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Proposed an Algorithm for Solving a Special Case of Multi-Index Multistage Transportation Problem

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Abstract

The transportation problem is a special case from applications of linear programming and plays a vital role in the logistics supply chain. Its primary objective is to minimize the total cost of transportation for a network that spans multiple sources and destinations while ensuring that the sources are available, and the destinations are fulfilled. This paper presents the mathematical formulations for four types of multi-index multistage transportation problems (MMTP1, MMTP2, MMTP3, and MMTP4), which cover the various conditions that apply to operating trucks fleets in real transportation networks. The algorithm introduced here is designed to solve MMTP1, a multi-index multistage transportation problem without any restrictions on transportation in the intermediate stages. The algorithm transforms the problem into a multi-index single-stage problem using dynamic programming to find the minimum transportation cost for each means of transport between the first stage's sources and the last stage's destinations. The LINGO software is then used to validate the optimal solution. An illustrative example is provided to demonstrate the effectiveness of the proposed algorithm.

KEYWORDS: Transportation Problems, Multi-Index Transportation Problem, Multistage Transportation Problem, Dynamic programming

1 Introduction

The classical transportation problem can be described as the problem of finding the optimal distribution of certain commodities from many sources with certain availabilities to many destinations with certain requirements to minimize the total transportation costs or/and any other required objective which could be minimizing transportation time, minimizing of deteriorations, or maximizing profit [1, 2]. “The transportation problem discovered by F. L. Hitchcock and presented by T. C. Koopmans for that it is known as Hitchcock Koopmans transportation problem” [3].

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It can be expressed as a linear programming problem. The key data needed to solve the problem include the availability of sources, the demand of destinations, and the unit transportation cost between the various sources and destinations. Several algorithms have been created to find the best solution to the transportation problem. Many computerized software programs have also been developed to solve the transportation problem more efficiently and in less time. For example, LINGO software has proven its effectiveness in solving many transportation problems and addressing large-scale problems in a simple and short time [4]. The transportation problem is characterized by two indices representing the sources and destinations of the network. In real-world scenarios, additional variables may be introduced, leading to a multi-index problem. This occurs when there are different types of commodities or transportation modes, resulting in a three-index or four-index problem, depending on the complexity of the variables involved.

There are many studies and investigations that have been made on multi-index transportation problems. K.B. Haley [5] presented the multi-index transportation problem and the necessary conditions required for solving the problem. Moravek and Vlach [6] presented the mathematical model for three-index transportation problems and illustrated the necessary conditions required for finding the problem solution. An algorithm for three-index problem based on reducibility of the problem to flow problem is proposed by Afraimovich [7]. Jasim and Aljanabi [8] introduced a solution method for solving the three-index transportation problem. Gurwinder Singh et al. [9] modified an optimization algorithm to solve the multi-index transportation problem.

In real world problems, the processes of transportation are done in more than one stage. So most real transportation problems are multistage problems. Ellaimony et al. [10] presented the bi-criteria multistage transportation problem with the mathematical model for all types of the problem as the author classified it to four types which are BMTP1, BMTP2, BMTP3 and BMTP4. An introduction to dynamic programming is also presented.

This paper presents formulations for various types of multi-index multistage transportation problems (MMTP1, MMTP2, MMTP3, and MMTP4). Additionally, it introduces an algorithm for solving MMTP1, a multi-index multistage transportation problem without transportation restrictions on intermediate stages. The algorithm utilizes dynamic programming in the first solution phase to identify the shortest route between the first stage' sources and the last stage' destinations for each mode of transport, transforming the problem into a single-stage multi-index transportation problem. Given the current trend towards using computer software for solving real problems, especially large-scale ones, to save time and effort compared to manual techniques, LINGO software was employed to solve the second stage of the problem. This decision was based on the software's proven efficiency in swiftly solving various transportation problems. The importance of this study summarized in introducing the formulations for different types of transportation problem which cover almost the majority of conditions that could be found in real transportation networks which help researchers to provide the optimal decisions for decision makers in several industrial and engineering sectors.

2 Methods/Experimental

2.1 The Aim

This study presents the formulations for different types of multi-index multistage transportation problems that are classified into four types which are MMTP1, MMTP2, MMTP3 and MMTP4 which cover many conditions that apply to operating trucks fleets in real transportation networks. An algorithm for solving the first type (MMTP1) introduced and applied on an illustrative example to prove its efficiency.

