Transient Analysis of a finite capacity Markovian queuing system with feedback, discouraged arrivals and retention of reneged customers

M.S. El – Paoumy

Department of Statistics, Faculty of Commerce, Al-Azhar University, Girls' Branch, Dkhlia , Egypt

Department of Mathematics, Faculty of Science, Prince Sattam bin Abdulaziz University, Al-Kharj, Saudi Arabia

E-mail: drmahdy_elpaoumy@yahoo.com

Abstract: This paper analyze a finite capacity Markovian feedback queue with discouraged arrivals, reneging and retention of reneged customers in which the inter-arrival and service times follow exponential distribution. The transient solution of the system, with results in terms of the eigenvalues of a symmetric tri-diagonal matrix. Feedback in queuing literature represents customer dissatisfaction because of inappropriate quality of service. In case of feedback, after getting partial or incomplete service, customer retries for service . After the completion of service , each customer may rejoin the system as a feedback customer for receiving another regular service with probability ξ_1 or he can leave the system with probability q_1 ($\zeta_1+q_1=1$). A reneged customer can be retained in many cases by employing certain convincing mechanisms to stay in queue for completion of service. Thus, a reneged customer can be retained in the queuing system with probability q_2 or he may leave the queue without receiving service with probability ζ_2 ($\zeta_2+q_2=1$). Expressing the Laplace transforms of the system of governing equations in matrix form and using the properties of symmetric tri-diagonal matrices, the steady state probabilities are derived and some important queuing models are derived as special cases of this model.

Keyword: Discouraged arrivals, Reneging, Feedback, retention of reneged customers, transient solution

1- Introduction

The study of feedback queuing systems with finite waiting space is very useful and important as such systems represent a commonly observed type of system in queuing theory. Therefore, many researchers have studied queuing systems with a finite waiting capacity. Sumeet and

Rakesh [7] studies a Markovian feedback queue with retention of reneged customers in steady –state. Udaya [9] discussed transient queue system with dependent servers and controllable arrivals. Rakesh and Sumeet [5] obtained the steady state solution of queuing system with discouraged arrivals, reneging and retention of reneged customers. Rakesh and Sumeet [5] study single – server, infinite capacity Markovian queue with feedback, retention of reneged customers and balking. Al-Seedy et al. [1] obtained the transient solution of finite waiting space single- server queue by using the matrix techniques. Vijayshree and Janani [10] considered an $M/M/c$ queuing model subject to multiple exponential working vacation wherein arrivals occur according to a Poisson distribution and obtained the time probabilities of the number in the system. Chandrasekarani And aravanarajan [2] obtained a transient solution for the system size probabilities in an $M/M/1$ queue with instantaneous Bernoulli feedback subject to catastrophe, erver Failure and Repairs using Continued Fractions method. Ayyappan and Shyamala [3] discussed an $M^{[X]}/G/1$ queue with Bernoulli feedback random server breakdowns, Bernoulli schedule server vacation and restricted admissibility policy. Ayyappan and Shyamala [4] studied a single server non-Markovian queuing system subject to catastrophes , server failure and repair. Sherif I. [8] obtained the transient solution for a two heterogeneous servers queue with impatient behavior.

In this paper transient and hence steady state results are derived for the queue with feedback, discouraged arrivals, reneging, and retention of reneged customers.

2- Model description

Consider an $M/M/1/N$ queue with feedback, discouraged arrivals, reneging and retention of reneged customers. Capacity of the system is taken as finite,(say N). Customers arrive at service station one by one according to Poisson stream with arrival rate λ_n depending on the number of customers present in the system at that time ($\lambda_n = \dfrac{\lambda}{n+1}$). There is a single server which provides service to all arriving customers. Service times are independently and identically distribution exponential random variables with parameter μ . After completion of each service, the customer can either join at the end of the queue with probability ξ_1 or he can leave the system with probability q_1 where $q_1+\zeta_1=1$. The customer both newly arrived and those that are fed back are served in order in which they join the tail of original queue. The queue discipline is first-in , first out (FIFO). The customer in queue (regular arrival or feedback arrival) may impatient when the service is not available for a long time T. This time T is a random variable with the following density function: $f(t)=\alpha e^{-\alpha t}$, $t\geq 0,\alpha>0,$ where α is the rate of time T. Also, T is reneging time of an individual customer after which customer either decide to abandon the queue with probability ζ_2 ($\zeta_2+q_2=1$) or never to return with complimentary probability.

