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Cosmological Evolution of Tsallis Holographic Dark Energy Model in The LTB Inhomogeneous Universe

Mohamed Abdelrashied[*1], Ayman Aly[2] and Mustafa Selim[1]

[1]Department of Physics, Faculty of Science, Damietta University, New Damietta City, P. O. 34517, Egypt.
[2]Department of Physics, Faculty of Science, Damanhour University, Damanhour City, P. O. 22511, Egypt

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[*]Corresponding author’s E-mail: mohamed.a.elrashied@gmail.com

Abstract

This work aims to construct the generalized form of holographic dark energy density which is known as the Tsallis holographic dark energy within an inhomogeneous universe scenario with interacting dark sectors. This holographic model has been investigated using the IR cutoff equals the Hubble horizon. The evolutionary behavior of some basic cosmological parameters, such as the Hubble parameter, the deceleration parameter, and the equation of state parameter is studied and plotted against the redshift. The stability of these models has been studied by investigating the evolution of the square of sound speed of the cosmological matter fluctuations against the redshift. In addition, the evolution of the diagnostic Om and Lm parameters is studied. The obtained results show a very good agreement between the recent values of some of the cosmological parameters with the observations. Some major differences and similarities between these models and previously studied inhomogeneous models have been shown. The features that distinguish these models from each other and the standard cosmological model have been studied by applying the state finder analysis. To investigate the viability of these models and how they are candidates to explain a realistic case of our universe, they have been tested with the violation of the second law of thermodynamics and the evolution of the sound spees squared parameter.

Keywords: Holographic dark energy; Tsallis holographic model; Inhomogeneous LTB universe.

Introduction

The fact that our universe has accelerated expansion has been confirmed over the last decades. This fact is supported using the observational data of type Ia of distant supernovae (Riess et al. 1998; Perlmutter et al. 1999) and the sky mapping of the cosmic microwave background (Ade et al. 2000; Huang et al. 2006). These observations provided also evidences of the existence of a mysterious component of the universe called Dark Energy (DE) which is assumed to be responsible for the observed accelerated expansion. This component constitutes around 70% of the matter content of the universe. Many measurements indicate that the cosmological constant, Λ, is the most suitable candidate to represent the DE. This constant is related to the concordance cosmological model called the ΛCDM model where CDM refers to another dark sector that is called the cold dark matter. The ΛCDM model is formulated based on the homogeneous and isotropic solution of General Relativity (Aly et al. 2020) which is called the Friedmann-Lemaitre-Robertson-Walker (FLRW) universe. However, this scenario has some difficulties. The first is the cosmological constant problem in which the magnitude of the expected vacuum energy, calculated according to quantum mechanics, is very large (about 120 orders of magnitude different from the observed value). Second, the coincidence problem where the DE density, ρ_D, is not only small but also in the same order of magnitude of matter density, ρ_m. These difficulties motivated many cosmologists to look for alternative scenarios to explain the accelerated expansion of the universe. Some of the other alternatives are the light scalar fields (Quintessence and K-essence), modified gravity models, higher-derivative gravity, etc. All these propositions are still debated. Another alternative proposes that the universe is isotropic but inhomogeneous, rather than homogeneous as in the standard cosmological models. This inhomogeneous model gained some interest as a scenario that can explain the observed accelerated expansion without invoking the DE component itself. The most common inhomogeneous scenarios are the Lemaitre−Tolman−Bondi (LTB) models. This is based on the idea that we are living close to the center of a huge spherically symmetric matter under-density with volume in the scale of Giga parsecs (Gpc). This idea faced some difficulties as it is very unlikely to observe such matter underdense (void) in any consistent large-scale cosmological observations. Instead, inhomogeneity can be assumed in the dark energy sector to explain the accelerated expansion without totally giving up the idea of dark energy and without facing the discrepancy problem from which the ΛCDM model is suffering. This scenario can also be investigated by assuming the holographic principle to represent the DE density. In previous work (Aly et al. 2020; Abd Elrashied et al. 2019) the normal holographic and Tsallis models of dark energy have been considered with the LTB inhomogeneous dark energy universe. The resultant evolution of some cosmological parameters against redshift was acceptable in comparison to the data of the recent observations of the type Ia supernovae and the results obtained with other models (Li et al. 2012; Jawad et al. 2018; Aditya et al. 2019. R´enyi and Tsallis holographic dark energy models are statistical generalizations of the ordinary holographic dark energy model in the case of the interacting systems. Many studies proposed a correspondence between thermodynamics and gravity, so generalizing one of the entropy or gravity concepts, as the black hole event horizon entropy idea, will change the other one. THDE model has been investigated with many other cosmological models and the results compared with the predictions of the ΛCDM model and with various types of observations. R´enyi holographic dark energy model (RHDE) is also studied with many cosmological scenarios. For example, the RHDE model has been explored with IR cut-off as the future and particle event horizons. The spatially homogeneous and anisotropic Bianchi $V\,I_0$ universe filled with RHDE with Granda-Oliveros and Hubble horizons as the IR cut-off have also been studied in the context of general relativity (Shekh 2023).