2.2 Processes and Methodologies

The multi-index multistage transportation problem studied in this paper as the formulation of four types of it which are MMTP1, MMTP2, MMTP3 and MMTP4 are represented.
An algorithm for solving MMTP1 is introduced in two phases. Dynamic programming technique applied in phase one to find the minimum transportation cost for each means of transport between the first stage sources and the last stage destinations which represents the shortest path in the network. In phase two, Lingo software applied to find the optimal distribution in the network after it was transformed into multi-index single stage problem using backward recursion dynamic programming technique.

2.3 Mathematical Formulation of Multi-index Transportation Problems

The multi-index transportation problems can be solved using common techniques used for solving the classical transportation problem such as the simplex method, Stepping-Stone Method, Modified Distribution Method (MODI) and other techniques that developed specially to solving the multi-index problems by many researchers [9, 11].

The mathematical formulation of the multi-index transportation problem represented as follow [12]:

Minimize:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} c_{ijk}x_{ijk} \tag{1}$$

Subject to:

$$\sum_{j=1}^{n} x_{ijk} = a_{ik}\ ,(i = 1,2,\dots,m);(k = 1,2,\dots,l) \tag{2}$$

$$\sum_{i=1}^{m} x_{ijk} = b_{jk}\ ,(j = 1,2,\dots,n);(k = 1,2,\dots,l) \tag{3}$$

$$\sum_{k=1}^{l} x_{ijk} = e_{ij}\ ,(i = 1,2,\dots,m);(j = 1,2,\dots,n) \tag{4}$$

Where:

$$x_{ijk} \geq 0\ , \tag{5}$$

$$\sum_{i=1}^{m} a_{ik} = \sum_{j=1}^{n} b_{jk}, \qquad (6)$$

$$\sum_{k=1}^{l} a_{ik} = \sum_{j=1}^{n} e_{ij}, \qquad (7)$$

$$\sum_{i=1}^{m} e_{ij} = \sum_{k=1}^{l} b_{jk} \qquad (8)$$

$Where$:

c_{ijk} $Is\ the\ cost\ of\ transporting\ one\ unit\ of\ one\ type\ of\ different\ products\ or\ by$ One type of different transport from source i to destination j.

x_{ijk} $Is\ the\ amount\ transported\ from\ source\ i\ to\ destiantion\ j\ for\ product\ k\ or\ by$ $transport$ k.

a_{ik} $Is\ the\ availablity\ at\ source\ i\ \ for\ product\ k.$

b_{jk} $Is\ the\ demand\ at\ destination\ j\ for\ product\ k.$

e_{ij} $Is\ the\ quantity\ of\ product\ k\ or\ the\ \ load\ capacity\ of\ transport\ k$

2.4 Formulations of Different Kinds of Multi-Index Multistage Transportation Problem

In this section four types of multi-index multistage transportation problem are presented with their mathematical forms. The four types almost cover the different real problems conditions for operating trucks fleet which give the researchers and decision makers an overview of the problem.

Multi-Index Multistage Transportation Problem of The First Kind (MMTP1).

This kind is a multi-index multistage transportation problem without any transportation restrictions on the intermediate stages which means that no availabilities and no requirements at the nodes of these stages.

The mathematical formula for this type represented as follow:

$$Minimize\ z = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} c_{ijk}\, x_{ijk} \qquad (9)$$

$Subject\ to$:

$$\sum_{j=1}^{n} x_{ijk} = a_{ik}\,, (i = 1, 2, \dots, m); (k = 1, 2, \dots, l) \qquad (10)$$

$$\sum_{i=1}^{m} x_{ijk} = b_{jk}\,, (j = 1, 2, \dots, n); (k = 1, 2, \dots., l) \qquad (11)$$

$$\sum_{k=1}^{l} x_{ijk} = e_{ij}\,, (i = 1, 2, \dots, m); (j = 1, 2, \dots, n) \qquad (12)$$

$$x_{ijk} \geq 0 \quad for\ all\ i, j, k. \qquad (13)$$

$Where$:

c_{ijk} $Is\ the\ minimum\ transportation\ costs\ between\ the\ first\ stage\ sources\ and\ the\ last\ stage\ destinations\ for\ different\ types\ of\ products\ or\ by\ different\ types\ of\ tarnsport.$

x_{ijk} $Is\ the\ amount\ transported\ from\ source\ i\ to\ destiantion\ j\ for\ product\ k\ or\ by\ transport$ k.

a_{ik} $Is\ the\ availabilities\ of\ the\ first\ stage\ sources\ for\ product\ k$

b_{ik} $Is\ the\ demands\ of\ the\ last\ stage\ destinations\ for\ product\ k$

e_{ij} $Is\ the\ quantity\ of\ product\ k\ or\ the\ load\ capacity\ of\ transport\ k$

Multi-Index Multistage Transportation Problem of The Second Kind (MMTP2).

