2 w}{\partial x^2}\right)^T\left(\frac{\partial u}{\partial x}\right)\right]dv \quad (47)$$

$$k_{22}=\int_v\left\{E\left[z^6\frac{16}{9h^4}\delta\left(\frac{\partial^2 w}{\partial x^2}\right)^T\left(\frac{\partial^2 w}{\partial x^2}\right)\right]+G\left[\begin{array}{l}\delta\left(\frac{\partial w}{\partial x}\right)^T\left(\frac{\partial w}{\partial x}\right)-z^2\frac{8}{h^2}\delta\left(\frac{\partial w}{\partial x}\right)^T\left(\frac{\partial w}{\partial x}\right)\\ +z^4\frac{16}{h^4}\delta\left(\frac{\partial w}{\partial x}\right)^T\left(\frac{\partial w}{\partial x}\right)\end{array}\right]\right\}dv$$

$$k_{23}=\int_v\left\{\begin{array}{l}E\left[-z^4\frac{4}{3h^4}\delta\left(\frac{\partial^2 w}{\partial x^2}\right)^T\left(\frac{\partial \phi_x}{\partial x}\right)+z^6\left(\frac{16}{9h^4}\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)^T\left(\frac{\partial \phi_x}{\partial x}\right)\right]+\\ G\left[\delta\left(\frac{\partial w}{\partial x}\right)^T\phi_x-z^2\frac{8}{h^2}\delta\left(\frac{\partial w}{\partial x}\right)^T\phi_x+z^4\frac{16}{h^4}\delta\left(\frac{\partial w}{\partial x}\right)^T\phi_x\right]\end{array}\right\}dv$$

$$k_{31}=E\int_{v}\left[z\delta\left(\frac{\partial\phi_x}{\partial x}\right)^{T}\left(\frac{\partial u}{\partial x}\right)-z^{3}\frac{4}{3h^{2}}\delta\left(\frac{\partial\phi_x}{\partial x}\right)^{T}\left(\frac{\partial u}{\partial x}\right)\right]dv$$

$$k_{32}=\int_{v}\left\{\begin{array}{l}E\left[-z^{4}\frac{4}{3h^{2}}\delta\left(\frac{\partial\phi_x}{\partial x}\right)^{T}\left(\frac{\partial^{2}w}{\partial x^{2}}\right)+z^{6}\left(\frac{16}{9h^{4}}\right)\delta\left(\frac{\partial\phi_x}{\partial x}\right)^{T}\left(\frac{\partial^{2}w}{\partial x^{2}}\right)\right]+\\ G\left[\delta\phi_x^{T}\left(\frac{\partial w}{\partial x}\right)-z^{2}\frac{8}{h^{2}}\delta\phi_x^{T}\left(\frac{\partial w}{\partial x}\right)+z^{4}\frac{16}{h^{4}}\delta\phi_x^{T}\left(\frac{\partial w}{\partial x}\right)\right]\end{array}\right\}dv$$

$$k_{33}=\int_{v}\left\{\begin{array}{l}E\left[z^{2}\delta\left(\frac{\partial\phi_x}{\partial x}\right)^{T}\left(\frac{\partial\phi_x}{\partial x}\right)-z^{4}\frac{4}{3h^{2}}\delta\left(\frac{\partial\phi_x}{\partial x}\right)^{T}\left(\frac{\partial\phi_x}{\partial x}\right)+z^{6}\left(\frac{16}{9h^{4}}\right)\delta\left(\frac{\partial\phi_x}{\partial x}\right)^{T}\left(\frac{\partial\phi_x}{\partial x}\right)\right]+\\ G\left[\delta\phi^{T}\phi_x-z^{2}\frac{8}{h^{2}}\left(\delta\phi_x^{T}\phi_x+z^{4}\frac{16}{h^{4}}\delta\phi_x^{T}\phi_x\right)\right]\end{array}\right\}dv$$

$$k_{41}=\tilde{e}_{31}\int_{v}\delta\left(\frac{\partial\varphi}{\partial z}\right)^{T}\left(\frac{\partial u}{\partial x}\right)dv$$

$$k_{42}=-\int_{v}\frac{4}{3h^{2}}z^{3}\delta\left(\frac{\partial\varphi}{\partial z}\right)^{T}\left(\frac{\partial^{2}w}{\partial x^{2}}\right)dv$$

$$k_{43}=\tilde{e}_{31}\int_{v}\left[\delta\left(\frac{\partial\varphi}{\partial z}\right)^{T}\left(\frac{\partial\phi_x}{\partial x}\right)-z^{3}\frac{4}{3h^{2}}\delta\left(\frac{\partial\varphi}{\partial z}\right)^{T}\left(\frac{\partial\phi_x}{\partial z}\right)\right]dv$$

$$k_{14}=\tilde{e}_{31}\int_{v}\delta\left(\frac{\partial u}{\partial x}\right)^{T}\left(\frac{\partial\varphi}{\partial z}\right)dv$$

$$k_{24}=-\int_{v}\frac{4}{3h^{2}}z^{3}\delta\left(\frac{\partial^{2}w}{\partial x^{2}}\right)^{T}\left(\frac{\partial\varphi}{\partial z}\right)dv$$

$$k_{34}=\tilde{e}_{31}\int_{v}\left[z\delta\left(\frac{\partial\phi_x}{\partial x}\right)^{T}\left(\frac{\partial\varphi}{\partial z}\right)-z^{3}\frac{4}{3h^{2}}\delta\left(\frac{\partial\phi_x}{\partial x}\right)^{T}\left(\frac{\partial\varphi}{\partial z}\right)\right]dv$$

$$k_{44}=-\tilde{\varepsilon}_{33}\int_{v}\delta\left(\frac{\partial\varphi}{\partial z}\right)^{T}\left(\frac{\partial\varphi}{\partial z}\right)dv$$

By substituting the shape functions equations (26), (29), (32) and (38) into Eqn. (47) and perform the integration over the beam volume yields the element stiffness matrix for isotropic beam.