In the present work, the properties of the THDE in the case of the interacting dark sectors have been studied in the case of the inhomogeneous but isotropic LTB universe. In section 2. we presented the basic formulas expressing the interacting THDE conservation in the LTB universe and the evolution of the corresponding fractional density with increasing the redshift. Specific LTB models have been chosen and their free parameters are specified based on our previous work on the same models in (Aly et al. 2020; Abd Elrashied et al. 2019). In section 3. the main cosmological parameters such as the holographic dark energy fractional density, the Hubble parameter, the deceleration parameter, and the equation of state (EoS) parameter have been calculated and plotted against the redshift. The evolution of these parameters shows an acceptable behavior in comparison with some observations of the universe in recent times. For example, The Hubble parameter came very close to $72 kms^{-1}Mpc^{-1}$ which is the most likely value estimated using the recent observations of the Ia-type supernovae (Riess et al. 1998; Perlmutter et al. 1999). In section 4, the stability of these models is analyzed using the sound speed squared parameter. In section 5, the distinction of the two holographic LTB models is investigated using the diagnostic parameters *Om* and *Lm*. In section 6, we applied the statefinder analysis by evaluating the statefinder parameters *r* and *s* and studied the evolution of the considered LTB models in the (*r,s*) plane to check the distinction of these models from each other and the ΛCDM model. In section 7, the conditions of satisfying the second law of thermodynamics have been explained for the LTB models in the case of the interacting RHDE representation of dark energy. This is done mainly by studying and comparing the evolution of the temperature of both dark sectors and the entropy of the holographic models. In the last section, some conclusion remarks are given.

Basic Equations

We consider the interacting scenario between the considered HDE and cold dark matter(CDM). Thus the energy conservation equation can given by (Jawad et al. 2018)

$$\dot{\rho}_m + 3H\rho_m = -Q, \quad (1)$$

and

$$\dot{\rho}_D + 3H(\rho_D + P_D) = Q, \quad (2)$$

where Q is the interaction term which represents the coupling between dark sectors. If Q is positive this means the energy transfers from the dark matter sector to the dark energy sector while it means the opposite if Q is negative. The evolution of the cosmological models including the interaction of the dark sector has been extensively studied for different holographic dark energy models (Saha et al. 2023; Som and Sil 2014; Nayak 2020; Landim 2022; Rodriguez-Benites et al. 2024. Many estimations are considered for the interaction term of the dark sector. We will adopt here a simple interaction function that can be given by

$$Q = 3H\xi\rho_D, \quad (3)$$

where ξ is the coupling parameter of the interaction. The late-time constraints on interacting dark energy that was investigated in (Benisty et al. 2024) revealed that the accepted values of ξ lay in $\xi \epsilon [-0.33,1]$.

Interacting Tsallis holographic dark energy in the LTB universe

The quantum aspects of gravity motivated generalized definitions of entropy. These generalized definitions introduced some new concepts to the entropy such as the nonadditivity and the non-extensivity. Tsallis entropy is among those generalized entropy measures that investigated widely and led to acceptable results in some cosmological setups. The Tsallis holographic dark energy has been studied for specific LTB models by (Abd Elrashied et al. 2019) where acceptable results have been obtained for some cosmological parameters in comparison with some of the late universe observations. In this work we are interested in studying the evolution of the basic cosmological parameters of the interacting THDE model in the LTB universe and investigate its thermal stability conditions. The Tsallis entropy is given by(Jawad et al. 2018)

$$S = \frac{1}{\delta}\sum_{i=1}^{n}\left(P_i^{1-\delta} - P_i\right), \quad (4)$$

where δ is the non-additivity parameter. The THDE density is given by

$$\rho_D = B\,H^{-2\delta+4}, \quad (5)$$

where B is a parameter which can be written as $B = 3c^2m^2{}_PH^{2\delta-2}$, and m_P is the Planck mass. The energy conservation equations (1) and (2) for the interacting holographic dark energy can be written for the LTB models as (Grande and Perivolaropoulos 2011)

$$\dot{\rho_m} + 2H\left(1 - \frac{P_t}{P_r}\right)\rho_m = -Q, \quad (6)$$

$$\dot{\rho_D} + 2H\left(1 - \frac{P_t}{P_r}\right)\rho_D = Q, \quad (7)$$

where P_t and P_r are the transverse and the radial components of the dark energy pressure in the case of the LTB inhomogeneous universe. From equations (3),(5), and (7) we can get

$$\left(\frac{P_t}{P_r} - 1\right) = \frac{3}{2}\xi + \frac{\dot{H}}{H^2}\left[2-\delta\right]. \quad (8)$$

Using equation (11) in (Grande and Perivolaropoulos) we can get

$$\left(\frac{P_t}{P_r} - 1\right) = \frac{\dot{H}}{H^2} + \frac{3}{2}\frac{\lambda^3\Omega_m}{\lambda^3\Omega_m + \Omega D}. \quad (9)$$