This kind represents multi-index multistage transportation problem at which any stage can be treated as an independent multi-index transportation problem.

The mathematical formula for this type represented as follow:

$$Minimize\ z^s = \sum_{i_s=1}^{m_s} \sum_{j_s=1}^{n_s} \sum_{k_s=1}^{l_s} c^s_{i_s j_s k_s}\, x^s_{i_s j_s k_s} \tag{14}$$

$Subject\ to$:

$$\sum_{j_s=1}^{n_s} x^s_{i_s j_s k_s} = a^s_{i_s k_s}\,, (i_s = 1, 2, \dots, m_s); (k_s = 1, 2, \dots, l_s) \tag{15}$$

$$\sum_{i_s=1}^{m_s} x^s_{i_s j_s k_s} = b^s_{j_s k_s}, (j_s = 1, 2, \dots, n_s); (k_s = 1, 2, \dots, l_s) \tag{16}$$

$$\sum_{k_s=1}^{l_s} x^s_{i_s j_s k_s} = e^s_{i_s j_s}\,, (i_s = 1, 2, \dots, m_s); (j_s = 1, 2, \dots, n_s) \tag{17}$$

$$x^s_{i_s j_s k_s} \geq o \quad for\ all\ i_s\,, j_s\,, k_s. \tag{18}$$

$Where$:

$$s = 1, 2, \dots N \qquad which\ represent\ the\ number\ of\ stage$$

Multi-Index Multistage Transportation Problem of The Third Kind (MMTP3).

This type represents multi-index multistage transportation problem with some additional restrictions on the intermediate stages which does not affect the problem formulation.

The mathematical formula for this type represented as follow:

$$Minimize\ z = \sum_{s=1}^{N} \sum_{i_s=1}^{m_s} \sum_{j_s=1}^{n_s} \sum_{k_s=1}^{l_s} c^s_{i_s j_s k_s}\, x^s_{i_s j_s k_s} \tag{19}$$

$Subject\ to$:

$$\sum_{j_s=1}^{n_s} x^s_{i_s j_s k_s} = a^s_{i_s k_s}, (i_s = 1,2,\dots,m_s); (k_s = 1,2,\dots,l_s) \tag{20}$$

$$\sum_{i_s=1}^{m_s} x^s_{i_s j_s k_s} = b^s_{j_s k_s}, (j_s = 1,2,\dots,m_s); (k_s = 1,2,\dots,l_s) \tag{21}$$

$$\sum_{k_s=1}^{l_s} x^s_{i_s j_s k_s} = e^s_{i_s j_s}, (i_s = 1,2,\dots,m_s); (j_s = 1,2,\dots,n_s) \tag{22}$$

$$F_{r_s}\left(x^{s-1}_{i_{s-1} j_{s-1}}, x^s_{i_s j_s}, x^{s+1}_{i_{s+1} j_{s+1}}\right) = b^0; \tag{23}$$

$$x^s_{i_s j_s k_s} \geq 0 \ for\ all\ i_s, j_s, k_s, \qquad s = 1,2,\dots,N. \tag{24}$$

Where:

F_{r_s} *are linear function representing the additional transportation restrictions at the N stages and* r_s *is the number of this linear function at the sth stage*

Multi-Index Multistage Transportation Problem of The Fourth Kind (MMTP4).

This type represents multi-index multistage transportation problem with transportation restrictions that affect the formulation of the problem at each stage as there is a difference between the input and the output of the commodities transported to sources (destinations) at each stage. The mathematical formula for this type represented as follow:

$$\textit{Minimize } z = \sum_{s=1}^{N} \sum_{i_s=1}^{m_s} \sum_{j_s=1}^{n_s} \sum_{k_s=1}^{l_s} c^s_{i_s j_s k_s} x^s_{i_s j_s k_s} \tag{25}$$