Case II: Anisotropic Beam:

The first variation of the strain energy equation (24) takes the form:

$$\delta U=\left[\begin{array}{l}\left[\tilde{Q}_{11}\left[\begin{array}{l}\left[\left(\frac{\partial u_0}{\partial x}\delta\frac{\partial u_0}{\partial x}\right)+z\left(\frac{\partial u_0}{\partial x}\delta\frac{\partial \phi_x}{\partial x}+\frac{\partial \phi_x}{\partial x}\delta\frac{\partial u_0}{\partial x}\right)+z^2\left(\frac{\partial \phi_x}{\partial x}\delta\frac{\partial \phi_x}{\partial x}\right)\right]\\ -\frac{4}{3h^2}z^3\left(\frac{\partial u}{\partial x}\delta\frac{\partial \phi_x}{\partial x}+\frac{\partial \phi_x}{\partial x}\delta\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\delta\frac{\partial^2 w}{\partial x}+\frac{\partial^2 w}{\partial x}\delta\frac{\partial u}{\partial x}\right)\\ -\frac{4}{3h^2}z^4\left(2\frac{\partial \phi_x}{\partial x}\delta\frac{\partial \phi_x}{\partial x}+\frac{\partial \phi_x}{\partial x}\delta\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial x}\delta\frac{\partial \phi_x}{\partial x}\right)\\ +\frac{16}{9h^4}z^6\left(\frac{\partial \phi_x}{\partial x}\delta\frac{\partial \phi_x}{\partial x}+\frac{\partial \phi_x}{\partial x}\delta\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\delta\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\delta\frac{\partial^2 w}{\partial x^2}\right)\end{array}\right]\right.\\ \left.+\tilde{Q}_{55}\left[\left(1-\frac{8}{h^2}z^2+\frac{16}{h^4}\right)\left((\phi_x\delta\phi_x)+\left(\phi_x\delta\frac{\partial w_0}{\partial x}+\frac{\partial w_0}{\partial x}\delta\phi_x\right)+\left(\frac{\partial w_0}{\partial x}\delta\frac{\partial w_0}{\partial x}\right)\right)\right]\right]\\ +\tilde{e}_{31}\left[\begin{array}{l}\left(\frac{\partial u_0}{\partial x}\delta\frac{\partial \varphi}{\partial z}+\frac{\partial \varphi}{\partial z}\delta\frac{\partial u_0}{\partial x}\right)+z\left(\frac{\partial \phi_x}{\partial x}\delta\frac{\partial \varphi}{\partial z}+\frac{\partial \varphi}{\partial z}\delta\frac{\partial \phi_x}{\partial x}\right)\\ -\frac{4}{3h^2}z^3\left(\frac{\partial \phi_x}{\partial x}\delta\frac{\partial \varphi}{\partial z}+\frac{\partial \varphi}{\partial z}\delta\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\delta\frac{\partial \varphi}{\partial z}+\frac{\partial \varphi}{\partial z}\delta\frac{\partial^2 w}{\partial x^2}\right)\end{array}\right]-\tilde{\varepsilon}^{s}_{zz}\left[\left(\frac{\partial \varphi}{\partial z}\delta\frac{\partial \varphi}{\partial z}\right)\right]\end{array}\right]dv \qquad (48)$$

Thus the element of the stiffness matrix can be given as:

$$k_{11}=A_{11}\int_A \delta\left(\frac{\partial u}{\partial x}\right)^T\left(\frac{\partial u}{\partial x}\right)dA$$

$$k_{12}=-E_{11}\int_A\left[\frac{4}{3h^2}\delta\left(\frac{\partial u}{\partial x}\right)^T\left(\frac{\partial^2 w}{\partial x^2}\right)\right]dA$$

$$k_{13}=\int_A\left[B_{11}\delta\left(\frac{\partial u}{\partial x}\right)^T\left(\frac{\partial \phi_x}{\partial x}\right)-E_{11}\frac{4}{3h^2}\delta\left(\frac{\partial u}{\partial x}\right)^T\left(\frac{\partial \phi_x}{\partial x}\right)\right]dA$$

$$k_{14}=\tilde{e}_{31}\int_v \delta\left(\frac{\partial u}{\partial x}\right)^T\left(\frac{\partial \varphi}{\partial z}\right)dv \qquad (49)$$

$$k_{21}=-\int_A\left[E_{11}\frac{4}{3h^2}\delta\left(\frac{\partial^2 w}{\partial x^2}\right)^T\left(\frac{\partial u}{\partial x}\right)\right]dA$$

$$k_{22}=\int_A\left\{\left[H_{11}\frac{16}{9h^4}\delta\left(\frac{\partial^2 w}{\partial x^2}\right)^T\left(\frac{\partial^2 w}{\partial x^2}\right)\right]+\left[\begin{array}{l}A_{55}\delta\left(\frac{\partial w}{\partial x}\right)^T\left(\frac{\partial w}{\partial x}\right)-D_{55}\frac{8}{h^2}\delta\left(\frac{\partial w}{\partial x}\right)^T\left(\frac{\partial w}{\partial x}\right)\\ +F_{55}\frac{16}{h^4}\delta\left(\frac{\partial w}{\partial x}\right)^T\left(\frac{\partial w}{\partial x}\right)\end{array}\right]\right\}dA$$

