

**Military Technical College  
Kobry El-Kobbah,  
Cairo, Egypt.**



**13<sup>th</sup> International Conference  
on Applied Mechanics and  
Mechanical Engineering.**

## **A NEW PROCEDURE OF ROBOTS SPATIAL ROTATIONS TERMINAL CONTROL**

MILNIKOV<sup>1</sup> A.

### **ABSTRACT**

In the article, using spinor representation of orthogonal transformations, the expressions between second order complex unitary transformations matrixes and real orthogonal matrixes of spatial rotations in three dimensional Euclidean space  $L^3$  are received, that allows easily calculating of corresponding Euler's angles. The obtained results have enabled reducing the actually three-dimensional problem of spatial motion control to the one-dimensional problem; control kinematical functions of Euler's angles and control spinor matrix of rotation were constructed, by means of which control process of spatial rotations is completely determine.

### **KEY WORDS**

spinors; control spinor matrix, hermitian functionals; orthogonal transformations, Euler's angles, spatial rotations; terminal control

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<sup>1</sup> Doctor of Sciences, Ph.D., Professor, Dean of the Faculty of Computer Technologies and Engineering of International Black Sea University (D. Aghmashenebelis Kheivani 13km. #2 Tbilisi, Georgia Tel:(+995 32) 595005 Fax: (+995 32) 595008, [www.ibsu.edu.ge](http://www.ibsu.edu.ge))

## STATEMENT OF THE PROBLEM

Methods of representation of three-dimensional rotations used in solving of various engineering problems are usually confined to the description of individual concrete rotations centered at the origin (zero center). Among these methods is in particular the well known method of orthogonal real matrices whose elements are functions of Euler angles [1, 2]. At the same time it should be said that the problem of describing so-called generalized rotations [3] evokes a much greater interest both from the theoretical standpoint and from the standpoint of applications (in the first place we mean an application in robotics and in particular in the planning of trajectories in the case of obstacles). Under generalized rotations we mean the set of all possible rotations with both zero and nonzero centers which transform the initial three-dimensional point to the finite one. The basic problem arising in this context can be formulated as follows: Given two three-dimensional points  $x(x^1, x^2, x^3)$  and  $y(y^1, y^2, y^3)$ , it is required to define the set of all possible transformations and centers of rotations which bring about the transformation of the point  $x$  to the point  $y$ . It is obvious that this problem can be easily extended to the case where instead of two points we consider two finite sets of points  $\{x_i(x_i^1, x_i^2, x_i^3)\}$  and  $\{y_i(y_i^1, y_i^2, y_i^3)\}$   $i=1,2,\dots,m$ , which corresponds to the case of rotations of a solid.

## EQUATIONS FOR GENERALIZED ROTATIONS

Let  $L^3$  be a linear Euclidean space with orthonormalized basis vectors  $e_1, e_2, e_3$ . To each vector  $x=x^1e_1+x^2e_2+x^3e_3$  of the space  $L^3$  we assign a traceless Hermitian matrix

$$X = \begin{vmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{vmatrix}, \quad (1)$$

whose elements are the so-called spinor components of the vector  $x$  [4]. When we pass from the usual Euclidean components of the vector  $x$  to the spinor ones, we thereby identify the vector  $x$  with Hermitian functionals on the two-dimensional linear space  $C^2$  over the field of complex numbers  $C$  [4]. Denote by  $L(C^2)$  the set of all Hermitian functionals on  $C^2$  and consider the following decomposition

$$X = x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3, \quad (2)$$

where

$$\sigma_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \sigma_2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \sigma_3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

are Pauli matrices [4].

From decomposition (2) it follows that the set  $L(C^2)$  is a linear three-dimensional space over the field of real numbers and thus it can be identified with  $L^3$ . Note that to each basis

vector of the two-dimensional space  $C^2$  we can assign the basis elements  $\sigma_1, \sigma_2, \sigma_3$  of the space  $L(C^2)$  (and also the orthonormalized basis vector  $e_1, e_2, e_3$  due to the identification of  $L^3$  and  $L(C^2)$ ): each of the matrices  $\sigma_i$  is represented as some linear combination of tensor products of basis vectors of the space  $C^2$  [4]. The foregoing reasoning implies that for any matrix  $C \in C^2$ , which is a matrix of transformation between two basis vectors of the space  $C^2$ , there also exists a transformation matrix of the corresponding orthonormalized basis vectors in the space  $L^3$ .