Substituting (9) in (8), the term $\frac{\dot{H}}{H^2}$ can be written as

$$\frac{\dot{H}}{H^2} = \frac{3}{2(\delta-1)}\left(\xi - \frac{\lambda^3(1-\Omega_D)}{\lambda^3(1-\Omega_D) + \Omega D}\right), \quad (10)$$

where $\lambda = \frac{R_0}{R}$, $R = R(r,t)$ is a function of space and time which is like the scale factor of the FRLW model of the homogeneous universe and $R_0 = R(r,0)$. By taking the derivative of equation (5) we can write

$$\Omega'_D = (4-2\delta)\,\frac{\dot{H}}{H^2}\Omega_D, \quad (11)$$

where $\Omega'_D = d\Omega_D/d\ln R = \dot{\Omega}_D/H$. From equations (10) and (11) we can write

$$\Omega'_D = 3\frac{\delta-2}{\delta-1}\left(\xi - \frac{\lambda^3(1-\Omega_D)}{\lambda^3(1-\Omega_D)+\Omega D}\right)\Omega_D. \quad (12)$$

The scale function R(r,t)

In this work we consider the same LTB models that have been studied in (Abd Elrashied et al. 2019) and (Aly et al. 2020) which are given by choosing the scale function $R(r,t)$. Since our main interest is in the case of the on-center observer, i.e, the observer who is arbitrarily close to the center of the assumed void region with dark energy overdense in the LTB universe as proposed in (Grande and Perivolaropoulos 2011), the scale function can only be chosen as as follows

$$R(r,t) = (t + \beta + \eta_0 r^q)^{2/3}. \quad (13)$$

The range of values of the parameters is given as $\beta \epsilon$ [0.5,4], q = 0.65, η_0 = 50 (Ribeiro 2008; Wang et al. 2000). The general formula of Hubble parameter as a function of the distance r from the center of the dark energy overdense and time t for the inhomogeneous LTB universe is given by (Aly et al. 2020; Enqvist 2008)

$$H(r,t) = H_0\sqrt{\lambda^3\,\Omega_m + \Omega_D}, \quad (14)$$

where $H_0 = H(r,0)$ and it can be given as a function of Ω_D as

$$H_0(r) = -2\frac{\left(\ln\left[\sqrt{1-\Omega_D}\sqrt{\Omega_D}\right] - \ln\left[\Omega_D + \sqrt{\Omega_D}\right]\right)}{3t_0\sqrt{\Omega_D}} \quad (15)$$

Evaluation of some cosmological parameters

It is more convenient to express the cosmological parameters in terms of the redshift. This could be done using the formula of expressing the time, t, in terms of the redshift, z as follows

$$t(z) = \frac{2}{H_0[1+(1+z)^2]} \quad (16)$$

The evolution of the THDE fractional density Ω_D can be obtained by solving (12) numerically at specific values of the space coordinates r. Considering the case of an observer is very close to the center of the dark energy overdense. The solutions for Ω_D are found for specific values of the coupling parameter, ξ = (−0.209,0.290) and for chosen values of the non-additivity parameter δ = {3.5,4.5,5.5}. These results are shown in figure 1.

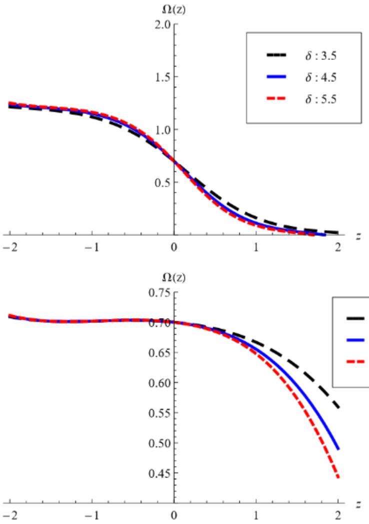

Fig. 1 The evolution of the dark energy fractional density of the THDS model with z for (a) $\xi = -0.209$ and (b) $\xi = 0.290$.

The evolution of the Hubble parameter can be evaluated from equation (14) by substituting with equation (13), and with the solution of (12). The evolutionary behavior of the Hubble parameter is shown in figure 2.

The evolution of the Hubble parameter shows that it is decreasing with the evolution of the universe and increasing the cosmic time. It is also converge to the value 72*km/sec.Mpc* which is in a good agreement with the average values obtained from the recent local observations of the recent universe such as the type Ia supernovae observations (Riess et al. 1998; Perlmutter et al. 1999).

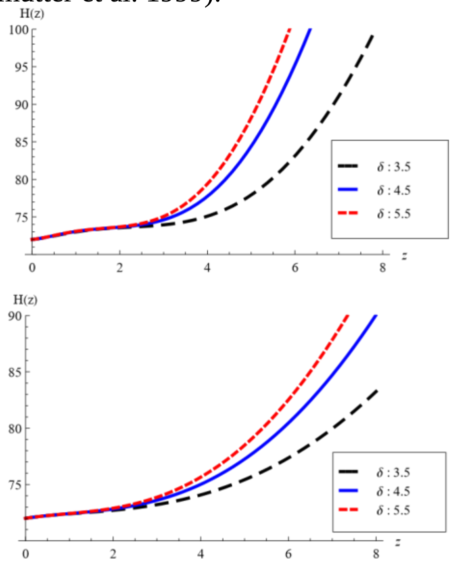

Fig. 2 The evolution of the Hubble parameter of the THDS model with z for (a) $\xi = -0.209$ and (b) $\xi = 0.290$.