$$k_{23} = \int_A \left\{ \begin{bmatrix} -F_{11}\dfrac{4}{3h^4}\delta\left(\dfrac{\partial^2 w}{\partial x^2}\right)^T\left(\dfrac{\partial \phi_x}{\partial x}\right) + H_{11}\left(\dfrac{16}{9h^4}\right)\delta\left(\dfrac{\partial^2 w}{\partial x^2}\right)^T\left(\dfrac{\partial \phi_x}{\partial x}\right) \end{bmatrix} + \\ \begin{bmatrix} A_{55}\delta\left(\dfrac{\partial w}{\partial x}\right)^T\phi_x - D_{55}\dfrac{8}{h^2}\delta\left(\dfrac{\partial w}{\partial x}\right)^T\phi_x + F_{55}\dfrac{16}{h^4}\delta\left(\dfrac{\partial w}{\partial x}\right)^T\phi_x \end{bmatrix} \right\} dA$$

$$k_{24} = -\int_v \frac{4}{3h^2} z^3 \delta\left(\frac{\partial^2 w}{\partial x^2}\right)^T\left(\frac{\partial \varphi}{\partial z}\right) dv$$

$$k_{31} = \int_A \left[B_{11}\delta\left(\frac{\partial \phi_x}{\partial x}\right)^T\left(\frac{\partial u}{\partial x}\right) - E_{11}\frac{4}{3h^2}\delta\left(\frac{\partial \phi_x}{\partial x}\right)^T\left(\frac{\partial u}{\partial x}\right)\right] dA$$

$$k_{32} = \int_A \left\{ \begin{bmatrix} -F_{11}\dfrac{4}{3h^2}\delta\left(\dfrac{\partial \phi_x}{\partial x}\right)^T\left(\dfrac{\partial^2 w}{\partial x^2}\right) + H_{11}\left(\dfrac{16}{9h^4}\right)\delta\left(\dfrac{\partial \phi_x}{\partial x}\right)^T\left(\dfrac{\partial^2 w}{\partial x^2}\right) \end{bmatrix} + \\ \begin{bmatrix} A_{55}\delta\phi_x^T\left(\dfrac{\partial w}{\partial x}\right) - D_{55}\dfrac{8}{h^2}\delta\phi_x^T\left(\dfrac{\partial w}{\partial x}\right) + F_{55}\dfrac{16}{h^4}\delta\phi_x^T\left(\dfrac{\partial w}{\partial x}\right) \end{bmatrix} \right\} dA$$

$$k_{33} = \int_A \left\{ \begin{bmatrix} D_{11}\delta\left(\dfrac{\partial \phi_x}{\partial x}\right)^T\left(\dfrac{\partial \phi_x}{\partial x}\right) - F_{11}\dfrac{4}{3h^2}\delta\left(\dfrac{\partial \phi_x}{\partial x}\right)^T\left(\dfrac{\partial \phi_x}{\partial x}\right) + H_{11}\left(\dfrac{16}{9h^4}\right)\delta\left(\dfrac{\partial \phi_x}{\partial x}\right)^T\left(\dfrac{\partial \phi_x}{\partial x}\right) \end{bmatrix} + \\ \begin{bmatrix} A_{55}\delta\phi^T\phi_x - \dfrac{8}{h^2}\left(D_{55}\delta\phi_x^T\phi_x + F_{55}\dfrac{16}{h^4}\delta\phi_x^T\phi_x\right) \end{bmatrix} \right\} dA$$

$$k_{34} = \tilde{e}_{31}\int_v \left[z\delta\left(\frac{\partial \phi_x}{\partial x}\right)^T\left(\frac{\partial \varphi}{\partial z}\right) - z^3\frac{4}{3h^2}\delta\left(\frac{\partial \phi_x}{\partial x}\right)^T\left(\frac{\partial \varphi}{\partial z}\right)\right] dv$$

$$k_{41} = \tilde{e}_{31}\int_v \delta\left(\frac{\partial \varphi}{\partial z}\right)^T\left(\frac{\partial u}{\partial x}\right) dv$$

$$k_{42} = -\int_v \frac{4}{3h^2} z^3 \delta\left(\frac{\partial \varphi}{\partial z}\right)^T\left(\frac{\partial^2 w}{\partial x^2}\right) dv$$

$$k_{43} = \tilde{e}_{31}\int_v \left[\delta\left(\frac{\partial \varphi}{\partial z}\right)^T\left(\frac{\partial \phi_x}{\partial x}\right) - z^3\frac{4}{3h^2}\delta\left(\frac{\partial \varphi}{\partial z}\right)^T\left(\frac{\partial \phi_x}{\partial z}\right)\right] dv$$

$$k_{44} = -\tilde{\varepsilon}_{33}\int_v \delta\left(\frac{\partial \varphi}{\partial z}\right)^T\left(\frac{\partial \varphi}{\partial z}\right) dv$$

Known that , A_{11} and A_{55} are the extensional stiffness, B_{11} is the bending stiffness, D_{11} and D_{55} are the bending stiffness, E_{11} is the warping extension stiffness, F_{11} and F_{55} are the warping bending stiffness, and H_{11} is the higher order warping bending coupling stiffness and given by [15]:

$$\begin{gathered} (A_{11}, B_{11}, D_{11}, E_{11}, F_{11}, H_{11}) = \int_{-h/2}^{h/2} (\tilde{Q}_{11})_k (1, z, z^2, z^3, z^4, z^6) dz \\ (A_{55}, D_{55}, F_{55}) = \int_{-h/2}^{h/2} (\tilde{Q}_{55})_k (1, z^2, z^4) dz \end{gathered} \tag{50}$$

Where; k is the layer number. The distance h is taken from the middle surface of the laminate to the outer and inner surfaces of the k^{th} lamina.