Subject to:

$$\sum_{j_1=1}^{n_1} x^1_{i_1 j_1 k_1} = a^1_{i_1 k_1}, (i_1 = 1,2,\dots,m_1); (k_1 = 1,2,\dots,l_1) \tag{26}$$

$$\sum_{i_{s-1}=1}^{m_{s-1}} x^{s-1}_{i_{s-1} j_{s-1} k_{s-1}} - \sum_{i_s=1}^{m_s} x^s_{i_s j_s k_s} = b^{s-1}_{j_{s-1} k_{s-1}} \tag{27}$$

$$, (j_{s-1} = 1,2,\dots,n_{s-1});\ (k_{s-1} = 1,2,\dots,l_{s-1});\ s = 1,2,\dots,N$$

$$\sum_{i_N=1}^{m_N} x^N_{i_N j_N k_N} = b^N_{j_N k_N} (, j_N = 1,2,\dots,n_N); (k_N = 1,2,\dots,l_N) \tag{28}$$

$$\sum_{k_s=1}^{l_s} x^s_{i_s j_s k_s} = e^s_{i_s j_s}, (i_s = 1,2,\dots,m_s); (j_s = 1,2,\dots,n_s) \tag{29}$$

$$x^{s}_{i_s j_s} \geq 0 \;\; for\; all\; i_s\,, j_s\,, k_s \qquad (30)$$

$$S = 1, 2, \dots, N$$

2.5 Solution Techniques of The Different Multi-Index Multistage Problem Types

The solution algorithm of MMTP1, MMTP2, MMTP3 and MMTP4 are mentioned in this section.

MMTP1 Solution Algorithm. This kind can be solved as a single stage multi-index transportation problem after applying the dynamic programming technique to find the minimum transportation cost for each mean of transport between the first stage's sources and the last stage's destinations which represent the shortest path in the network.

MMTP2 Solution Algorithm. This type can be solved as N multi-index single stage problem and as the minimum transportation cost for each stage could be obtained individually then the minimum total transportation cost of the problem obtained as follow:

$$z = \sum_{s=1}^{N} z^{s} \qquad (31)$$

Where:

z^{s} *is the optimal solution (minmum transportation cost) for each stage s.*

MMTP3 Solution Algorithm. The decomposition technique used for solving this kind.

MMTP4 Solution Algorithm. This type can be solved using any linear programming solution techniques.

2.6 Dynamic Programming.

Dynamic programming is an optimization technique that used for solving many optimization problems and making a sequence for correlating decisions. It doesn't have a standard mathematical formulation as it considers as an optimization strategy, it divides the problem into subproblems which allow for obtaining the optimum solution in a simpler way after solving each subproblem.

The solution technique of the dynamic programming relies on a recursive nature. Recursive computations carried out in one of two ways which are forward recursion and backward recursion. Dynamic programming used backward recursion as it is more efficient so the solution of problems using D.P. obtained by moving from the end of problem toward the beginning which results in dividing the problem into smaller and simpler subproblems.

We applied the dynamic programming for solving MMTP1 and finding the shortest path between the first stage sources and the last stage destinations which represent the minimum transportation cost for the transportation network as the technique prove its efficiency in solving the problem over any other technique beside it requires lower number of iterations compared to other techniques. The number of problems required to be solved using backward D.P. equal the total number of destinations (n) at the last stage which results in lower number of computations to solve the problem compared to other techniques that requires solving (m*n) problems where m is the total number of sources at the first stage. As a result, dynamic programming is an efficient, easy, and fast optimization technique.

2.7 The Proposed Algorithm for Solving MMTP1.

The proposed solution algorithm for solving MMTP1 consist of two solution phases. In phase one the problem transformed into single stage problem by finding the minimum transportation cost for each mean of transport between the first stage sources and the last destinations as we can apply the backward recursion dynamic programming technique for finding these values where the solution started from the last stage destinations and moving towards the beginning of the problem and the number of shortest route problem depend on the number of the last stage destinations. After that the problem is transformed to a multi-index single stage problem and in this phase LINGO software applied to find the optimal network distribution that achieve the minimum transportation cost in the network.

Dynamic Programming Backward Recursive Equation. The dynamic programming backward recursive equation for the shortest route problem is illustrated in equation (32) which calculate the minimum transportation cost to a certain destination from all the sources at the last stage, and equation (33) which calculate the minimum transportation cost at all destinations from all sources at all other stages expect the last stage as follow:

$$\boldsymbol{For\ all\ (j_N = 1, 2, 3, \dots, n_N) : calculate}$$

$$\boldsymbol{F^N(i_N) = c^N_{i_N j_N}\ , i_N = 1, 2, \dots, m_N} \tag{32}$$

$$\boldsymbol{F^s(i_s) = min.\left\{c^s_{i_s j_s} + F^{s+1}(i_s)\right\};} \tag{33}$$

$$\boldsymbol{j_s = 1, 2, \dots, n_s\ ; i_s = 1, 2, \dots, m_s\ ; s = N-1, N-2, \dots, 1}$$