This value is also in a good agreement with the recent observation data combination (CMB + BAO + SN + H0 and CMB + BAO + SN + H0 +WL+RSD) of Hubble parameter (H0 ≈ 69 km/s.Mpc)

The deceleration parameter is given by

$$q = -1 - \frac{\dot{H}}{H^2}, \tag{17}$$

which can be re-written as

$$q = -1 - \frac{3}{2(\delta - 1)}\left[\xi - \frac{\lambda^3(1-\Omega_D)}{\lambda^3(1-\Omega_D) + \Omega D}\right]. \tag{18}$$

The evaluation of the deceleration parameter against the redshift is shown in figure 3. Note that the acceleration parameter is decreasing by evolving the universe to its recent age. The two scenarios of the interacting THDE show that the LTB universe is experienced a phase transition from the decelerated expansion phase to the accelerated expansion phase at z = 3.5 for the ξ = −0.209 case and at z = 5 for the ξ = 0.290 case. Both cases show a phase of accelerated expansion of the LTB univere in the near future times, i.e. $-2 \leq z \leq 0$ as shown in figure 3. Although these values still imply that some transition has occurred, they are much larger than the value, z = 0.46, which is obtained by constraining the luminosity distances of a high redshift sample of Ia supernovae and more consistent with the ΛCDM model (Riess et al. 2001, 2004).

The equation of energy conservation for the interacting dark energy sector can be given in terms of the equation of state parameter as

$$\dot{\rho}_D + 2H(1 + \omega_D)\rho_D = Q\,. \tag{19}$$

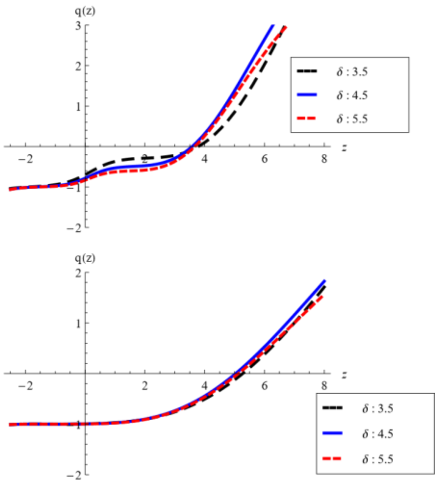

Fig. 3 The evolution of the deceleration parameter of the THDS model with z for (a) ξ = −0.209 and (b) ξ = 0.290.

By comparing this equation to equation (7) given for the inhomogeneous LTB universe, the equation of state parameter ω_D is given by

$$\omega_D = -1 - \frac{2}{3}\left(\frac{P_t}{P_r} - 1\right). \tag{20}$$

Combining this equation with equations (8) and (10), the EoS parameter can be written as

$$\omega_D = -1 - \xi + \frac{\delta - 2}{\delta - 1}\left[\xi - \frac{\lambda^3(1-\Omega_D)}{\lambda^3(1-\Omega_D) + \Omega D}\right]. \tag{21}$$

The evolution of the EoS parameter is shown in figure4. We can notice that the evoluted EoS parameter has negative values through the past times, the present and

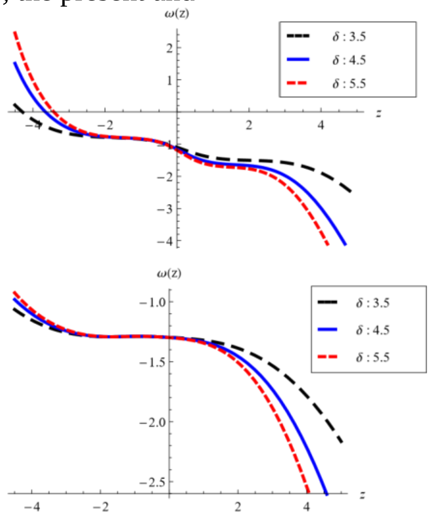

Fig. 4 The evolution of EoS parameter of the THDS model with z for (a) ξ = −0.209 and (b) ξ = 0.290.

even in the near future times $-2 \leq z \leq 0$. This refers to that the LTB universe with interacting THDE dark energy had a phantom-like expansion in the past and in its recent time. This because the EoS parameter is always has negative values for both interacting cases at $z \geq 0$. It is also clear that the first case with ξ = −0.209 the LTB universe will get rid of this phantom behavior and pass through the value $\omega_D = -1$ of the ΛCDM model in the very near future then evolve to higher values later. On the other hand, the second interacting case with ξ = 0.209 will staying longer, even in the near future times, in this phantom state.

Stability

In this section, the stability of the interacting THDE model of dark energy is discussed by deriving the sound speed squared parameter, v_s^2 for the considered LTB inhomogeneous models of the universe. The positive value of v_s^2 (and so a real sound speed) implies a regular propagating mode for the density perturbations in the dark energy fluid. This implies a stable HDE model, while the negative v_s^2 (and so an imaginary sound speed) implies an unstable HDE model. The sound speed squared is given by (Moradpour et al. 2018)

$$V^2 = \frac{dP_D}{d\rho_D} = \frac{\rho_D\ \omega_D}{\dot{\rho_D}} + \omega_D$$

$$V^2 = \frac{dP_D}{d\rho_D} = \frac{\rho_D\ \omega_D}{\dot{\rho_D}} + \omega_D \qquad (22)$$

Substituting equations (2),(3), (5) and (19) in the above equation then solving it numerically, the evolutionary behavior of V^2 against z can be given as shown in figure 5. We can observe that the V^2 parameter has only negative values in the recent and near future times of the universe. This shows that the LTB universe with interacting THDE dark energy model has instabilities in the perturbation level. It has irregular modes of superluminal propagation in which the density perturbations have an exponential mode of growing (Myung 2007; Kim et al. 2008; Sharma and Dubey 2022). This behavior is unlike the stability behavior has been shown by the LTB model of the universe with non-interacting THDE model studied in detain in (Aly et al. 2020).