By substituting the shape functions equations (26), (29), (32) and (38) into Eqn. (49) and perform the integration over the beam with length L, width b, and height h yields the element stiffness matrix for anisotropic beam.

EQUATION OF MOTION

The equation of motion of the whole structure system is represented by:

$$\begin{bmatrix} M_{qq} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \ddot{\varphi} \end{Bmatrix} + \begin{bmatrix} K_{qq} & K_{q\varphi} \\ K_{\varphi q} & K_{\varphi\varphi} \end{bmatrix} \begin{Bmatrix} q \\ \varphi \end{Bmatrix} = \begin{Bmatrix} F \\ G \end{Bmatrix} \tag{51}$$

where M_{uu} is the global mass matrix of the structure and $\{q\}$ is the global nodal generalized displacements coordinates vector, $\{\varphi\}$ is the global nodal generalized electric coordinates vector describing the applied voltages at the actuators [20-23], $\{F\}$ is the applied mechanical load vector, and $\{G\}$ is the electric excitation vector. In case of the piezoelectric layer working as sensor, the electric excitation applied to the sensor layer is zero **(**$G = 0$**),** the voltage from the sensor layer can be written as:

$$\{\varphi\} = -\left[K_{\varphi\varphi}\right]^{-1}\left[K_{\varphi q}\right]\{q\} \tag{52}$$

VALIDATION EXAMPLES

In the present study, a MATLAB code is constructed to perform the finite element analysis of isotropic and orthotropic smart beams with piezoelectric materials using a simple higher order shear deformation theory. The static deformation and the fundamental natural frequency are calculated for beams subjected to different kinds of mechanical and electrical loads.

The geometry of the smart beam, structure substrate, adhesive layer, and piezoelectric layer are shown in Figure 3. The length and width of the beam are L = 0.1524 m, $b = 2.54\times10^{-2} m$ respectively. The thicknesses of the substrate, PZT and adhesive are h =0.01524 m, $h_p = 1.524\times10^{-3} m$, $h_a = 0.254\times10^{-3} m$, respectively.

Model Convergences

The convergence of the present model is checked for a given beam with the material properties given in Table (1) for the beam substrate, adhesive and piezoelectric layers. A constant electric potential of 12.5 k-volts was applied on the upper surface of the PZT-4 layer, while the lower surface was grounded.

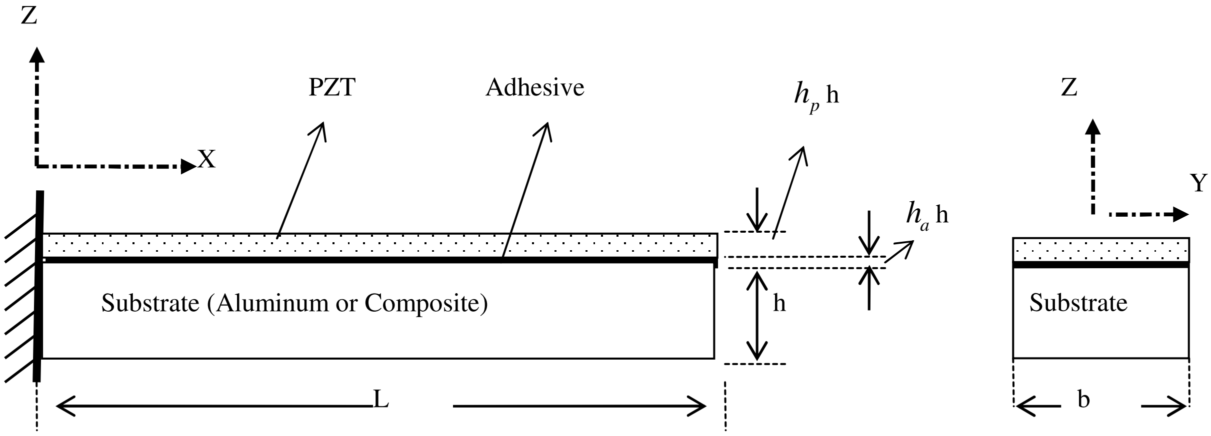

Figure 3: Smart Beam with PZT layer.

Table 1: Material properties of the smart structure.

Property	**Aluminum**	**Adhesive**
Young modulus	$E = 70.3 \quad GPa$	$E = 6.9 \quad GPa$
Shear modulus	$G = 27.6 \quad GPa$	$G = 2.46 \quad GPa$
Poisson ratio	$v = 0.345$	$v = 0.4$
Density	$\rho = 2769 \quad kg/m^3$	$\rho = 1662 \quad kg/m^3$
Piezoelectric		
Stiffness coefficient Q_{11}		13.9 10^{10} Pa
Stiffness coefficient Q_{12}		7.78 10^{10} $Pa.$
Stiffness coefficient Q_{33}		11.5 10^{10} $Pa.$
Stiffness coefficient Q_{13}		7.43 10^{10} $Pa.$
Stiffness coefficient Q_{44}		2.56 10^{10} $Pa.$
Piezoelectric constant e_{31}		-5 C/m^2
Piezoelectric constant e_{33}		15 C/m^2
Piezoelectric constant ε_{33}^{s}		5.6198 10^{-9} F/m
Density ρ		7600 kg/m^3

Figure 4 shows the effect of number of elements on the transverse displacement of an aluminum beam with piezoelectric actuator. The beam is composed of two layers of aluminum, one layer of adhesive and one layer of PZT-4 actuator. It can be seen that the transverse deflection reduces until it reaches an asymptotic value at reasonable number of elements which proves convergence of the finite element solution.