**Proposition** The matrix of transformation of the basis elements in  $C^2$  is unitary.

**Proof.** If on the space  $C^2$  we consider Hermitian functionals of the form

$$X = \begin{vmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{vmatrix},$$

then they will correspond to the four-dimensional vectors of a pseudo-Euclidean space with signature (1,3) and with basis vectors

$$\sigma_0 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}; \sigma_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}; \sigma_2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}; \sigma_3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}. \quad (3)$$

Now, transformations of the basis vectors of the two-dimensional space  $C^2$  lead to transformations of the basis vectors (3), while the transformation matrices remain the same as in the case of functionals of form (1). The orthogonal complement  $\perp\sigma_0$  of the first basis vector  $\sigma_0$  is an anti-Euclidean space (because of the pseudo-Euclidean property of the space defined by vectors (3)) and, after changing the signs of the scalar products, a three-dimensional Euclidean space that coincides with  $L(C^2)$ . The narrowing of the action of matrices of basis vector transformation in  $C^2$  to the subspace  $\perp\sigma_0$  means that these matrices satisfy the condition  $\bar{C}^T \sigma_0 C = \sigma_0$ , i.e.  $C^{-1} = \bar{C}^T$ , Q.E.D.

The problem posed in Subsection 1 can be now reformulated in terms of the spinor space  $C^2$ : Given two traceless matrices of Hermitian functionals

$$X = \begin{vmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{vmatrix} \text{ и } Y = \begin{vmatrix} y^3 & y^1 - iy^2 \\ y^1 + iy^2 & -y^3 \end{vmatrix},$$

it is required to define:

- 1) a set of unitary matrices  $C = \begin{vmatrix} \bar{\alpha} & -\beta \\ \beta & \alpha \end{vmatrix}$  which satisfy the equality

$$Y = \bar{C}^T X C; \quad (4)$$

- 2) one-dimensional subspaces which are invariant with respect to transformations represented by matrices  $C$  (i.e. a set of respective rotation centers).

Note that since the transformation  $C$  is unitary, the vector norms defined by the determinants of matrices of the Hermitian functionals  $X$  and  $Y$  coincide and therefore (4) defines rotation.

From equality (4) we can obtain the following system of linear homogeneous equations with respect to the unknown variables  $\alpha$  and  $\beta$ :

$$\begin{aligned} x_3\alpha + (x_1 + ix_2)\beta &= y_3\alpha - (y_1 - iy_2)\bar{\beta} \\ (x_1 - ix_2)\alpha - x_3\beta &= y_3\beta + (y_1 + iy_2)\bar{\alpha} \end{aligned} \quad (5)$$

For arbitrary  $\alpha$ , a solution of (5) is given by

$$\beta = \frac{(x_1 - ix_2)\alpha - (y_1 - iy_2)\bar{\alpha}}{x_3 + y_3} \quad (6)$$

From (6) we have

$$\operatorname{Re} \beta = \beta_1 = \frac{\alpha_1(x_1 - y_1) + \alpha_2(x_2 + y_2)}{x_3 + y_3} \quad \text{and} \quad \operatorname{Im} \beta = \beta_2 = \frac{\alpha_2(x_1 + y_1) - \alpha_1(x_2 - y_2)}{x_3 + y_3} \quad (7)$$

Using the one of the properties of unitarity of the matrix  $C$  ( $\det C = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$ ), we can define either  $\alpha_1 = \operatorname{Re} \alpha$  or  $\alpha_2 = \operatorname{Im} \alpha$ . Note that one of these parameters remains arbitrary. Thus (6) defines rotation for  $\alpha \neq 0$  and  $x_3 + y_3 \neq 0$ .

The invariance of the rotation center  $z(z_1, z_2, z_3)$  with respect to the transformation  $C$  is written as a condition

$$\bar{C}^T Z C = Z,$$

whence we obtain

$$\beta_1 z_1 - \beta_2 z_2 = 0; \quad \alpha_2 z_2 - \beta_1 z_3 = 0; \quad \alpha_1 z_1 - \beta_2 z_3 = 0.$$

It is not difficult to verify that the determinant of this system considered for the unknown values  $z_1, z_2$  and  $z_3$  is identically zero and therefore for given  $\alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  ( $\alpha \neq 0$  and  $x_3 + y_3 \neq 0$ ) there always exist nontrivial solutions written in the form

$$z_1 = \frac{\beta_2}{\alpha_2} z_3; \quad z_2 = \frac{\beta_1}{\alpha_2} z_3, \quad (8)$$

where  $z_3$  is arbitrary.