3- Governing Equations

Let $p_n(t)$ be the probability that at time t , the system size is n

($n=0,1,\ldots,N$). We assume that there are i customers in the system at $t=0$.

The system is governed by the following set of differential-difference equations:

$$p_0'(t)=-\lambda\, p_0(t)+\mu\, q_1 p_1(t) \tag{3.1}$$

$$p_n'(t)=-\left[\left(\frac{\lambda}{n+1}\right)+\mu\, q_1+(n-1)\,\alpha\,\zeta_2\right]p_n(t)+\left[\mu\, q_1+n\,\alpha\,\zeta_2\right]p_{n+1}(t)+\left(\frac{\lambda}{n}\right)p_{n-1}(t),\;\; n=1,2,\ldots,N-1 \tag{3.2}$$

$$p_N'(t)=-\left[\mu\, q_1+(N-1)\,\alpha\,\zeta_2\right]p_N(t)+\left(\frac{\lambda}{N}\right)p_{N-1}(t) \tag{3.3}$$

The Laplace transform of the differential difference equations (3.1)-(3.3) are

$$(s+\lambda)\,p^*{}_0(s)=\mu q_1\, p^*{}_1(s)+p_0(0),\quad n=0 \tag{3.4}$$

$$\left[s+\left(\frac{\lambda}{n+1}\right)+\mu\, q_1+(n-1)\alpha\,\zeta_2\right]p^*{}_n(s)=\left(\frac{\lambda}{n}\right)p^*{}_{n-1}(s)+\left[\mu q_1+n\,\alpha\,\zeta_2\right]p^*{}_{n+1}(s)+p_n(0),$$
$$1\leq n\leq N-1 \tag{3.5}$$

$$\left[s+\mu q_1+(N-1)\,\alpha\,\zeta_2\right]p^*{}_N(s)=\left(\frac{\lambda}{N}\right)p^*{}_{N-1}(s)+p_N(0),\; n=N \tag{3.6}$$

The equations (3.4)-(3.6) can be expressed in matrix as follows:

$A(s)P(s)=P(0)$

Where

$P(s)=\left(p^*{}_0(s),p^*{}_1(s),...,p^*{}_N(s)\right)^T$, $P(0)=\left(\delta_{i0},\delta_{i1},...,\delta_{iN}\right)^T$,

$p_n(0)=\delta_{in}$ (δ_{in} is the usual Kronecker delta)

$$\text{and} \quad p^*{}_n(s)=\int_0^\infty e^{-st}\,p_n(t)dt.$$

The matrix $A(s)$ with order $N+1$ is given as following

$$\begin{bmatrix} s+\lambda_0 & -\mu_1 & 0 & 0\dots & 0 & 0 \\ -\lambda_0 & s+\lambda_1+\mu_1 & -\mu_2 & 0\dots & 0 & 0 \\ 0 & -\lambda_1 & s+\lambda_2+\mu_2 & -\mu_3\dots & 0 & 0 \\ 0 & 0 & -\lambda_2 & s+\lambda_3+\mu_3\dots & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0\dots & s+\lambda_{N-1}+\mu_{N-1} & -\mu_N \\ 0 & 0 & 0 & 0\dots & -\lambda_{N-1} & s+\mu_N \end{bmatrix},$$

Where

$\mu_n=\mu q_1+(n-1)\alpha\zeta_2, n=1,2,...,N$; $\lambda_n=\dfrac{\lambda}{n+1}$, $n=0,1,...,N-1$.