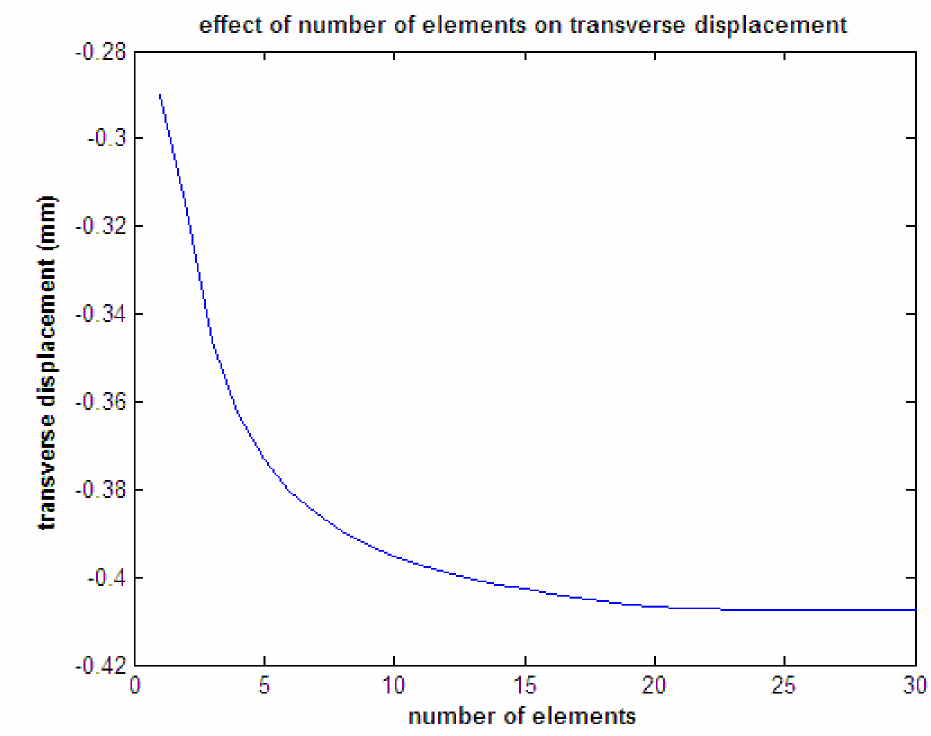

Figure 4: Effect of number of elements on the transverse displacement of Aluminum beam with piezoelectric actuator.

Case I: Isotropic Beam

Static validation

The transverse displacements obtained by the proposed model for cantilever beam composed of two layers of aluminum, and one layer of adhesive, and one layer of piezoelectric actuators are given in Figure (4) and compared with the results given in references [23-26-27].

The predicted mid-span deflection of cantilever beam shown in Table 2 is compared with other references as shown in Table 2. The obtained results are found almost identical however the Sarvano and Heyliger model [26] was based on layer wise method, and Elshafei and Bendary [33], whose models were based on the classical beam theory and Chee model [27] developed his model based on the third-order beam theory.

The difference between the static deflection calculated with Sar/Hey [26], and Chee [27] are found to be 8.2 % and 1.03 %, respectively.

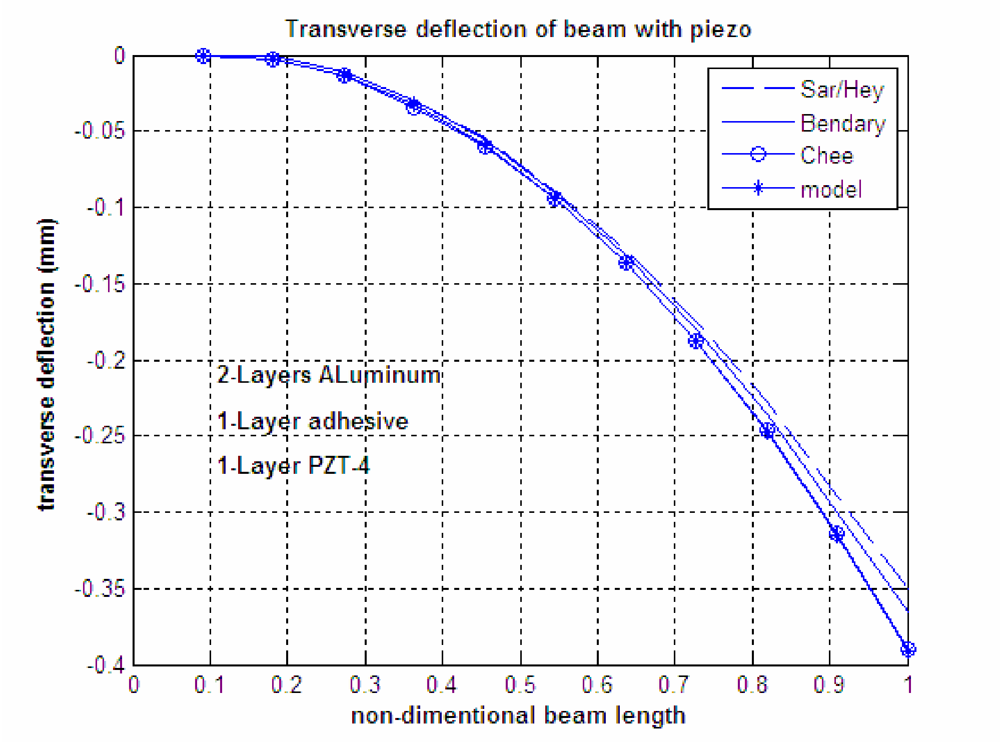

Figure 4: Transverse deflection of cantilever beam.

Table 2: mid-span deflection (mm) of cantilever beam.

References	**Ref. [23]**	**Ref. [26]**	**Ref. [27]**	**Present Model**
Mid-span deflection	0.093	0.089	0.094	0.0937

The effect of the applied voltage on the transverse displacement of the aluminum beam is shown in Figure 5. It is seen that as the voltage increases, the transverse displacement linearly increases.