Thus, (7) together with the normalization property define a generalized rotation transforming  $(x^1, x^2, x^3)$  to  $(y^1, y^2, y^3)$  with respect to the set of centers which is defined by (8).

### RELATIONS BETWEEN TRANSFORMATIONS IN $C^2$ AND $L^3$

We can establish the correspondence between the elements of the transformation matrix

$$C = \begin{vmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{vmatrix} \text{ in } C^2 \text{ and the elements of the orthogonal real matrix of rotation } A \text{ in } L^3.$$

The matrix  $A$  is, by definition, the matrix of transformation between two orthonormalized basis vectors of the space  $L^3$  and its rows are decompositions of the new basis vectors in terms of the initial basis vectors. Hence due to the identification of the spaces  $L(C^2)$  and  $L^3$  we have

$$\bar{C}^T \sigma_i C = a_i^{i'} \sigma_{i'} \quad (i, i' = 1, 2, 3), \tag{9}$$

where  $\sigma_i$  are the Pauli matrices corresponding to the initial basis,  $\sigma_{i'}$  are the Pauli matrices of the new basis, and  $a_i^{i'}$  are the elements of the matrix  $A^{-1}$ .

Equation (9) can be written explicitly in the form of three matrix equalities

$$a_1^1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + a_2^1 \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} + a_3^1 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = \begin{vmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{vmatrix} * \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} * \begin{vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{vmatrix},$$

$$a_1^2 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + a_2^2 \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} + a_3^2 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = \begin{vmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{vmatrix} * \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} * \begin{vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{vmatrix}.$$

$$a_1^3 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + a_2^3 \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} + a_3^3 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = \begin{vmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{vmatrix} * \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} * \begin{vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{vmatrix}$$

which readily yield the following expressions for calculating the elements of the matrix  $A$  by the elements of the matrix  $C$ :

$$\begin{aligned} a_1^1 &= (\alpha_1^2 - \alpha_2^2) - (\beta_1^2 - \beta_2^2); & a_2^1 &= 2(\alpha_1 \alpha_2 + \beta_1 \beta_2); & a_3^1 &= 2(\alpha_2 \beta_2 - \alpha_1 \beta_1); \\ a_1^2 &= 2(\beta_1 \beta_2 - \alpha_1 \alpha_2); & a_2^2 &= (\alpha_1^2 - \alpha_2^2) + (\beta_1^2 - \beta_2^2); & a_3^2 &= 2(\alpha_1 \beta_2 + \alpha_2 \beta_1); \\ a_1^3 &= 2(\alpha_1 \beta_1 + \alpha_2 \beta_2); & a_2^3 &= 2(\alpha_2 \beta_1 - \alpha_1 \beta_2); & a_3^3 &= (\alpha_1^2 + \alpha_2^2) - (\beta_1^2 + \beta_2^2). \end{aligned} \tag{10}$$

Expressions (10) enable to calculate the elements of the matrix  $A$  through the given coordinates of three points (initial, terminal and the center) which define rotation. On the other hand, taking into account that the matrix  $A$  can be written in the form [1]

$$A = \begin{vmatrix} \cos \varphi \cos \psi - \cos \theta \sin \varphi \sin \psi & -\cos \varphi \sin \psi - \cos \theta \sin \varphi \sin \psi & \sin \varphi \sin \theta \\ \sin \varphi \cos \psi + \cos \theta \cos \varphi \sin \psi & -\sin \varphi \sin \psi + \cos \theta \cos \varphi \cos \psi & -\cos \varphi \sin \theta \\ \sin \psi \sin \theta & \cos \psi \sin \theta & \cos \theta \end{vmatrix}, \quad (11)$$

where  $-\pi < \varphi \leq \pi$ ,  $0 \leq \theta \leq \pi$  и  $-\pi < \psi \leq \pi$  are Euler angles, it easily follows that expressions (10) enable to define Euler angles as well

$$\cos \theta = a_{33}; \quad \sin \varphi \sin \theta = a_{31} \quad \text{and} \quad \sin \psi \sin \theta = a_{13} \quad (12)$$

### CONTROL OF SPATIAL ROTATIONS

Having expressions (7) and (12), it is easy to calculate the Euler angles which ensure rotation of the point  $x(x^1, x^2, x^3)$  to the point  $y(y^1, y^2, y^3)$ . If it is assumed that to the initial point  $x(x^1, x^2, x^3)$  there correspond the zero Euler angles  $\theta_0 = \phi_0 = \psi_0 = 0$ , then the control of rotation consists in making a time-dependent change of the Euler angles from the initial values  $\theta_0; \phi_0; \psi_0$  to the terminal values  $\theta_f; \phi_f; \psi_f$  calculated by equations (12).