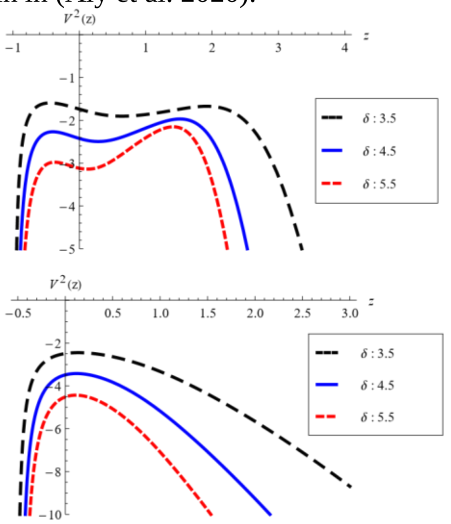

Fig. 5 The evolution of V^2 parameter of the THDS model with z for (a) $\xi = -0.209$ and (b) $\xi = 0.290$.

Diagnostic parameters

Diagnostic Omega parameter

The diagnostic parameters are mainly used to distinguish the dark energy models without precise knowledge of the fractional densities. The easiest diagnostic parameter that can be used is the omega parameter, *Om*, which is given by

$$Om(z) = \frac{-1 + \left(\frac{-H(z)}{H_0}\right)^2}{-1 + \left(\frac{2 - H_0 t(z)}{H_0 t(z)}\right)^{\frac{3}{2}}}. \qquad (23)$$

The two LTB models of the inhomogeneous universe exhibit the same evolutionary behavior of the *Om* parameter with z for the given values of δ and ξ free parameters by considering the RHDE model of dark energy. This shows how the *Om* parameter is independent on δ and ξ for the two LTB models in the case of $r = 0$. This is shown in figure 6 The two coupling cases of the THDE model of dark energy in the inhomogeneous universe exhibit the same evolutionary behavior of the *Om* parameter with z for the given values of δ free parameter. This shows how the *Om* parameter is independent on δ and ξ for the two scenarios in the case of the on-center observer, i.e. $r = 0$. This behavior indicates that the two coupling scenarios are indistinguishable relative to an observer located at the center of the overdense of the interacting dark energy. So, it is reasonable to try using another diagnostic parameter to distinguish the two cases of the interacting holographic dark energy.

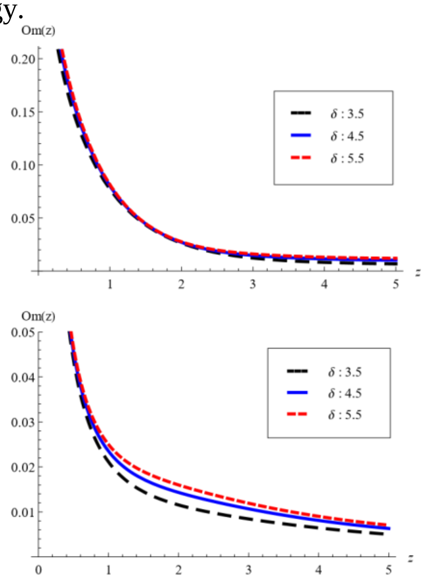

Fig. 6 The evolution of the diagnostic parameter *Om*(*z*) of the THDS model with z for (a) $\xi = -0.209$ and (b) $\xi = 0.290$.

Diagnostic L_m parameter

The L_m parameter is the first derivative of the *Om* parameter, and it is given by (Aly 2019)

$$L_m(z) = 2z\left(z^2+3z+3\right)h\frac{dh}{dz} + 3\left(z+1\right)^2\left(1+h^2\right), \quad (24)$$

where $h = H(z)/H_0$. The evolutionary behavior of the L_m diagnostic paremeter with z is shown in figure 7. It is clear that both scenarios of the interacting THDE model have almost the same evolutionary of the diagnostic parameter $Lm(z)$ as shown in figure 7. So, the two models are also using this parameter. We can conclude that the two parameters Om and Lm are not enough to distinguish the two interacting THDE models of dark energy in the LTB universe. This motivates us to look for another more powerful diagnostic tool as one that will be discussed in the next section.

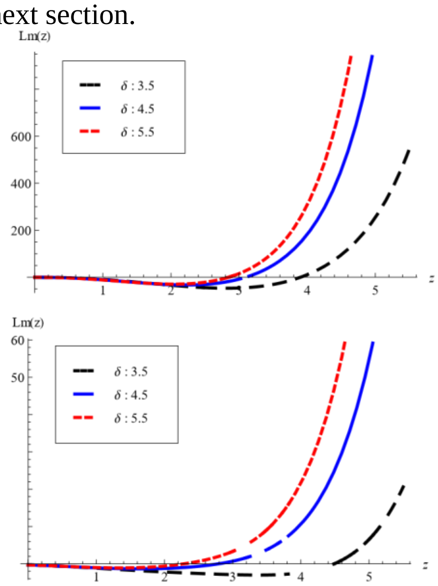

Fig. 7 The evolution of the diagnostic parameter $Lm(z)$ of the THDS model with z for (a) $\xi = -0.209$ and (b) $\xi = 0.290$.