Figure 5 presents the effect of the applied voltage on the transverse displacement of the aluminum beam. It is seen that as the voltage increases, the transverse displacement increases almost linearly.

Figure 6 presents the effect of applied voltages with different values on the transverse displacement of an aluminum beam with PZT-4 actuator under clamped-free boundary conditions. It is shown that as the voltage increases the transverse displacement increases.

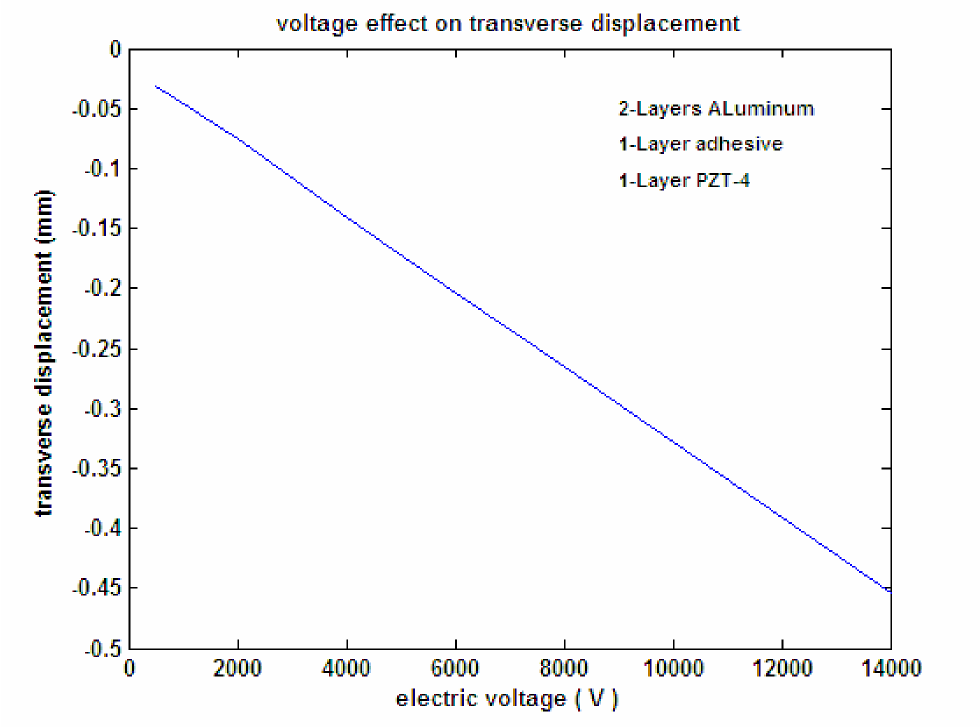

Figure 5: Applied voltage vs. transverse displacement.

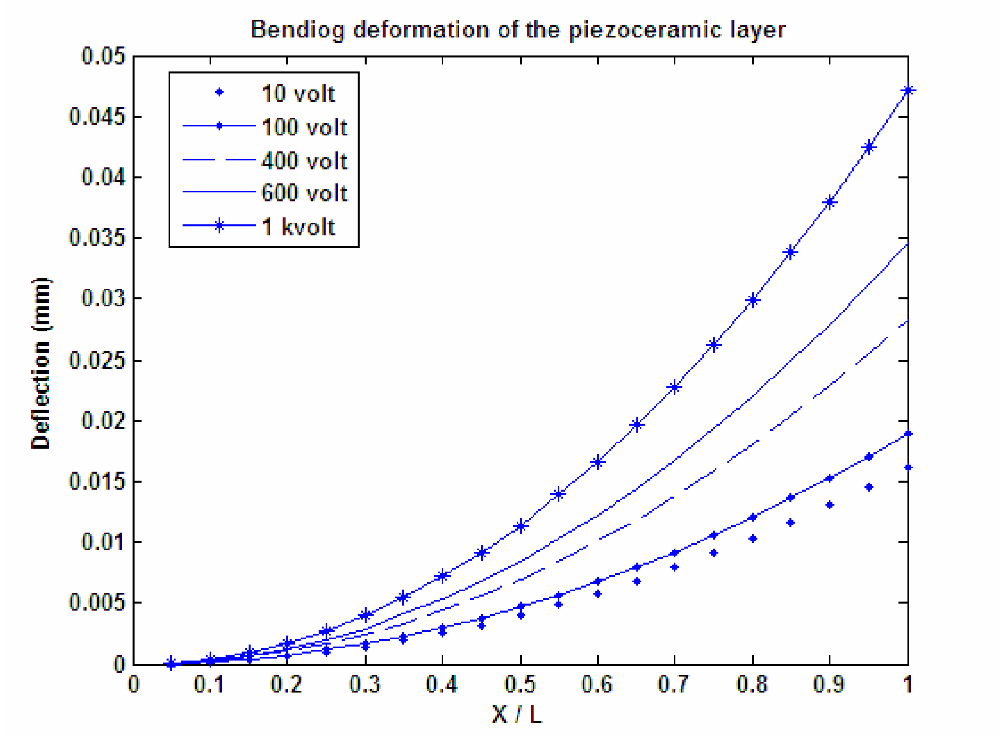

Figure 6: The effect of applied voltage value on transverse displacement.

Dynamic validation

Table 3 shows the first four natural frequencies (Hz) obtained by the proposed model compared with the results given in reference [26] for cantilever beam composed of two layers of aluminum, one layer of adhesive and one layer of PZT-4 actuator with material properties shown in Table 1. The obtained results are computed for different number of elements.

Table 3: First four natural frequencies (Hz) of aluminum beam compared with Ref.[26].