In a general form, the control process can be represented as change functions of the Euler angles  $\theta(t); \phi(t); \psi(t)$  which must satisfy the conditions

$$\begin{aligned} \theta(t_0) &= 0; \quad \phi(t_0) = 0; \quad \psi(t_0) = 0, \\ \theta(t_f) &= \theta_f; \quad \phi(t_f) = \phi_f; \quad \psi(t_f) = \psi_f, \end{aligned}$$

where  $t_0$  and  $t_f$  are the initial and terminal moments of time.

The above-said naturally implies the problem on defining the control functions  $\theta(t); \phi(t); \psi(t)$ . It should be emphasized that dependences  $\theta(t); \phi(t); \psi(t)$  have a kinematics character, since they take into account neither moments, nor elastities nor any other dynamic characteristics of the process and therefore, after defining them, there arises a problem of synthesizing – on the basis of these functions – the dynamic adaptive control. This issue is discussed in [5].

Figure 1 shows the fixed vectors  $x(x^1, x^2, x^3); y(y^1, y^2, y^3)$  and the intermediate rotating vector  $\xi(\xi^1, \xi^2, \xi^3)$  which at the initial moment of time  $t = t_0$  coincides with the initial rotation vector  $x(x^1, x^2, x^3)$  and, at the terminal moment of time  $t = t_f$ , with the terminal vector  $y(y^1, y^2, y^3)$ . The moving angle  $\gamma$  between the vectors  $x(x^1, x^2, x^3)$  and  $\xi(\xi^1, \xi^2, \xi^3)$  is equal, at the initial moment of time  $t = t_0$ , to zero and, at the moment of time  $t = t_f$ , to  $\gamma = \gamma_f$ , where

$$\gamma_f = \ar \cos\left(\frac{(x, y)}{|x|*|y|}\right) = \ar \cos\left(\frac{(x, y)}{|x|^2}\right)^2; (x, y) \text{ is the dot product of the vectors } x \text{ and } y. \text{ It is}$$

obvious that the moving angle between the vectors  $y(y^1, y^2, y^3)$  and  $\xi(\xi^1, \xi^2, \xi^3)$  is equal to  $\gamma_f - \gamma$ .

Let us define the coordinates of the vector  $\xi(\xi^1, \xi^2, \xi^3)$  assuming that it forms the angles  $\gamma$  and  $\gamma_f - \gamma$  with the vectors  $x(x^1, x^2, x^3)$  and  $y(y^1, y^2, y^3)$  and is located in their plane. To this end, we introduce the vector  $r = (x^2y^3 - x^3y^2, x^3y^1 - x^1y^3, x^1y^2 - x^2y^1)$  which is the cross product of the vectors  $x$  and  $y$ . Then the above conditions can be written in the form of the following system of linear equations:

$$\begin{aligned} (\xi, r) &= 0; \\ (\xi, x) &= |x|^2 \cos \gamma; \\ (\xi, y) &= |x|^2 \cos(\gamma_f - \gamma). \end{aligned} \tag{13}$$

It is not difficult to see that the vector  $\xi(\xi^1, \xi^2, \xi^3)$  defined from system (13) satisfies the following conditions:

1. for  $\gamma = 0$ ,  $\xi(\xi^1, \xi^2, \xi^3) = x(x^1, x^2, x^3)$ , which follows from the second equation of system (2.12), since in this case  $(\xi, x) = |x|^2$ , which is possible only provided that  $\xi(\xi^1, \xi^2, \xi^3) = x(x^1, x^2, x^3)$ ;
2. for  $\gamma = \gamma_f$ ,  $\xi(\xi^1, \xi^2, \xi^3) = y(y^1, y^2, y^3)$ , which follows from the third condition of system (13), since in this case  $(\xi, y) = |x|^2$ , which is possible only provided that  $\xi(\xi^1, \xi^2, \xi^3) = y(y^1, y^2, y^3)$ ;
3.  $|\xi| = |x| = |y|$ , which follows from the second and third equation of system (13).