Applying some elementary row and column transformations and using the properties of tri-diagonal determinants on $|A(s)|$ we obtain $|A(s)|=s\,|B(s)|$, where $B(s)$ is a symmetric tri-diagonal matrix of order N with negative off diagonal elements as presented below.

$$\begin{bmatrix} s+\lambda_0+\mu_1 & -\sqrt{\lambda_1\mu_1} & 0 & 0\ldots & 0 & 0 \\ -\sqrt{\lambda_1\mu_1} & s+\lambda_1+\mu_2 & -\sqrt{\lambda_2\mu_2} & 0\ldots & 0 & 0 \\ 0 & -\sqrt{\lambda_2\mu_2} & s+\lambda_2+\mu_3 & -\sqrt{\lambda_3\mu_3}\ldots & 0 & 0 \\ 0 & 0 & -\sqrt{\lambda_3\mu_3} & s+\lambda_3+\mu_4\ldots & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0\ldots & s+\lambda_{N-2}+\mu_{N-1} & -\sqrt{\lambda_{N-1}\,\mu_{N-1}} \\ 0 & 0 & 0 & 0\ldots & -\sqrt{\lambda_{N-1}\,\mu_{N-1}} & s+\lambda_{N-1}+\mu_N \end{bmatrix}$$

The roots of the polynomial $|B(s)|$ are the negative eigenvalues of the matrix B, obtained by putting $s=0$ in the matrix $B(s)$. Since the matrix B is positive definite, real and symmetric, the eigenvalues of B are real, distinct and positive. Hence the roots of the polynomial $|A(s)|$ are real, distinct and negative. Let the roots are $s_0(=0), s_1, \ldots, s_N$, then we get:

$$p^*_n(s) = \frac{|A_{n+1}(s)|}{s(s-s_1)\ldots(s-s_N)}. \tag{3. 7}$$

Where $|A_{n+1}(s)|$ is the determinants of the matrix obtained by replacing the $(n+1)^{\text{th}}$ column of $A(s)$ by the column vector $P(0)$.

Table 1. The roots of the matrix B,

$N=15,\ i=4,\ \lambda=2,\ \mu=4,\ q_1=0.3, \zeta_2=0.4$ and $\alpha=0.2$.
-4.1046
-3.3387
-3.2742
-3.1367
-2.9505
-2.7281
-2.4802
-2.2189
-1.9563
-1.7049
-1.4756
-1.2693
-1.0619
-8.2223e-001
-5.1436e-001

Let $T_r(s)$ and $B_r(s)$ be the determinants of the top left and bottom right $r \times r$ matrices obtained from the matrix $A(s)$. Then, the determinants $T_r(s)$ and $B_r(s)$ are verify the recurrence relations

$$B_r(s) - (s + \lambda_{N-r+1} + \mu_{N-r+1}) B_{r-1}(s) + \lambda_{N-r+1}\, \mu_{N-r+1}\, B_{r-2}(s) = 0,\; r = 2,3,\dots,N$$

$$T_r(s) - (s + \lambda_{r-1} + \mu_{r-1}) T_{r-1}(s) + \lambda_{r-2}\, \mu_{r-2} T_{r-2}(s) = 0,\; r = 2,3,\dots,N$$

With initial conditions

$$B_0(s) = 1,\; B_1(s) = s + \mu_N\;,\; T_0(s) = 1, \quad \text{and} \quad T_1(s) = s + \lambda_0$$

Then, we will derive an expression of $|A_{n+1}(s)|$ in terms of $T_r(s)$ and $B_r(s)$.

$$|A_{n+1}(s)| = \begin{cases} T_i(s) c_i\; c_{i+1} \dots c_{n-1} B_{N-n}(s), n \ge i \\ T_n(s) b_n\; b_{n+1} \dots b_{i-1} B_{N-i}(s), n < i \end{cases}$$

or

$$|A_{n+1}(s)| = \begin{cases} T_i(s) \dfrac{i\,!\, \lambda^{n-i}}{n\,!} B_{N-n}(s), & n \ge i \\ T_n(s) \prod_{k=n}^{i-1} [\mu q_1 + k\; \alpha \zeta_2]\; B_{N-i}(s), & n < i \end{cases}, \tag{3.8}$$

Where :

$c_i = \dfrac{\lambda}{i+1}$ and $b_n = [\mu q_1 + n\, \alpha \zeta_2]$.