$\boldsymbol{Where}$:

$\boldsymbol{F^N(i_N)}$ $represents\ the\ optimum\ value\ of cost\ for\ each\ source\ at\ the\ last\ stage\ (N)$

$\boldsymbol{F^s(i_s)}$ $represents\ the\ optimum\ value\ of\ cost\ for\ each\ source\ at\ any\ stage$

$expect\ the\ last\ stage$

$\boldsymbol{c^N_{i_N j_N}}$ $is\ the\ transportation\ cost\ at\ the\ last\ stage (N)\ from\ its\ sources\ i_N\ to\ its$

$destunations\ j_N$

$\boldsymbol{c^s_{i_s j_s}}$ $is\ the\ trasnportation\ cost\ at\ any\ stage\ (s)\ from\ its\ source\ i_k\ to\ its$

$destinations\ j_k$

$\boldsymbol{i_N}$ $is\ the\ source\ number\ at\ the\ last\ stage\ (N)$

$\boldsymbol{j_N}$ $is\ the\ destination\ number\ at\ the\ last\ stage\ (N)$

$\boldsymbol{i_s}$ $is\ the\ source\ number\ at\ any\ stage\ (s)\ expect\ the\ last\ stage$

$\boldsymbol{j_s}$ $is\ the\ destination\ number\ at\ any\ stage\ (s)\ expect\ the\ last\ stage$

$\boldsymbol{m_N}$ *is the total number of sources at the last stage* (N)

$\boldsymbol{n_N}$ *is the total number of destinations at the last stage* (N)

$\boldsymbol{m_s}$ *is the total number of sources at any stage expect the last stage* (s)

$\boldsymbol{n_s}$ is the total number of destinations at any stage expect the last stage (s)

3 Illustrative Example.

3.1 The Problem.

The following figure represents the transportation network of the problem. A three stages transportation network with three main sources at the first stage H_1, H_2 *and* H_3 with availabilities equal those the values of $(a_1, a_2$ *and* $a_3)$. The requirements of the last stage's destinations H_8 *and* H_9 equal the values of $(b_1$ *and* $b_2)$ while the middle stages have no transportation restrictions on the availabilities or requirements. Two means of transport used K_1 *and* K_2 with load capacity 50 tons for the first one and 70 tons for the second. The values of the transportation costs per unit while transporting the product from any source i_s to any destination j_s through any stage (s) using the two means of transport are presented in tables 1, 2 and 3 as $(c_{i_s j_s k_1}, c_{i_s j_s k_2})$. It is required to find the optimal network distribution that minimize the total transportation costs.

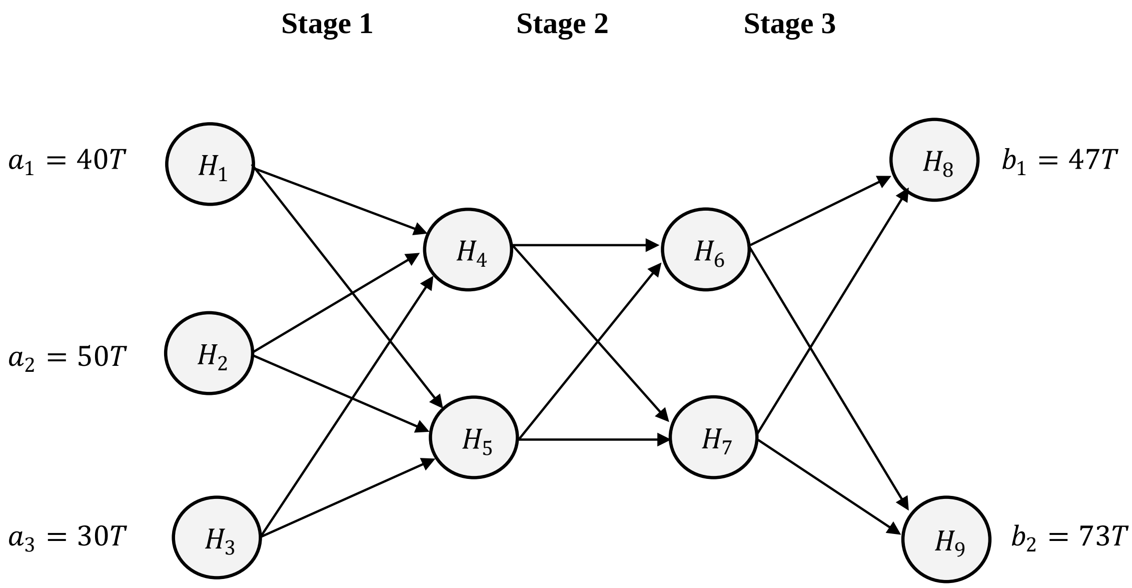

Fig. 1. The illustrative example transportation network.