Therefore the vector  $\xi(\xi^1, \xi^2, \xi^3)$  defined from system (13) corresponds to Fig. 1, i.e. it can actually be regarded as the vector rotating (condition 3) from the vector  $x(x^1, x^2, x^3)$  (condition 1) to the vector  $y(y^1, y^2, y^3)$  (condition 2). Note that in this case the angle  $\gamma$  changes in within  $0 \leq \gamma \leq \gamma_f$ .

The equations of system (13) can be written in the coordinate form as follows:

$$\begin{aligned} \xi^1 r^1 + \xi^2 r^2 + \xi^3 r^3 &= 0 \\ \xi^1 x^1 + \xi^2 x^2 + \xi^3 x^3 &= |x|^2 \cos \gamma \\ \xi^1 y^1 + \xi^2 y^2 + \xi^3 y^3 &= |x|^2 \cos(\gamma_f - \gamma). \end{aligned}$$

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<sup>2</sup>  $|x|*|y| = |x|^2$ , since for rotation  $|x| = |y|$ .

It is not difficult to see that its determinant is equal to

$$\Delta = \begin{vmatrix} r^1 & r^2 & r^3 \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix} = |r|^2.$$

Other determinants of Kramer's equations for system (13) will be equal to

$$\Delta_1 = \begin{vmatrix} 0 & r^2 & r^3 \\ |x|^2 \cos \gamma & x^2 & x^3 \\ |x|^2 \cos(\gamma_f - \gamma) & y^2 & y^3 \end{vmatrix} = |x|^2 ((\cos(\gamma_f - \gamma)(r^2 x^3 - r^3 x^2) - \cos \gamma(r^2 y^3 - r^3 y^2)));$$

$$\Delta_2 = \begin{vmatrix} r^1 & 0 & r^3 \\ x^1 & |x|^2 \cos \gamma & x^3 \\ y^1 & |x|^2 \cos(\gamma_f - \gamma) & y^3 \end{vmatrix} = |x|^2 ((\cos(\gamma_f - \gamma)(r^3 x^1 - r^1 x^3) - \cos \gamma(r^3 y^1 - r^1 y^3));$$

$$\Delta_3 = \begin{vmatrix} r^1 & r^2 & 0 \\ x^1 & x^2 & |x|^2 \cos \gamma \\ y^1 & y^2 & |x|^2 \cos(\gamma_f - \gamma) \end{vmatrix} = |x|^2 ((\cos(\gamma_f - \gamma)(r^1 x^2 - r^2 x^1) - \cos \gamma(r^1 y^2 - r^2 y^1)).$$

If we introduce the new vectors  $r_x = (r^2 x^3 - r^3 x^2; r^3 x^1 - r^1 x^3; r^1 x^2 - r^2 x^1)$  and  $r_y = (r^2 y^3 - r^3 y^2; r^3 y^1 - r^1 y^3; r^1 y^2 - r^2 y^1)$  which equal to the vector products  $[r \times x]$  and  $[r \times y]$ , respectively, then the coordinates of the rotating vector will be presented in the following form:

$$\xi^i(t) = \frac{|x|^2}{|r|^2} (\cos(\gamma_f - \omega t) r_x^i - \cos \omega t r_y^i) \quad (i = 1, 2, 3) \quad (14)$$

In these expressions, the angle  $\gamma$  is an independent variable and can be treated as time function, which means that the coordinates of the vector  $\xi(\xi^1, \xi^2, \xi^3)$  are also time functions. We would like to emphasize that the problem of synthesis of spatial motion control thus reduces to defining a function  $\gamma(t)$  of the concrete form, which is connected with the rotation process dynamics and was discussed in [5]. Here we assume that  $\gamma(t)$  is sufficiently smooth and satisfies the conditions  $\gamma(t = t_o) = 0$  u  $\gamma(t = t_f) = \gamma_f$ . For definiteness, we assume that  $\gamma(t) = \omega t$ , where  $\omega = 2\pi f$  is the constant angular velocity.