From equation (3.7) and using partial fraction, we get :

$$p^*_n(s) = \frac{p_n}{s} + \sum_{k=1}^{N} \frac{d_{n,k}}{s - s_k} \tag{3.9}$$

Where

$p_n = \lim\limits_{s \to 0} s\, p^*_n(s)$; $d_{n,k} = \lim\limits_{s \to s_k} (s - s_k)\, p^*_n(s)$ and

$$d_{n,k} = \frac{T_i(s_k)\, i!\, \lambda^{n-i} B_{N-n}(s_k)}{n!\, s_k \prod_{j=1, j\neq k}^{N} (s_k - s_j)}, \; i \le n < N$$

$$= \frac{T_n(s_k) \prod_{k=n}^{i-1} [\mu q_1 + k\,\alpha\,\zeta_2] B_{N-i}(s_k)}{n!\, s_k \prod_{j=1, j\neq k}^{N} (s_k - s_j)}, \; 0 \le n < i.$$

Applying Laplace inverse transform on equation (3.9), we obtain:

$$p_n(t) = p_n + \sum_{k=1}^{N} d_{n,k}\, e^{s_k t} \tag{3.10}$$

4- Expected queue length for customers

Let $Q(t)$ be the random variable denoting queue length and $E[Q(t)]$ be its expected value. Then,

$$E[Q(t)] = \sum_{n=1}^{N} n\, p_n(t) \; = \sum_{n=1}^{N} n\, p_n + \sum_{n=1}^{N} n \sum_{k=1}^{N} d_{n,k}\, e^{s_k t}$$

$$= \sum_{n=1}^{N} \left(\frac{\lambda^n}{(n-1)! \prod_{k=1}^{n} [\mu q_1 + (k-1)\alpha\,\zeta_2]} \right) p_0 + \sum_{k=1}^{N} E(k,t) \sum_{n=1}^{i-1} n \prod_{j=n}^{i-1} [\mu q_1 + j\,\alpha\,\zeta_2]\, T_n(s_k)\, B_{N-i}(s_k)$$

$$+ \sum_{k=1}^{N} E(k,t) \sum_{n=i}^{N} \frac{i!\,\lambda^{n-i}}{(n-1)!} T_i(s_k)\, B_{N-n}(s_k), \tag{4.1}$$

Where:

$$E(k,t) = \frac{e^{s_k t}}{\prod_{m=1, m\neq k}^{N} (s_k - s_m)}.$$

5- Steady state probabilities

Let be the steady state probability of n customers being in the system. Then from equation (3.7) we get:

$$p_n = \lim_{s\to 0} s\, p^*_n(s) = \frac{|A_{n+1}(0)|}{\prod_{i=0}^{N} (-s_i)}, n = 0,1,\ldots N \; . \tag{5.1}$$

From the definitions of $T_r(s)$, $B_r(s)$ and the difference equations, we get:

$$T_n(0) = \frac{\lambda^n}{n!}, \quad B_{N-n}(0) = \prod_{K=1}^{N-n}[\mu q_1 + (N-k)\alpha\zeta_2] \text{ and } T_0(0) = 1.$$

For $n = 0$, $p_0 = \frac{|A_1(0)|}{\prod_{i=0}^{N}(-s_i)} \Rightarrow \quad p_n = \frac{|A_{n+1}(0)|}{|A_1(0)|} p_0, n = 0,1,...N$,

$$p_n = \frac{|A_{n+1}(0)|}{|A_1(0)|} p_0 = \frac{T_n(0) b_n b_{n+1} ... b_{i-1} B_{N-i}(0)}{T_0(0) b_0 b_1 .. b_{i-1} B_{N-i}(0)} p_0, \; n < i ,$$

$$p_n = \frac{\lambda^n}{n! \prod_{k=1}^{n}[\mu q_1 + (k-1)\alpha\zeta_2]} p_0, \; n = 1,...,i,...,N \quad . \tag{5.2}$$

Using (3.8), (3.9), (3.10) and (5.1) one can prove that the expression in (5.2) is independent of the position of i .