Table 1. Costs of transportation in dollars per ton kilometer for K_1 *and* K_2 respectively & sources availabilities and destinations requirements in tons for stage 1.

Destinations / Sources	H_4	H_5	Availabilities
H_1	(16,8)	(3,13)	40
H_2	(4,10)	(5,5)	50
H_3	(7,6)	(5,3)	30
Requirements	No limit	No limit	

Table 2. Costs of transportation in dollars per ton kilometer for K_1 and K_2 respectively & sources availabilities and destinations requirements in tons for stage 2.

Destinations / Sources	H_6	H_7	Availabilities
H_4	(6,8)	(5,2)	No limit
H_5	(14,7)	(13,15)	No limit
Requirements	No limit	No limit	

Table 3. Costs of transportation in dollars per ton kilometer for K_1 and K_2 respectively & sources availabilities and destinations requirements in tons for stage 3.

Destinations / Sources	H_8	H_9	Availabilities
H_6	(13,9)	(10,3)	No limit
H_7	(5,12)	(8,12)	No limit
Requirements	47	73	

3.2 Problem Solution (Phase 1).

In this example backward dynamic programming technique is applied to find the minimum transportation cost between the first stage main sources and the last stage destinations for each mean of transport so the problem can be solved as a single stage problem after that. Since the number of the last stage destinations in this example is two destinations so we need to solve two D.P. problems for each transport (K). For more simplification, we applied the D.P. solution steps to find the shortest routes for just one destination H_8 to sources H_1, H_2 and H_3 by K_1 and K_2 in tables 4 and 5, where the same procedures is applied to find the shortest routes from destination H_9 to sources H_1, H_2 and H_3 by K_1 and K_2 after that. Solution steps showed as follow:

Table 4. Dynamic programming technique steps to find the shortest routes from $(H_1, H_2$ and $H_3)$ to H_8 by transport K_1 .

Stage 3 for destination H_8 and transport K_1:

State (S_3)	Alternatives	Optimal	solution
	H_8	$F_3(X_3)$	D_3^*
H_6	13	13	H_8
H_7	5	5	H_8

Stage 2 for destination H_8 and transport K_1:

State (S_2)	Alternatives		Optimal	solution
	H_6	H_7	$F_2(X_2)$	D_2^*
H_4	6+13=19	5+5=10	10	H_7
H_5	14+13=27	13+5=18	18	H_7

Stage 1 for destination H_8 and transport K_1:

State (S_1)	Alternatives		Optimal	solution
	H_4	H_5	$F_1(X_1)$	D_1^*
H_1	16+10=26	3+18=21	21	H_5
H_2	4+10=14	5+18=23	14	H_4
H_3	7+10=17	5+18=23	17	H_4

Table 5. Dynamic programming technique steps to find the shortest routes from $(H_1, H_2 \; and \; H_3)$ to H_8 by transport K_2 .

Stage 3 for destination H_8 and transport K_2:

State (S_3)	Alternatives	Optimal	solution
	H_8	$F_3(X_3)$	D_3^*
H_6	9	9	H_8
H_7	12	12	H_8

Stage 2 for destination H_8 and transport K_2:

State (S_2)	Alternatives		Optimal	solution
	H_6	H_7	$F_2(X_2)$	D_2^*
H_4	8+9=17	2+12=14	14	H_7
H_5	7+9=16	15+12=27	16	H_6

Stage 1 for destination H_8 and transport K_2:

State (S_1)	Alternatives		Optimal solution	
	H_4	H_5	$F_1(X_1)$	D_1^*
H_1	8+14=22	13+16=29	22	H_4
H_2	10+14=24	5+16=21	21	H_5
H_3	6+14=20	3+16=19	19	H_5

After applying phase 1, the three stages transportation network is converted into a single stage network and the problem now can be solved as a multi-index single stage transportation problem. The results of solving the four dynamic programming problems and the equivalent transportation network outlined in table 6 and figure 2 as follow:

Table 6. Dynamic programming results for finding the shortest path for destinations H_8 *and* H_9 by the two transports K_1 *and* K_2.