As has already been noted, the vector  $\xi(\xi^1, \xi^2, \xi^3)$  is a rotating vector and therefore at each moment of time it can be considered as a terminal vector of the current moment of the rotation process. If in equations (7) we replace the coordinates of the point  $y(y^1, y^2, y^3)$



by expressions (14), then we obtain representations of the parameters of the spinor matrix  $C$ , the orthogonal matrix  $A$  (11) and Euler angles (12) in terms of time functions. Thus, we obtain a time-dependent (kinematics) representation of the rotation of the point  $x(x^1, x^2, x^3)$  to the point  $y(y^1, y^2, y^3)$ . However first we should predetermine the matrix  $C$  so that at the initial moment of time the spinor equations of rotation (4) would have the form  $X = \bar{C}^T X C$ , which is evidently possible only if  $C$  is a unit matrix. This can be done by an appropriate choice of the parameters  $\alpha_1$  and  $\alpha_2$ .

Indeed, setting  $\alpha_1 = 1; \alpha_2 = 0$  we obtain

$$\operatorname{Re} \beta = \beta_1 = \frac{x^1 - y^1}{x^3 + y^3}; \operatorname{Im} \beta = \beta_2 = \frac{y^2 - x^2}{x^3 + y^3}.$$

By substituting the last equations and (14) for corresponding elements of the matrix  $C$ , it will take the following form:

$$C(t) = \frac{1}{1 + |\beta|^2} \begin{vmatrix} 1 & \frac{((\xi^1(t) - x^1) + i(\xi^2(t) - x^2))}{x^3 + \xi^3(t)} \\ \frac{((x^1 - \xi^1(t)) - i(x^2 - \xi^2(t)))}{x^3 + \xi^3(t)} & 1 \end{vmatrix}, \quad (15)$$

where  $|\beta|^2 = \frac{(x^1 - \xi^1(t))^2 + (x^2 - \xi^2(t))^2}{(x^3 + \xi^3(t))^2}$ . Note that  $\det C(t) = 1$  for any  $t$ .

It is obvious that at the initial moment of time  $t_0$  the matrix  $C(t = t_0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ , since in that case  $\gamma(t_0) = 0$  and  $\xi(\xi^1; \xi^2; \xi^3) = x(x^1, x^2, x^3)$ .

For  $t = t_f$  we have  $\gamma(t_f) = \gamma_f$ ,  $\xi(\xi^1; \xi^2; \xi^3) = y(y^1, y^2, y^3)$ .

From the above-said it follows that the obtained spinor matrix of rotation (15) is defined correctly. But in that case the Euler angles (12), too, are defined correctly. They also turn out to be the functions of time

$$\begin{aligned} \theta(t) &= \arccos\left(\frac{(x^3 + \xi^3(t))^2 - (x^1 - \xi^1(t))^2 - (x^2 - \xi^2(t))^2}{(x^3 + \xi^3(t))^2}\right); \\ \varphi(t) &= \arcsin\left(\frac{2(x^1 - \xi^1(t))}{(x^3 + \xi^3(t)) \sin \theta(t)}\right); \\ \psi(t) &= \arcsin\left(\frac{2(\xi^1(t) - x^1)}{(x^3 + \xi^3(t)) \sin \theta(t)}\right). \end{aligned} \quad (16)$$

Expressions (16) solve the problem we have formulated on defining the kinematics functions  $\theta(t); \phi(t); \psi(t)$ . On the other hand, it should be noted that the proposed theory allows one to reduce an actually three-dimensional problem of spatial motion control to a one-dimensional problem. Indeed, for this it is sufficient to synthesize in one way or another function  $\gamma(t)$  satisfying the corresponding boundary conditions [5]. Then it is obvious that the control process is completely defined by the spinor matrix of rotation (15) and the Euler angle functions (16).

## A NUMERICAL EXAMPLE

Let us consider a numerical example illustrating the above reasoning. Assume that the initial vector  $x(10, -45, 30)$  and the terminal vector  $y(1, 20, 51.225)$  are given arbitrarily.

The angle between them is equal to  $\gamma_f = \arccos\left(\frac{(x, y)}{|x|^2}\right) = 77.65^\circ$ . Assuming for the sake of

simplicity that  $\omega = 1$  with a step equal to  $\frac{\gamma_f}{3}$ , lets us calculate in two different ways three intermediate positions of the rotating vector  $\xi(\xi^1; \xi^2; \xi^3)$ . Using equations (14), we obtain the following coordinates of the rotating vector for three angle values (Table 1).