Mode	No of elements	Present Model	Reference [26]
1	10	556	567.1
	20	555	544.2
	30	555	544.1
2	10	3425	3287
	20	3419	3242
	30	3413	3232
3	10	7635	7254
	20	7629	7616
	30	7628	7614
4	10	9545	8976
	20	9327	8496
	30	9288	8428

The difference between the first three natural frequencies Table 3 was 1.94 %, 5.17 %, 0.17 %, respectively.

Case II: orthotropic Beam

Static Validation

The static deflection is performed for laminated graphite epoxy composite beam with the following material properties: $E_1 = 144.80$ GPa, $E_2 = 9.65$ GPa, $G_{12} = 4.14$ GPa, $\gamma_{12} = 0.3$ and $\rho = 1389.23 Kg / m^3$.

The effect of ply-orientation angle on the transverse displacement of cantilever graphite epoxy composite beams with PZT-4 given in Table 1 is shown in Figure 7. A constant electric potential of 500 volt was applied and the number of twenty elements is taken for this example. Results are reported.

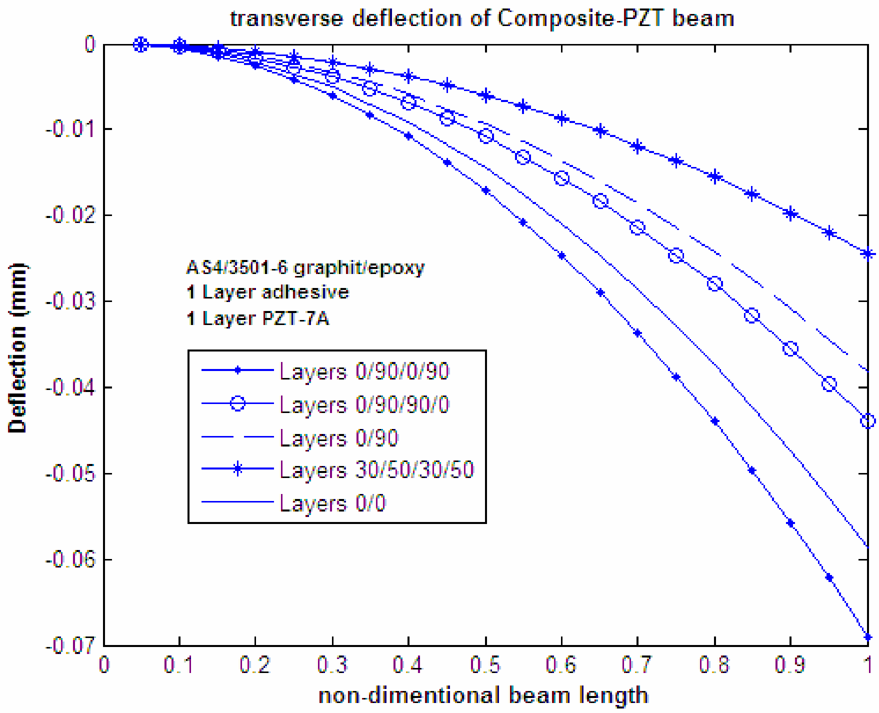

Figure 7: Effect of ply-orientation angle on smart orthotropic beams with PZT.

Dynamic Validation

The free vibration validation is performed for a graphite/epoxy cantilever beam of [0/90] stacking sequences for beam with various types of piezoelectric materials with properties shown in Table (4). The first three natural frequencies (Hz) for the beams are given in Table (5).

Table (4). Piezoelectric Characteristics.

Property	PZT-2	PZT-5A	PZT-8A	PZT-5H
Q_{11} $10^{10}\,Pa.$	13.5	12.1	14.9	12.6
Q_{12} $10^{10}\,Pa.$	6.79	7.54	8.11	7.95
Q_{33} $10^{10}\,Pa.$	11.3	11.1	13.2	11.7
Q_{13} $10^{10}\,Pa.$	6.81	7.52	8.11	8.41
Q_{44} $10^{10}\,Pa.$	2.22	2.11	3.13	2.3
e_{31} C/m^2	-1.86	-5.4	-4.1	-6.55
e_{33} C/m^2	9	15.8	14	23.3
$\varepsilon_{33}^{s}/\varepsilon_{\circ}$	260	830	600	1470
ρ kg/m^3	7600	7750	7600	7500

where; $\varepsilon_{\circ} = 8.85*10^{-12}\left(\dfrac{c^2}{N-m^2}\right)$

Table (5) Effect the type of PZT-Material on natural frequency.

Type of Material	First Mode	Second Mode	Third Mode
PZT-2	560	3450	7661
PZT-5A	535	3294	7499
PZT-8A	562	3464	7675
PZT-5H	533	3285	7491

CONCLUSION

A finite element model was proposed to predict the static and the free vibration characteristics of laminated aluminum and fiber reinforced composite beams with piezoelectric materials using a simple higher order shear deformation theory made by Reddy. The following conclusions have been drawn:

1. The good agreement was found between the present model predictions using a simple higher order shear deformation theory, and the corresponding predicted results of other investigators using Euler-Bernoulli's beam theory, layer wise theory with different models, and HODT modeling, proved the predictive capabilities of such model with less computational effort.
2. The developed displacements model can explain the parabolic distribution of the transverse shear stresses as an advantage over the classical laminated theory which neglects the effects of transverse shear stresses.
3. The proposed model using Reedy method did not suffer from the shear correction factors which are problematic in the first order shear deformation theory.
4. The number of degrees of freedom of the element in the present model is one third of the number of degrees of freedom of the element in the model developed by the higher order shear deformation theory, which, of course, save the computational time.
5. The validity of representing the electric potential shape function at each node as a function of the thickness and the length of the beam in the proposed finite element model using the Reddy theory.
6. The proposed finite element model results were obtained at resizable number of elements
7. As the applied voltage increases, both the transverse and axial displacements increase, respectively.
8. As the number of layers increases, the transverse deflection and the natural frequencies decrease.
9. The types of the piezoelectric materials affect the obtained natural frequencies of the beams.
10. The model can be extended to the following:
    (a) Studying the forced vibration analysis for both mechanical and electrical loads applied to the beams using the same theory.
    (b) Taking into account the geometric nonlinearities in the finite element model which may improve the obtained results.