Statefinder Analysis

In cosmology there are many dark energy models that need to be compared. Such a comparison is useful in measuring the advantages of a specific model over the standard ΛCDM model or any other existing model. The statfinder dignostic is a pair of two geometrical dimensionless variables $\{r,s\}$ that is introduced in (Saha et al. 2023). These geometrical parameters can be constructed from the higher derivative orders of the scale factor with cosmic time. The state finder parameters can be written in terms of the parameters $q(z)$ and $H(z)$ as follow

$$r(z) = 2q(z)^2 + q(z) - \frac{dq(z)/dz}{H(z)\,(dt/dz)}, \quad (25)$$

$$s(z) = \frac{r(z)-1}{3\left(q(z)-\frac{1}{2}\right)} \quad (26)$$

The statefinder parameter $r(z)$ is given in terms of the Hubble parameter $H(z)$ and the deceleration parameter $q(z)$ and hence it is given in terms of the scale function $R(r,t)$ of the LTB model and its higher derivatives. The other statefinder parameter $s(z)$ is a combination of $r(z)$ and $q(z)$ parameters. The evolutionary behavior of the two statefinder parameters against the redshift is shown in figure 8. We can notice

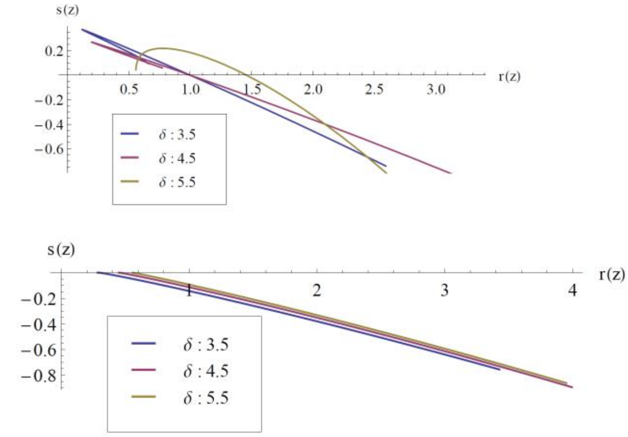

Fig. 8 The evolution of the statefinder diagnostic parameters $\{r,s\}$ with z for the THDE model for (a) $\xi = -0.209$ and (b) $\xi = 0.290$.

that both THDE models show almost the same proportionality behavior between the evolved statefinder parameters, specifically for the values of $\delta = 3.5$ and $\delta = 4.5$. However, the first interacting model with negative ξ is passing the characteristic value $\{1, 0\}$ of the ΛCDM model at $\delta = 3.5$ and $\delta = 4.5$, while the other model with positive ξ not passing that point. This difference can be considered as a distinction between the two models of interacting THDE models. It is noticed that this behavior is different from that of the statefinder analysis of the ordinary holographic dark energy model in the inhomogeneous LTB universe that has been studied in (Abd Elrashied et al. 2019). In that work the statefinder pair {r, s} do not pass the characteristic value {1, 0} of the ΛCDM model at any value of the parameter δ within the timeline of the obervational universe. The same attidude is given for the Chaplygin gas model that has been studied for the holographic Ricci dark energy model of FRW universe in (Feng 2008). As it is thought that the evolution of the observable universe has to converge to the ΛCDM model in the future times, we can conclude that the THDE model within an inhomogeneous universe has more advantage than some other models to represent the recent universe.

Thermodynamics and Thermal Analysis

The standard cosmological model assumes the adiabatic expansion of the observable universe where each of its components are conserved separately and independently (Melendres 2021). The separate conservation of the non-interacting dark matter and dark energy components is given by equations (6) and (7) with $Q = 0$ for the considered LTB inhomogeneous models. In that case, the second law of thermodynamics tells us that the entropy of each component is evolving and not a constant. Thermodynamics can be used to investigate the viability of the given models which must have an increasing entropy with time and thus the second law of thermodynamics would not be violated. To establish that, the second law of thermodynamics can be written for each dark sector as.

$$TdS = d[(\rho + p)V] - V\,dp, \qquad (27)$$

where T, ρ, and p are the temperature, density, and pressure of each dark sector and V is a volume element within a comoving coordinates system. In the case considered here of the interacting dark sector fluids, where $Q \neq 0$, the Q term represents the time rate of change of the energy densities of the dark sector. Following the argument in (Saha et al. 2023), and using

$$Q = \frac{T_D}{V}\frac{dS_D}{dt} = -\frac{T_m}{V}\frac{dS_m}{dt}, \qquad (28)$$

where S_D and S_m stand for the entropy of dark energy and dark matter respectively. This equation leads to the following

$$d\left(S_D + S_m\right) = \left(1 - \frac{T_m}{T_D}\right) dS_m \neq 0, \qquad (29)$$

and so $S_D + S_m \neq$ *constant*. So, the case of interacting with dark sector does not imply an adiabatic evolution of the universe. Equation (29) shows that the second law of thermodynamics will be satisfied if the dark matter is always cooler than dark energy ($T_m < T_D$) because $\frac{dS_m}{dt} > 0$. From equation (28) the time evolution of the total entropy can be given as

$$\frac{d}{dt}\left(S_D + S_m\right) = -\left(\frac{1}{T_m} - \frac{1}{T_D}\right) QV. \qquad (30)$$