Destination	Transport (K_1)		Transport (K_2)	
	Shortest Route	Min. Cost	Shortest Route	Min. Cost
H_8	$H_1 - H_5 - H_7 - H_8$ $H_2 - H_4 - H_7 - H_8$ $H_3 - H_4 - H_7 - H_8$	21 14 17	$H_1 - H_4 - H_7 - H_8$ $H_2 - H_5 - H_6 - H_8$ $H_3 - H_5 - H_6 - H_8$	22 21 19
H_9	$H_1 - H_5 - H_7 - H_9$ $H_2 - H_4 - H_7 - H_9$ $H_3 - H_4 - H_7 - H_9$	24 17 20	$H_1 - H_4 - H_6 - H_9$ $H_2 - H_5 - H_6 - H_9$ $H_3 - H_5 - H_6 - H_9$	19 15 13

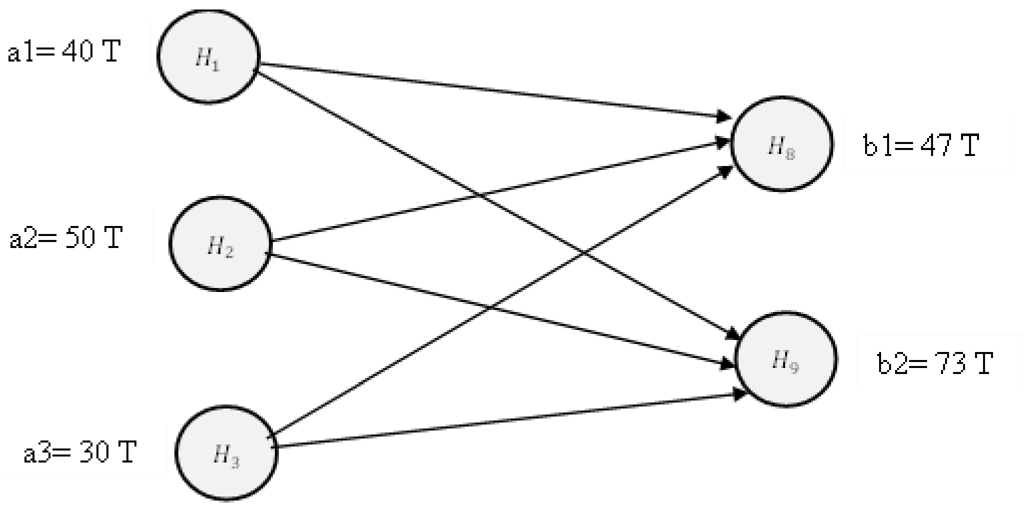

Fig. 2. The equivalent single stage transportation network.

3.3 Problem Solution (Phase 2).

Problem Mathematical Model.

Objective function:

$$\boldsymbol{Minimize\ z} = \sum_{i=1}^{3}\sum_{j=1}^{2}\sum_{k=1}^{2} \boldsymbol{c_{ijk}x_{ijk}} \qquad (\mathbf{34})$$

Objective function extended form:

$$Min.Z = 21x_{111} + 22x_{112} + 24x_{121} + 19x_{122} + 14x_{211} + 21x_{212} + 17x_{221} + 15x_{222} + 17x_{311} + 19x_{312} + 20x_{321} + 13x_{322}$$

Sources constraints:

$$\sum_{j=1}^{3} \boldsymbol{x_{ijk}} = \boldsymbol{a_{ik}} \qquad (\mathbf{35})$$

$$x_{111} + x_{112} + x_{121} + x_{122} = 40$$
$$x_{211} + x_{212} + x_{221} + x_{222} = 50$$
$$x_{311} + x_{312} + x_{321} + x_{322} = 30$$

Destinations constraints:

$$\sum_{i=1}^{2} \boldsymbol{x_{ijk}} = \boldsymbol{b_{jk}} \qquad (\mathbf{36})$$

$$x_{111} + x_{112} + x_{211} + x_{212} + x_{311} + x_{312} = 47$$
$$x_{121} + x_{122} + x_{221} + x_{222} + x_{321} + x_{322} = 73$$

Transports constraints:

$$\sum_{k=1}^{2} \boldsymbol{x_{ijk}} = \boldsymbol{e_{ij}} \qquad (\mathbf{37})$$

$$x_{111} + x_{121} + x_{211} + x_{221} + x_{311} + x_{321} = 50$$
$$x_{112} + x_{122} + x_{212} + x_{222} + x_{312} + x_{322} = 70$$

LINGO Code.

```
min=21*x111+22*x112+24*x121+19*x122+14*x211+21*x212+17*x221+15*x222+17*x311+
19*x312+20*x321+13*x322;

x111+x112+x121+x122=40;
x211+x212+x221+x222=50;
x311+x312+x321+x322=30;
x111+x112+x211+x212+x311+x312=47;
x121+x122+x221+x222+x321+x322=73;
x111+x121+x211+x221+x311+x321=50;
```

x112+x122+x212+x222+x312+x322=70;

LINGO Output.