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Appendix A

The elastic coefficients c_{ij} are given by [21]:

$$c_{11} = \frac{1-\upsilon_{23}\upsilon_{32}}{E_2 E_3 \Delta} \qquad c_{12} = \frac{\upsilon_{21}+\upsilon_{31}\upsilon_{23}}{E_2 E_3 \Delta} = \frac{\upsilon_{12}+\upsilon_{32}\upsilon_{13}}{E_1 E_3 \Delta} \quad \textbf{(A-1)}$$

$$c_{13} = \frac{\upsilon_{31}+\upsilon_{21}\upsilon_{32}}{E_2 E_3 \Delta} = \frac{\upsilon_{13}+\upsilon_{12}\upsilon_{23}}{E_1 E_2 \Delta} \qquad c_{22} = \frac{1-\upsilon_{13}\upsilon_{31}}{E_1 E_3 \Delta}$$

$$c_{23} = \frac{\upsilon_{32}+\upsilon_{12}\upsilon_{31}}{E_1 E_3 \Delta} = \frac{\upsilon_{23}+\upsilon_{21}\upsilon_{13}}{E_1 E_2 \Delta} \qquad c_{33} = \frac{1-\upsilon_{12}\upsilon_{21}}{E_1 E_2 \Delta}$$

where; $c_{44} = G_{23}$ $\qquad c_{55} = G_{31}$ $\qquad c_{66} = G_{12}$

$$\Delta = \frac{1-\upsilon_{12}\upsilon_{21}-\upsilon_{23}\upsilon_{32}-\upsilon_{31}\upsilon_{13}-2\upsilon_{21}\upsilon_{32}\upsilon_{13}}{E_1 E_2 E_3}$$

The transformed reduced stiffness coefficients $\overline{Q}_{ij}$ are represented by [14]:

$$\overline{\mathrm{Q}}_{11} = Q_{11}\cos^4\theta + 2(\mathrm{Q}_{12}+2Q_{44})\cos^2\theta\sin^2\theta + Q_{22}\sin^4\theta \qquad \text{(A-2)}$$

$$\overline{\mathrm{Q}}_{12} = (Q_{11}+Q_{22}-4Q_{44})\cos^2\theta\ \sin^2\theta + Q_{12}(\cos^4\theta+\sin^4\theta)$$

$$\overline{Q}_{14} = (Q_{11} - 2Q_{44} - Q_{12})\cos^3\theta\sin\theta + (Q_{12} - Q_{22} + 2Q_{44})\cos\theta\sin^3\theta$$
$$\overline{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{44})\cos^2\theta\sin^2\theta + Q_{22}\cos^4\theta$$
$$\overline{Q}_{24} = (Q_{11} - 2Q_{44} - Q_{12})\cos\theta\sin^3\theta + (Q_{12} - Q_{22} + 2Q_{44})\cos^3\theta\sin\theta$$
$$\overline{Q}_{44} = (Q_{11} + Q_{22} - 2Q_{44})\cos^2\theta\sin^2\theta + Q_{44}(\cos^4 + \sin^4\theta)$$
$$\overline{Q}_{55} = Q_{55}\cos^2\theta + Q_{66}\sin^2\theta$$
$$\overline{Q}_{56} = (Q_{55} - Q_{66})\cos\theta\sin\theta$$
$$\overline{Q}_{66} = Q_{55}\sin^2\theta + Q_{66}\cos^2\theta$$

The transformed piezoelectric modules $\bar{e}_{ij}$. And the transformed dielectric oefficients $\bar{\varepsilon}^s_{xx}, \bar{\varepsilon}^s_{yy}$, and $\bar{\varepsilon}^s_{zz}$, are represented by [14]:

$$\bar{e}_{31} = e'_{31}\cos^2\theta + e'_{32}\sin^2\theta \qquad \bar{\varepsilon}^s_{xx} = \varepsilon'^s_{11}\cos^2\theta + \varepsilon'^s_{22}\sin^2\theta \tag{A-3}$$
$$\bar{e}_{32} = e'_{31}\sin^2\theta + e'_{32}\cos^2\theta \qquad \bar{\varepsilon}^s_{yy} = \varepsilon'^s_{11}\sin^2\theta + \varepsilon'^s_{22}\cos^2\theta$$
$$\bar{e}_{33} = e'_{33} \qquad \bar{\varepsilon}^s_{xy} = \left(\varepsilon'^s_{11} - \varepsilon'^s_{22}\right)\sin\theta\cos\theta$$
$$\bar{e}_{36} = \left(e'_{31} - e'_{32}\right)\sin\theta\cos\theta \qquad \bar{\varepsilon}^s_{zz} = \bar{\varepsilon}'_{33}$$
$$\bar{e}_{14} = \left(e'_{15} - e'_{24}\right)\sin\theta\cos\theta$$
$$\bar{e}_{24} = e'_{24}\cos^2\theta + e'_{15}\sin^2\theta$$
$$\bar{e}_{15} = e'_{15}\cos^2\theta + e'_{24}\sin^2\theta$$
$$\bar{e}_{25} = \left(e'_{15} - e'_{24}\right)\sin\theta\cos\theta$$