This equation implies that there is another condition on Q to get the second law of thermodynamics to be satisfied when $Q < 0$. This condition shows that the interacting dark sector models with negative coupling parameter, $\xi < 0$, are the more realistic models from the perspective of thermodynamics. In these models, the energy always flows from dark energy to dark matter. To investigate the first thermal condition that viable models should have ($T_m < T_D$), the temperature of dark sectors has to be evaluated. With Q that given in (3) and following the derivation arguments in (C´ardenas et al. 2019), the temperature of dark sectors can be written as

$$T_m(z) = T_m(0)\,(1+z)^{3\omega_m} \exp\left[3\xi \int_0^z \frac{1-\Omega_D}{\Omega_D}\, d\ln(1+z)\right], \qquad (31)$$

and

$$T_D(z) = T_D(0)(1 + z)^{3(\omega_D - \xi)}, \qquad (32)$$

where $T_m(0)$ and $T_D(0)$ are the values of the dark matter and dark energy temperature at the recent time of the universe with $z = 0$. The dark energy EoS parameter ω_D is given by equation (21). The cold dark matter density can be represented in terms of $\mathbf{\Omega}_D$ as

$$\rho_m = \Omega_m \rho_c = (1 - \Omega_D)\,\frac{3H^2}{8\pi G}. \qquad (33)$$

Using this equation with equations (6),(8), (9) and (10), the EoS parameter of the cold dark matter sector is given by

$$\omega_m(z) = -\frac{3}{2}\xi + \frac{3}{2}\frac{\delta - 2}{\delta - 1}\left[\xi - \frac{\lambda^3(1-\Omega_D)}{\lambda^3(1-\Omega_D) + \Omega D}\right]. \qquad (34)$$

By substituting this equation and (21) in equations (31) and (32), the temperature ratios $T_m(z)/T_m(0)$ and $T_D(z)/T_D(0)$ are numerically evaluated and plotted aganist each other with z in figure 9 at $\delta = 3.5$. As it is clear from the figure 9 the temperature ratio of dark matter is almost greater than that ratio for dark energy at $z \geq 0$, while at the near future times, i.e.$z \leq 0$, the dark energy temperature ratio starts to raise over that of the dark matter. Considering this and the condition of $T_m(z) < T_D(Z)$ should be always satisfied and we also have $T_m(0) < T_D(0)$, we can say that this condition is satisfied in the past and the second law of thermodynamics was always satisfied in the past times of the LTB universe with interacting THDE models. However, in the recent and future time of this models predicts the violation of the temperature condition and hence the violation of the second law of thermodynamics.

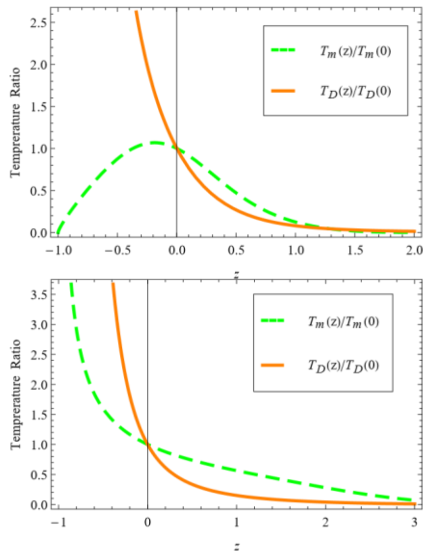

Fig. 9 The evolution of the temperature ratio for the two dark sectors with *z* for the THDE model for (a) $\xi = -0.209$ and (b) $\xi = 0.290$.

This is agreed with the facts revealed before that the two models of interacting THDE have a phantom behavior of their evolution and irregular mode of density perturbations. This conclusion can be further asserted by investigating the evolved entropy of both models as shown in figure 10. The entropy function is increasing with time untill reach a maximum value in the near future times then start to decrease and this clearly violates the second law of thermodynamics.

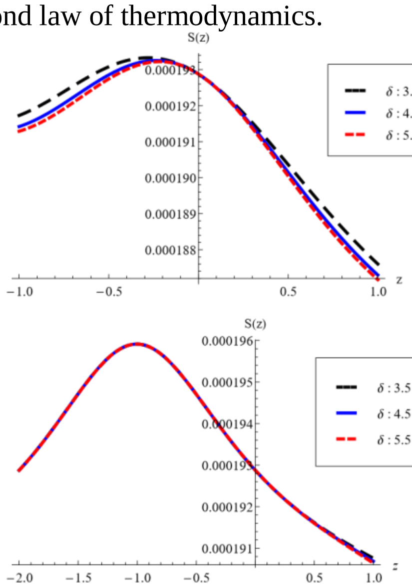

Fig. 10 The evolution of the entropy function for the THDE model with *z* for (a) $\xi = -0.209$ and (b) $\xi = 0.290$.