Substituting the expressions of p_n, $n = 0,1,...,N$, in the normalized equation, the expression for p_0 can be derived as follows.

$$p_0 = \left[1 + \sum_{n=1}^{N} \frac{\lambda^n}{n! \prod_{k=1}^{n}[\mu q_1 + (k-1)\alpha\zeta_2]} \right]^{-1} . \tag{5.3}$$

6- Numerical Results

In this section, we discuss the effect of the reneging rate(α), feedback rate (q_1) , and retention of reneged rate(ζ_2) on the system as shown by plotting in the below figures .

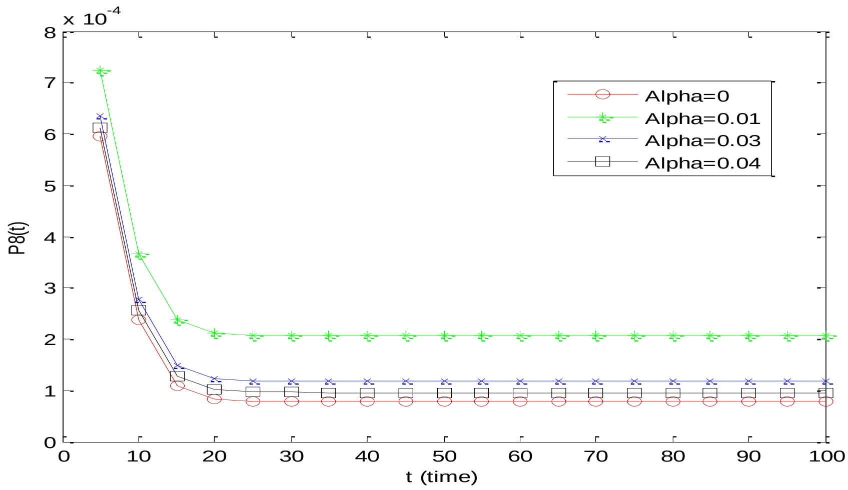

Figure 1. The effect of reneging rate, α on the probability of the system

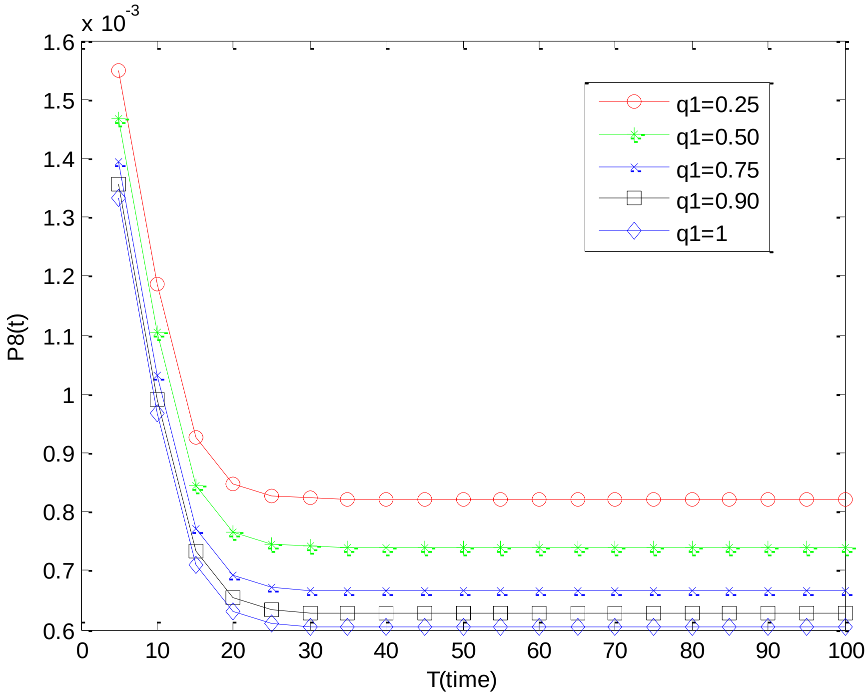

Figure 2. The effect of feedback rate, q_1 on the probability of the system

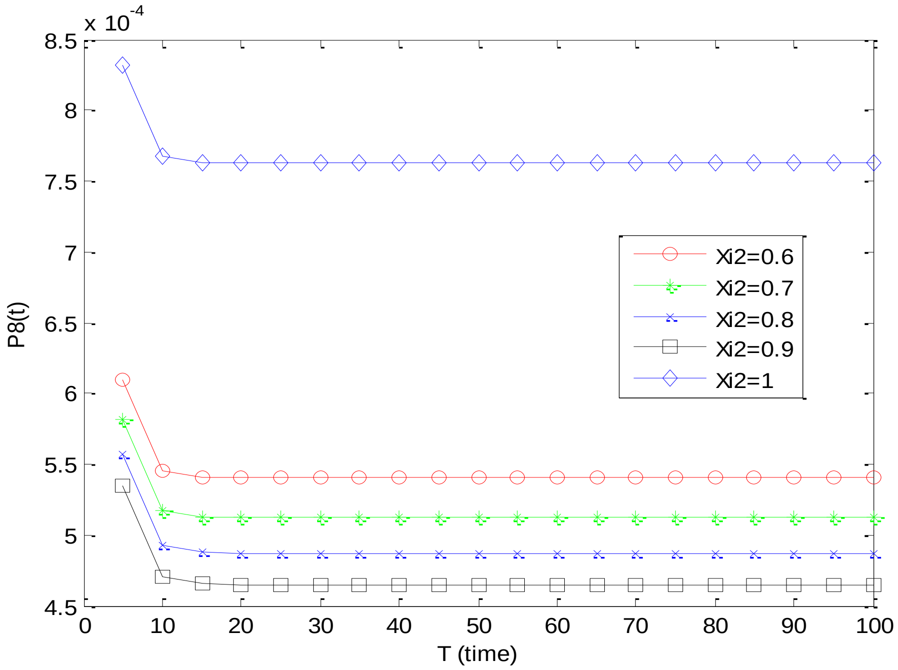

Figure 3. The effect of f retention of reneged rate, ζ_2 on the probability of the system

As demonstrated in the figures , $p_n(t)$ is an increasing with increasing values of α and its values are constant at reaches the steady-state, also $p_n(t)$ decreasing with increasing values of q_1, ζ_2 and its values are constant at reaches the steady-state. The system reaches the steady-state at t=40.

7- Special Cases

1- In the above described model selecting $q_1 = 1, 0 < \zeta_2 < 1,$ and $t \to \infty$, we shall reach to an $M/M/1/N$ queuing model with reneging and retention of reneged customers as studied by Rakesh Kumar and Sumeet [2].

2- At $q_2 = 0, 0 < q_1 < 1,$ and $t \to \infty$ the queuing system reduces to $M/M/1/N$ queuing model with feedback and reneging.

3- For $q_2 = 1(\alpha = 0), 0 < q_1 < 1,$ and $t \to \infty$ the queuing system reduces to $M/M/1/N$ queuing model with feedback.

4- selecting $q_2 = 1(\alpha = 0), q_1 = 1,$ and $t \to \infty$ the model will be approached which would be the system $M/M/1/N$ without any concepts.

7-Conclusions:

This paper studies a single server queuing model with feedback, discouraged arrivals, reneging and retention of reneged customers. We obtain the transient solution and steady state in explicit form by using a computable matrix technique. Some queuing systems are derived as special cases of this model. Some numerical results have been obtained to display the effect of some system parameters as reneging rate(α), feedback rate(q_1), and retention of reneged rate(ζ_2) on the probabilities of the system.

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