The procedure of verifying whether the Euler angles have been calculated consists in the following: using the obtained coordinates of the intermediate positions of the vector  $\xi(\xi^1; \xi^2; \xi^3)$ , for each of three angle values from Table 1 we should calculate the Euler angles by equations (16) and the three-dimensional orthogonal matrix  $A$  of the basic representation<sup>3</sup> (11) and then again the intermediate coordinates of the rotating vector by the equation  $\xi = Ax$ , where  $x$  is the initial rotation vector. The obtained values should coincide with those given in the Table 1. The results of the corresponding calculations are presented in Table 2.

The matrix  $A$  given in Eqn. (11) was calculated for the Euler angle values by equations (16) and given in column 2 of Table 2. The coordinate values of the rotating vector  $\xi(\xi^1; \xi^2; \xi^3)$  were calculated by multiplying matrix  $A$  by the initial rotation vector  $x(10, -45, 30)$ :  $\xi = Ax$ . From columns 4÷6 of Table 2 we see that the coordinates of the rotating vector coincide with the coordinates calculated by equations (14) (Table 1).

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<sup>3</sup> The term “basic representation” came from groups’ representation theory: describing physical rotation group by means of  $3 \times 3$  orthogonal matrices is called basic representation, meanwhile spinor representation used in the article is representation of weight  $\frac{1}{2}$  [6].

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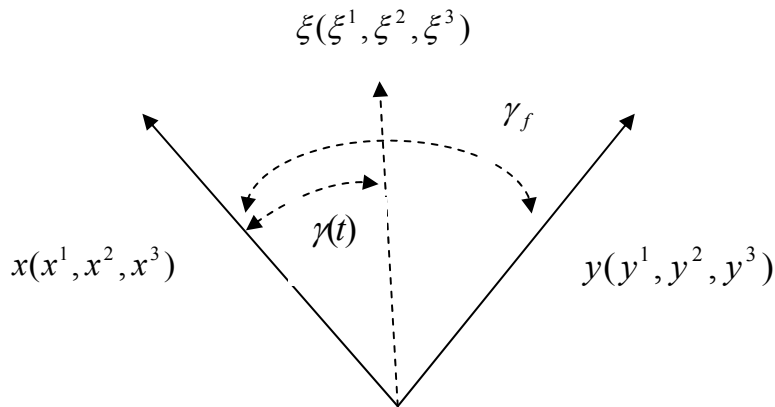
**Tables and Figures:**

**Table 1.** Calculation of spatial rotation states by means of spinor approach

Angle $\gamma$	Coordinates of rotating vector (equations (14))			Norm of $\xi$
	$\xi^1$	$\xi^2$	$\xi^3$	
$\frac{\gamma_f}{3} = 25.88^\circ$	8.48	-27.25	47.02	55
$\frac{2\gamma_f}{3} = 51.77^\circ$	5.2	-4.03	54.60	55
$\gamma_f = 77.65^\circ$	1	20	51.23	55

**Table 2.** Check-up calculation of spatial rotation states

Angle $\gamma$	Euler angles (equations (16))	Orthogonal matrix $A$ (equations (12) and (16))	Coordinates of rotating vector ( $\xi=Ax$ )		
			$\xi^1$	$\xi^2$	$\xi^3$
1	2	3	4	5	6
$\frac{\gamma_f}{3} = 25.88^\circ$	$\theta = 26.05^\circ$ $\varphi = 4.87^\circ$ $\psi = -4.87^\circ$	$A_1 = \begin{pmatrix} 0.999 & 8.59 \times 10^{-3} & 0.037 \\ 8.59 \times 10^{-3} & 0.899 & -0.438 \\ -0.037 & 0.438 & 0.898 \end{pmatrix}$	8.48	-27.25	47.02
$\frac{2\gamma_f}{3} = 51.77^\circ$	$\theta = 56.98^\circ$ $\varphi = 6.58^\circ$ $\psi = -6.58^\circ$	$A_2 = \begin{pmatrix} 0.995 & 0.044 & 0.09 \\ 0.044 & 0.621 & -0.783 \\ -0.09 & 0.783 & 0.616 \end{pmatrix}$	5.27	-4.03	54.60
$\gamma_f = 77.65^\circ$	$\theta = 77.87^\circ$ $\varphi = 7.88^\circ$ $\psi = -7.88^\circ$	$A_3 = \begin{pmatrix} 0.985 & 0.107 & 0.134 \\ 0.107 & 0.225 & -0.968 \\ -0.134 & 0.968 & 0.21 \end{pmatrix}$	1	20	51.23



**Fig.1.** The initial, terminal and intermediate vectors of spatial rotation.