Conclusion

In this work, we consider the interacting Tsallis model of holographic dark energy. This model is one of the statistical generalizations of the entropy formula of the black hole which was introduced first by Bekenstein and Hawking and studied in more detail in (Tavayef et al. 2018).This Tsallis holographic dark energy model has been investigated in the case of the generalized LTB model of the inhomogeneous universe which was studied previously in the non-interacting case in (Aly et al. 2020; Abd Elrashied et al. 2019; Grande and Perivolaropoulos 2011). To clarify our model, the main cosmological parameters are calculated as functions of redshift, *z*, at two different values of the coupling parameter, ξ between dark matter and dark energy, and three different values of the non-additivity parameter δ. By considering the IR cutoff equal to he Hubble horizon, the Hubble parameter, the deceleration parameter, and the equation of state parameter are calculated for both holographic LTB models. The evolution of these parameters with *z* is studied by considering an observer located at the center of the dark energy overdense, i.e. $r = 0$. The evolutionary behavior of these cosmological parameters is plotted and compared with the previous work at certain values of some free parameters for the proposed models. Sometimes, the behavior of these cosmological parameters is in close agreement with the concordance ΛCDM model of the observed universe. Therefore, these results can be compared and tested against the current observations of the recent observable universe. For example, the recent values of the Hubble parameter, the deceleration parameter, and the fractional density of the holographic dark energy show coincidence with the measurable values obtained using the recent Ia supernovae pantheon data. Maybe the most distinguished feature is the existence of a phase transition form the decelerated expansion phase to an accelerated expansion phase at a particular redshift. However, there are some major deviations from the ΛCDM model such as the phantom-like behavior that has been shown in both LTB models in the recent epoch of the universe with a small decrease of their the EoS parameter below −1. Further, analyzing the stability of these models using the sound speed squared parameter and the evolution of this parameter shows that it is close to a small negative value at $z = 0$ which indicates a negative sound speed propagation of density fluctuation and so unstable evolutionary behavior. Using the diagnostic parameters shows that these two LTB models with the THDE are distinguishable only through the evolution of the statefinder diagnostic which is not the case with the HDE and THDE models studied previously in (Aly et al. 2020). Studying the evolution of these models in the state finder plane can distinguish these models from each other and the standard cosmological model. This is so helpful as we have a huge number of candidate models in the literature. Regarding the viability of each of these models, the thermal analysis is very useful to establish that. This could be done by studying two conditions for satisfying the second law of thermodynamics, which implies a real physical evolution of the observable universe. This analysis has shown that the two models are not always satisfying the basic condition that the dark matter should be kept cooler than dark energy. This refers to the feature that both LTB scenarios with THDE model of dark energy have a decreased entropy in the near future times and hence a violation of the second law of thermodynamics. So, we can state that both scenarios are not thermally stable.

Finally, we can conclude that this work investigates a new scenario of the inhomogeneous universe with a non-homogeneous distribution of the THDE. The major similarities and differences between this model and the other homogeneous and nonhomogeneous holographic models are discussed. Besides the previous studies that investigated the same scenario with different statistical generalizations of the holographic principle, this work uncovers the features of the inhomogeneous universe as a possible alternative to the standard homogeneous one. This could be very helpful in tackling big cosmological problems such as the Hubble crisis from a different point of view in both the analytical and the observational perspectives.

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الملخص العربي

عنوان البحث: : التطور الكونى لنموذج تساليس الهولوجرافى للطاقة المظلمة فى كون لوميتر – بوندى – تولمان الغير متجانس

محمد عبد الرشيد*[١]، أيمن على[٢]، مصطفى سليم[١]

[١] قسم الفيزياء – كلية العلوم – جامعة دمياط – مصر
[٢] قسم الفيزياء – كلية العلوم – جامعة دمنهور – مصر

هذه المقالة تهدف الى درسة النموذج العام للطاقة المظلمة الهولجرافية وهذهة الحالة العامة تسمى بنموذج تسالس للطاقة المظلمة الهولوجرافية ويتم دراسة الحالة العامة لهذا النموذج الكونى الغير متجانس فى حالة وجود طاقة مظلمة و مادة مظلمة فى حالة تفاعل متبادل. عند افتراض ان أفق الكون المنظور هو أفق هابل يمكن دراسة تطور هذا النموذج الكونى مع الزمن عن طريق

حساب عدد من المعاملات الكونية الاساسية مثل معامل هابل و معامل التباطؤ و معادلة الحالة و دراسة سلوك و تطور تلك المعاملات مع مقدار الانزياح نحو الاحمر للاشعاع الكونى المرصود و الذي يعتمد على مقدار توسع الكون. تم دراسة استقرار هذا النموذج ايضا بحساب بعض المعاملات التشخيصية الاخرى مثل مربع سرعة الصوت فى المادة الكونية الأولى. تم أيضا دراسة الاستقرار الثيرموديناميكى لهذا النموذج الكونى بدراسة مدى تحقيقه للقانون الثانى للديناميكا الحرارية خلال تطوره مع الزمن الكونى.