Global optimal solution found.

Objective value:	1859.000
Infeasibilities:	0.000000
Total solver iterations:	4
Elapsed runtime seconds:	3.63
Model Class:	LP
Total variables:	12
Nonlinear variables:	0
Integer variables:	0
Total constraints:	8
Nonlinear constraints:	0
Total nonzeros:	48
Nonlinear nonzeros:	0

Variable	**Value**	**Reduced Cost**
X111	0.000000	1.000000
X112	0.000000	6.000000
X121	0.000000	1.000000
X122	40.00000	0.000000
X211	47.00000	0.000000
X212	0.000000	11.00000
X221	3.000000	0.000000
X222	0.000000	2.000000
X311	0.000000	3.000000
X312	0.000000	9.000000
X321	0.000000	3.000000
X322	30.00000	0.000000

Row	**Slack or Surplus**	**Dual Price**
1	1859.000	-1.000000
2	0.000000	-6.000000
3	0.000000	0.000000
4	0.000000	0.000000
5	0.000000	-10.00000
6	0.000000	-13.00000
7	0.000000	-4.000000
8	0.000000	0.000000

4 Results and Discussion.

After applying the dynamic programming technique on the problem, we found the shortest routes between the sources of the first stage and the destinations of the last stage which represents the minimum transportation costs between these sources and destinations by the two transports K_1 *and* K_2 and as a result the problem transformed from multistage to single stage transportation problem. LINGO software applied to find the optimal distribution of the network after that then we found the optimum solution as the value of the objective function which represents the minimum total transportation cost of the transportation network (Z) equal 1859 Dollars/Km. The optimum network distribution summarized in table 7:

Table 7. Illustrative example optimum network distribution.

Destinations / Sources	H_8		H_9		Sources availabilities	Shortest routes (Path)
Transport (K)	K_1	K_2	K_1	K_2	(50 T, 70 T)	Transport load
H_1				40 T	40 T	$H_1 - H_4 - H_6 - H_9$
H_2	47 T		3 T		50 T	$H_2 - H_4 - H_7 - H_8$ $H_2 - H_4 - H_7 - H_9$
H_3				30 T	30 T	$H_3 - H_5 - H_6 - H_9$
Destinations requirements	47 T		73 T		Total = 120 T	$z_{min,} = 1859$

5 Conclusions.

This paper introduced the mathematical formulations and solution methods for various types of multi-index multistage transportation problems, namely MMTP1, MMTP2, MMTP3, and MMTP4. To solve MMTP1, a two-phase solution algorithm is proposed. In the first phase, dynamic programming technique is used to convert the multistage problem into a single stage problem, as it has shown its effectiveness for such problems where intermediate stage restrictions are absent. In the second phase, LINGO software is employed to solve the resulting multi-index single stage transportation problem efficiently and quickly, as it has demonstrated its competence in solving transportation problems. In conclusion, this study introduced an algorithm for solving a special case of multi-index multistage transportation problem which is a multi-index multistage problem without any transportation restrictions on the intermediate stages. The proposed algorithm applied on an illustrative example to prove its efficiency which show that the algorithm helps in solving and finding the optimal solution for such type of problems. Also using LINGO software aims to find the optimal solution in a fast and easy manner.

6 List of Abbreviations.

MMTP1	Multi-index multistage transportation problem of the first kind

MMTP2	Multi-index multistage transportation problem of the second kind
MMTP3	Multi-index multistage transportation problem of the third kind
MMTP4	Multi-index multistage transportation problem of the fourth kind
BMTP1	Bi-criteria multistage transportation problem of the first kind
BMTP2	Bi-criteria multistage transportation problem of the second kind
BMTP3	Bi-criteria multistage transportation problem of the third kind
BMTP4	Bi-criteria multistage transportation problem of the fourth kind
D.P.	Dynamic programming

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